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A topological duality for M_3-lattices

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Abstract. In this article we determine a topological duality for M_3-lattices, introduced by A. V. Figallo in the journal Rev. Colombiana de Matemática, XXI, 1987 ([3]). By means of this duality we describe the congruences and the subdirectly irreducible M_3-lattices and reach some of Figallo's results in a different way.

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1. Introduction

In this work, we extend the duality obtained by H. A. Priestley for bounded distributive lattices (see [8] and [9]), known as *Priestley duality*, to the case of bounded M_3-lattices, showing that there exists a duality between the category whose objects are the bounded M_3-lattices and whose morphisms are the homomorphisms in the variety of the bounded M_3-lattices, and the category of M_3-spaces and M_3-functions.

By means of this duality we have managed to characterize the congruence lattice of an M_3-lattice in terms of certain closed subsets of its associated M_3-space, showing that there is an isomorphism between the lattice of the congruences and the dual lattice of certain closed subsets of its associated Priestley space, more precisely the closed and $\triangle$-involutive subsets.

Given that any variety of algebras is determined by its subdirectly irreducible algebras and what Birkhoff's Theorem states, that *Every non-trivial algebra A is isomorphic to a subdirect product of subdirectly irreducible algebras, each of which is a homomorphic image of A*, it is important to have their characterization. In this work we determine the simple and subdirectly irreducible M_3-lattices by using the characterization of the congruence lattice obtained and reach the same results as those achieved by Figallo in an algebraic way.

This article has been organized as follows. In Section 2 we introduce the definition and properties of M_3-lattices given by Figallo as well as some basic definitions of Priestley's duality. In Section 3 we describe a duality for M_3-lattices, starting with a study of the properties of M_3-lattice prime spectrum, which later allowed us to define the category of M_3-spaces and M_3-morphisms. Section 4 is devoted to the study of congruences and the determination of the simple and the subdirectly irreducible algebras, concluding that these algebras coincide, for which reason the variety is semisimple.

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2. Preliminaries

The consideration of the class of M_3−lattices, closely related to the class of trivalent Lukasiewicz algebras, was motivated by its possible application to the study of the behavior of certain trivalent switching electric circuits and was defined by A. V. Figallo in *Los M_3−Reticulados* [3], Rev. Colombiana de Matemática, XXI, 1987, as follows.

An M_3−lattice is an algebra $\langle L, \wedge, \vee, \sim, \triangle, 0\rangle$ of type $(2,2,1,1,0)$ such that the reduct $\langle L, \wedge, \vee, 0\rangle$ is a distributive lattice with first element 0 which satisfies the following identities:

(M1) $\triangle(x \wedge \sim x) = 0$,
(M2) $\sim\sim x = x$,
(M3) $x = \triangle x \vee \sim \nabla x$, where ∇x is an abbreviation of $x \vee \sim x$,
(M4) $\triangle x = \triangle x \vee \sim \triangle x$,
(M5) $\triangle \nabla x = \nabla x$,
(M6) $\triangle(x \vee y) = \triangle x \vee \triangle y$,
(M7) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

If $\langle L, \wedge, \vee, \sim, \triangle, 0\rangle$ is an M_3−lattice such that the reduct $\langle L, \wedge, \vee\rangle$ is a distributive lattice with last element, we will say that it is a bounded M_3−lattice.

We will denote the variety of bounded M_3−lattices by $\mathbf{M_3}$ and, where no doubt should arise, we will represent any M_3−lattice by its support set.

The following properties of the M_3−lattices were proved by A. V. Figallo in [3], some of which are directly derived from the axioms:

(M8) $\triangle x \leq x$, (M9) $\sim \nabla x \leq x$,
(M10) $\sim \triangle x \leq \triangle x$,
(M11) $x \leq \nabla x$,
(M12) $\sim x \leq \nabla x$,
(M13) $\triangle(x \wedge \sim x) = 0$ is the first element of the lattice $\langle L, \wedge, \vee\rangle$,
(M14) $\nabla \sim x = \nabla x$,
(M15) $\nabla\nabla x = \nabla x$,
(M16) $\nabla \triangle x = \triangle x$,
(M17) if $x \leq y$, then $\triangle x \leq \triangle y$ and $\nabla x \leq \nabla y$,
(M18) $\triangle \sim \nabla x = 0$,
(M19) $\triangle \sim \triangle x = 0$,
(M20) $\triangle\triangle x = \triangle x$,
(M21) $\nabla x = \triangle x \vee \triangle \sim x$,
(M22) $\sim x = x$, if and only if, $x = 0$,
(M23) $\triangle 0 = \nabla 0 = \sim 0 = 0$.

In the same work, A. V. Figallo defined the notion of invariant element as follows: an element a of an M_3−lattice L is said to be invariant if it verifies that $\triangle a = a$, and represented the set of all invariant elements of an M_3−lattice L by $K(L)$.

He also showed that the set $K(L)$ is closed under the operations $\wedge$, $\vee$, $\triangle$ and ∇, and proved the following properties:

(M24) $\nabla(x \vee y) = \nabla x \vee \nabla y$,
(M25) $\triangle(x \wedge y) = \triangle x \wedge \triangle y$,
(M26) (Principle of determination) if $\triangle x = \triangle y$ and $\nabla x = \nabla y$, then $x = y$,
(M27) if $\triangle x \leq \triangle y$ and $\nabla x \leq \nabla y$, then $x \leq y$,
(M28) $\sim (x \vee y) \leq \sim x \vee \sim y$,

(M29) $\sim x \wedge \sim y \leq \sim (x \wedge y)$.

Then, for the purposes of finding a representation theorem for M_3-lattices, he introduced the notion of (prime) n-ideal of an M_3-lattice L as a (prime) ideal N of L which verifies whether $x \in N$, then $\sim x \in N$, or, similarly, $x \in N$ implies $\nabla x \in N$, and proved that, if $I(X)$ and $N(X)$ represent the ideal and the n-ideal generated by a subset X of L, respectively, then the following properties are verified:

(M30) $N(X) = I(K(X))$, where $K(X) = \{\nabla x : x \in X\} = \{\triangle x : x \in X\}$.

(M31) If M is an $n-$ideal of L and $a \in L$, then $N(M \cup \{a\}) = \{z \in L : z \leq \nabla(u \vee a)$ for some $u \in M\}$.

He also proved that the set $E(L) = \{M \subseteq L : M$ is a prime n-ideal$\}$, called the prime spectrum of L, verifies that $\bigcap\{M : M \in E(L)\} = \{0\}$. He gave, in addition, a characterization of the prime n-ideals as follows:

(M32) M is a prime n-ideal of L, if and only if, there exists a prime ideal I such that $M = I \cap \sim I$, where $\sim I = \{\sim x : x \in I\}$, and proved the following results:

(M33) L is a simple M_3-lattice; then, the only prime n ideal of L is $\{0\}$ and, therefore, L is isomorphic to $\langle \mathrm{T}, \wedge, \vee, \sim, \triangle, 0\rangle$, where T is the three-element chain $\{0, 1/2, 1\}$ with $0 \leq 1/2 \leq 1$, the operations $\sim$ and $\triangle$ being defined in the following chart:

x	$\sim x$	$\triangle x$
0	0	0
1/2	1	0
1	1/2	1

(M34) Representation Theorem: Every non-trivial M_3-lattice L is isomorphic to an M_3-sublattice of $\mathrm{T}^{E(L)}$, $\mathrm{T}^{E(L)}$ being the M_3-lattice of all the functions of $E(L)$ in T, where the operations are defined pointwise and T is the three-element chain given as in (M33).

In a later work ([5]) A. V. Figallo defined:

(M35) $x|y = x \wedge (\sim (x \wedge \nabla y) \vee \sim (y \vee \sim x))$,

and proved that if $\langle L, \wedge, \vee, \sim, , 0\rangle$ is an M_3-lattice, then $\langle L, \wedge, \vee, 0, 1\rangle$ is a Browerian algebra, which allowed him to characterize the congruences as follows:

(M36) If N is an n-ideal of an M_3-lattice L, then $R(N) = \{(x, y) \in L^2 : (x|y) \vee (y|x) \in N\}$ is an $\mathbf{M_3}-$congruence and, conversely, for each $\mathbf{M_3}-$congruence R of L, there is an n-ideal N such that $R = R(N)$.

In this same work he also highlighted the importance of the lattice $K(L)$ when he proved that, if L is an M_3-lattice, then $K(L)$ is a generalized Boolean algebra. These results allowed him to prove that:

(M37) In the M_3-lattices the notions of maximal $n-$ideal, prime $n-$ideal, irreducible $n-$ideal and completely irreducible $n-$ideal coincide.

Another important result to point out in [5] is that, if in $\langle \mathrm{T}, \wedge, \vee, \sim, , 0\rangle$, the algebra indicated in (M33), the operations $\to$ and $\neg$ are defined by the formulas that follow:

(i) $x \to y = \triangle \sim (x \vee \sim 1) \vee y$

(ii) $\neg x = \triangle \sim (\nabla x \vee \sim 1)$,

the corresponding charts of which are:

$\to$	0	1/2	1
0	1	1	1
1/2	1	1	1
1	0	1/2	1

x	$\neg x$
0	1
1/2	0
1	0

and from which the operations $\Diamond$ and $-$ are defined by the formulas:
(iii) $\Diamond x = \neg\neg x$,
(iv) $-x = (\neg\neg x \to x) \wedge (x \to \neg x)$.

The algebra $\langle \mathrm{T}, \wedge, \vee, -, \Diamond, 0, 1\rangle$ is a trivalent Lukasiewicz algebra in the sense of [7].

From this result, taking into account the representation theorem given in (M34), we can insure that, if $\langle L, \wedge, \vee, \sim, , 0\rangle$ is an M_3−lattice with last element 1, and $\Diamond$ and $-$ are the operations indicated in (iii) and (iv), respectively, then $\langle L, \wedge, \vee, -, \Diamond, 0, 1\rangle$ is a centered trivalent Lukasiewicz algebra, with center $c = \sim 1$.

The definitions and results used in this volume with respect to the notion of distributive lattice and universal algebra may be extended in [1] and [2, 6], respectively. For the purposes of facilitating the reading of the text and establishing the notions we will use, we present some general notions. If X is a set, we will indicate the power set of X by $\mathcal{P}(X)$. If $(X, \leq)$ is a partially ordered set and $C \subseteq X$, by $(C]$ ($[C)$) we will represent the set of all the elements $x \in X$ such that $x \leq y$ ($y \leq x$) for some $y \in C$, and we will say that C is decreasing (increasing) if $C = (C]$ ($C = [C)$. By $max\, X$ ($min\, X$) we will indicate the set of the maximal elements (minimal elements) of X. Some of the properties of the increasing and decreasing sets we will use in this article are the following:

(i) C is increasing (decreasing), if and only if, $X \setminus C$ is decreasing (increasing).
(ii) If $\{C_i\}_{i \in I}$ is a family of increasing (decreasing) sets, then $\bigcap\limits_{i\in I} C_i$ and $\bigcup\limits_{i \in I} C_i$ are increasing (decreasing) sets.

In general, we will denote the set of all the congruences on an algebra A by $Con(A)$. If $\mathbf{K}$ is a class of algebras and $A \in \mathbf{K}$, we will indicate the set of the congruences on A by $Con_{\mathbf{K}}(A)$ or call them $\mathbf{K}$−congruences in order to highlight the class of algebras we are considering.

Even though the theory of Priestley spaces and its relation with distributive lattices with first and last element is well known, we will present some results. For more details about this subject the reader is referred to [8] and [9].

A totally order-disconnected topological space is a triple $(X, \tau, \leq)$ such that $(X, \leq)$ is an ordered set, (X, τ) is a topological space and, given $x, y \in X$ such that $x \not\leq y$, there exists an open, closed and increasing set $U \subseteq X$ such that $x \in U$ and $y \notin U$.

A Priestley space is a compact topological space and totally order-disconnected.

Let I be any set. If we consider the two-element chain $\mathbf{2} = \{0, 1\}$ with the discrete topology, then by Tychonoff's theorem, $\mathbf{2}^I = \{f : I \longrightarrow \mathbf{2}\}$ is a Priestley space with the product topology and the natural order of functions ($f \leq g$ iff $f(x) \leq g(x)$ for every $x \in I$).

If L is a bounded distributive lattice, it is not difficult to prove that the set $Hom(L, \mathbf{2})$ of the bounded homomorphisms of L onto $\mathbf{2}$ is a closed subset of $\mathbf{2}^L$ and, as a consequence, a Priestley space. From this result, it can be deduced that $\mathcal{I}_p(L)$, the set of the prime ideals of L, ordered by the inclusion relation and endowed with the topology τ, which has as sub-basic set the elements of the set $\sum = \{\sigma_L(a) : a \in L\} \cup \{\mathcal{I}_p(L) \setminus \sigma_L(a) : a \in L\}$, where

(A1) $\sigma_L(a) = \{I \in \mathcal{I}_p(L) : a \notin I\}$, for each $a \in L$,

is a Priestley space, called the Priestley space of L or dual of L. In addition, the function $\sigma_L : L \longrightarrow D(\mathcal{I}_p(L))$ is an isomorphism of bounded lattices.

If X is a Priestley space and we denote the bounded distributive lattice of the open, closed and decreasing subsets of X by $D(X)$, the function $\varepsilon_X : X \longrightarrow \mathcal{I}_p(D(X))$, defined by

(A2) $\varepsilon_X(x) = \{V \in D(X) : x \notin V\}$, for each $x \in X$,

is a homeomorphism and an order isomorphism.

This indicates that every bounded distributive lattice can be considered as the lattice of the open, closed and decreasing sets of a Priestley space and that every Priestley space can be considered as the set of the prime ideals of a bounded lattice.

If we denote by $\boldsymbol{\mathcal{P}}$ the category whose objects are the Priestley spaces or $P-$spaces and whose morphisms are the continuous and increasing functions or $P-$functions, and by $\boldsymbol{\mathcal{L}}$, the category whose objects are the bounded distributive lattices and whose morphisms are the lattice bounded homomorphisms, then, in the language of the category theory, the isomorphisms mentioned above define a dual equivalence between $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{P}}$, which is usually called *Priestley duality.*

The following results will be frequently used in this work. The proof of (i) may be consulted in [11].

(i) Let X be a Priestley space and S, a closed subset of X. For each $x \in S$, there exists $z \geq x$ $(z \leq x)$ such that $z \in maxS$ $(z \in minS)$, in particular, $maxS \neq \emptyset$ $(minS \neq \emptyset)$ if $S \neq \emptyset$.

(ii) Let X be a Priestley space and $x \in X$. The sets $U_z = \{y \in X : y \not\leq x\}$ and $V_z = \{y \in X : y \not\geq x\}$ are open.

3. A topological duality for M_3-lattices

In this section we extend Priestley duality for bounded distributive lattices to the case of bounded M_3-lattices.

3.1. Properties of the prime spectrum of an M_3-lattice.

Now we will see some properties of the prime spectrum of an M_3-lattice used in the development of the section in order to obtain the duality. In what follows we will denote the family of the prime n-ideals of an M_3- lattice L by $\mathcal{NI}_p(L)$.

Lemma 3.1. *Let L be an M_3-lattice and, for each $I \in \mathcal{I}_p(L)$ let $I_\nabla = I \cap \sim I$ and $I_\triangle = \triangle^{-1}(I)$. Then, the following properties are verified:*

(I1) $I_\nabla \in \mathcal{NI}_p(L)$,

(I2) $I_\triangle \in \mathcal{I}_p(L)$,

(I3) $I_\nabla \subseteq I \subseteq I_\triangle$,

(I4) *if* $I \in \mathcal{NI}_p(L)$, *then* $I_\nabla = I \subset I_\triangle$, *and, in addition,* $I_\triangle \in \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L)$,

(I5) *if* $I \in \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L)$, *then* $I_\nabla \subset I$.

Proof.

(I1) It immediately results from (M32). On the other hand, it is easy to prove (I2) taking into account the axioms for M_3-lattice.

(I3) From the definition of I_∇ it is immediate that $I_\nabla \subseteq I$. In addition, if $x \in I$, then by (M8), $\triangle x \in I$; therefore, $x \in I_\triangle$, from which we conclude that $I \subseteq I_\triangle$.

(I4) If $I \in \mathcal{NI}_p(L)$, then by (M32), $I = I_\nabla$, and as I is a proper n-ideal, by (M2), there exists $a \in L \setminus I$ such that $\sim a \notin I$; in addition, as I is a prime ideal, (1) $a \wedge \sim a \notin I$. On the other hand, taking into account (M1), $\triangle(a \wedge \sim a) \in I$ is verified, then (2) $a \wedge \sim a \in I_\triangle$. Therefore, from (I3), (1) and (2), we have that

$I \subset I_{\triangle}$, and as by(M37), I is a maximal n-ideal, $I_{\triangle} \in \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L)$ must be verified; otherwise, $I_{\triangle} = L$ should hold, which would contradict the fact that $I_{\triangle}$ is a proper ideal.

(I5) If $I \in \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L)$, then there exists $x \in I$ such that $\sim x \notin I$ and, therefore, $x \notin I_{\nabla}$. Then, $I_{\nabla} \subset I$ results by (I3).

□

Lemma 3.2. *If L is an M_3-lattice, then the following properties are verified:*
(I6) $min\,\mathcal{I}_p(L) = \mathcal{NI}_p(L)$,
(I7) $max\,\mathcal{I}_p(L) = \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L)$,
(I8) $I \in max\,\mathcal{I}_p(L)$, *if and only if,* $I = I_{\triangle}$.

Proof.
(I6) Let $I \in min\,\mathcal{I}_p(L)$, and suppose that I is not an n-ideal, then by (M32), $M = I \cap \sim I$ is a prime ideal of L, such that $M \subset I$; this contradicts that $I \in min\,\mathcal{I}_p(L)$, then $I \in \mathcal{NI}_p(L)$.
Conversely, let $N \in \mathcal{NI}_p(L)$, and suppose that $N \notin min\,\mathcal{I}_p(L)$, then there exists $I \in \mathcal{I}_p(L)$ such that $I \subset N$. Let $M = I \cap \sim I$, then by (M32), $M \in \mathcal{NI}_p(L)$, and as by (M37), M is a maximal n-ideal such that $M \subseteq I \subset N$, then $N = L$ must be verified, which contradicts the fact that N is proper.

(I7) If $I \in \mathcal{NI}_p(L)$, then by (I4) and (I2), $I \notin max\,\mathcal{I}_p(L)$, from which $max\,\mathcal{I}_p(L) \subseteq \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L)$ follows. Conversely, let $I \in \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L)$, then by (I5), $I_{\nabla} \subset I$. Suppose that $I \notin max\,\mathcal{I}_p(L)$, $R \in \mathcal{I}_p(L)$ such that (1) $I \subset R$. Let $N = N(I)$; then from (1), there exists (2) $t \in R \setminus I$, such that (3) $t \notin N$. Indeed, if $t \in N$, then by (M30), there exists $x \in I$ such that $t \leq \triangle x$, from which, by (M8), $t \in I$, which contradicts (2). Therefore, from (2) and (3), $I \subset N$ results. On the other hand, from (M37), (I1) and (I5), we have that I_{∇} is a maximal n-ideal such that $I_{\nabla} \subset I \subset N$; from this it follows that $N = L$, which contradicts (3).

(I8) If $I \in max\,\mathcal{I}_p(L)$, then by (I2) and (I3), $I = I_{\triangle}$. Conversely, let now $I \in \mathcal{I}_p(L)$ such that (1) $I = I_{\triangle}$. Suppose that $I \notin max\,\mathcal{I}_p(L)$, then there exists $R \in \mathcal{I}_p(L)$ such that (2) $I \subset R$; as a consequence, we obtain(3) $I_{\nabla} \subseteq R_{\nabla}$, and there exists (4) $x \in R \setminus I$, from which we conclude that (5) $\triangle x \notin I$, taking into account (1). On the other hand, by (M1) and (M25), $\triangle x \wedge \triangle \sim x \in I$, then from (5) and, as I is a prime ideal, $\triangle \sim x \in I$ is verified, from which $\sim x \in I_{\triangle}$ results; therefore, from (1), (2) and (M2), we have (6) $x \in \sim R$.
Taking into account (4) and (6), we have that $x \in R_{\nabla} \setminus I_{\nabla}$; consequently, from (3), it follows that $I_{\nabla} \subset R_{\nabla}$, which contradicts, by (I1) and (M37), that I_{∇} is a maximal n-ideal.

□

Theorem 3.3. *Every prime ideal I of an M_3-lattice is a member of one, and only one, chain of two-element prime ideals, which is precisely $I_{\nabla} \subset I_{\triangle}$.*

Proof. If $I \in \mathcal{I}_p(L)$, then by (I4), (I7) and (I8), we have that $I = I_{\triangle}$ or $I = I_{\nabla}$, with $I_{\nabla} \subset I_{\triangle}$; in addition, $I_{\nabla} \in min\,\mathcal{I}_p(L)$, and $I_{\triangle} \in max\,\mathcal{I}_p(L)$. If $R \in \mathcal{I}_p(L)$ is such that (1) $I \subset R$, then by (I6), $R \notin \mathcal{NI}_p(L)$, from which, by (I7) and (I8), (2) $R = R_{\triangle}$ results. On the other hand, from (1), $I_{\triangle} \subseteq R_{\triangle}$, and as $I_{\triangle}$ is maximal on $\mathcal{I}_p(L)$, (3) $I_{\triangle} = R_{\triangle}$ is deduced. Therefore, from (2) and (3), $R = I_{\triangle}$is obtained in this case. If (4) $R \subset I$, then by (I7), $R \notin \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L)$, from which it follows, by (I4), that

(5) $R = R_\nabla$; in addition, (6) $R_\nabla \subseteq I_\nabla$. As, by (I1) and (I6), I_∇ is minimal element in $\mathcal{I}_p(L)$, then from(5) and (6), $I_\nabla = R$. Therefore, I is a member of one, and only one, two-element chain, which is $I_\nabla \subset I_\triangle$. □

Corollary 3.4. *If L is an M_3-lattice, then the set $\mathcal{I}_p(L)$, ordered by the relation of inclusion, is the cardinal sum of totally ordered sets, each of which has two elements.*

Proof. It is a direct consequence of Theorem 3.3. □

3.2. The category of M_3-spaces and M_3-morphisms.

With the objective of making clear a topological duality for bounded M_3-lattices and taking into account that described in Section 2 for bounded distributive lattices, in this section we will introduce the category $\mathfrak{M}_3$, define its objects and morphisms and give some of their properties.

Definition 3.1. An M_3-space is a triple $(X, \tau, \leq)$ such that:
(MP1) $(X, \tau, \leq)$ is a $P-$space,
(MP2) $(X, \leq)$ is the cardinal sum of a family of chains, each of which has exactly two elements,
and for each $U \in D(X)$ the following are satisfied:
(MP3) $(M_X U]$ is open and closed in X,
(MP4) $[m_X U) \setminus M_X U$ is open and closed in X,
where $M_X U = max\, X \cap U$ y $m_X U = min\, X \cap U$.

Remark 3.1. If $(X, \tau, \leq)$ is an M_3-space, then from (MP1) and (MP2) we have that:
(i) $min\ X \cup max\ X = X$,
(ii) $min\ X \cap max\ X = \emptyset$,
(iii) the order-connected components have two elements.

Definition 3.2. Let $(X, \tau, \leq)$ and $(X', \tau', \leq')$ be M_3-spaces. An M_3-function of $(X, \tau, \leq)$ in $(X', \tau', \leq')$ is a $P-$function $h : X \longrightarrow X'$ such that for every $V \in D(X')$ the following is verified:
(i) $(M_X h^{-1}(V)] = h^{-1}((M_{X'} V])$,
(ii) $[m_X h^{-1}(V)) \setminus M_X h^{-1}(V) = h^{-1}([m_{X'} V) \setminus M_{X'} V)$.

We will denote the category of M_3-spaces and M_3-functions by $\mathfrak{M}_3$, and the category of bounded M_3-lattices and the $\mathbf{M_3}-$homomorphisms by $\boldsymbol{\mathcal{M}_3}$.

Lemma 3.5. *Let $(X, \tau, \leq)$ be an object in $\mathfrak{M}_3$. If, for each $U \subseteq X$, we define:*
(D) $\triangle^* U = (M_X U]$,
(N) $\neg U = [m_X U) \setminus M_X U$,
(B) $\nabla^* U = U \cup \neg U$,
then, for every $U, V \in D(X)$, the following are verified:
(MP5) $\triangle^* U, \neg U, \nabla^* U \in D(X)$,
(MP6) $\triangle^* U \subseteq U$,
(MP7) $U \subseteq \nabla^* U$,
(MP8) $\neg U = m_X U \cup \{x \in X \setminus U : \text{there exists } t \in U \text{ and } \ t < x\}$,
(MP9) $m_X \neg U = m_X U$,
(MP10) $m_X U = U \cap \neg U$,
(MP11) $M_X U = \{y \in X \setminus \neg U : \text{there exists } v \in \neg U \text{ and } v < y\}$,

(MP12) $\{x \in X \setminus \triangle^* U : \text{there exists } t \in \triangle^* U \text{ and } t < x\} = \emptyset$,
(MP13) $\{x \in X \setminus \nabla^* U : \text{there exists } t \in \nabla^* U \text{ and } t < x\} = \emptyset$,
(MP14) $U \cup \neg U = U \cup \{x \in X \setminus U : \text{there exists } t \in U \text{ and } t < x\}$,
(MP15) $\neg \nabla^* U = m_X U$,
(MP16) $M_X U \subseteq M_X \nabla^* U$,
(MP17) $U \subseteq V \Rightarrow \nabla^* U \subseteq \nabla^* V$.

Proof. (MP5): If $U \in D(X)$, then by (MP3) and (MP4), $\triangle^* U, \neg U, \nabla^* U$ are open and closed in X. It only remains to prove that $\neg U$ is decreasing. Let $u \in \neg U$ and $v \in X$ such that (1) $v < u$. Then by (MP2), $u \notin min\ X$, from which $u \notin m_X U$ follows, and by (N) there exists (2) $t \in m_X U \setminus M_X U$ such that (3) $t < u$. From (1), (3) and (MP2), $t = v$ results, from which we obtain that $v \in \neg U$, taking into account (2) and (N).
(MP6): If $x \in \triangle^* U$, then by (D) there exists $u \in M_X U$ such that $x \leq u$. As U is decreasing, we have that $x \in U$.
(MP7) and (MP8): they are immediate consequences of (B) and (N), respectively.
(MP9): By (MP8) $m_X U \subseteq \neg U$, and as $m_X U \subseteq min\, X$, then it is immediate that $m_X U \subseteq m_X \neg U$. On the other hand, from (MP8) and (MP2), it follows that $m_X \neg U \subseteq m_X U$.
(MP10): It is a consequence of (MP8).
(MP11): Let $m \in M_X U$, then by (MP8) and (MP2), $m \in U \setminus \neg U$, and there exists $z \in min\, X$ such that $z < m$. As U is decreasing, then $z \in m_X U$, from which it follows, by (MP8), that $z \in \neg U$. From this we conclude that $m \in \{y \in X \setminus \neg U : \text{there exists } v \in \neg U \text{ y } v < y\}$.

The other inclusion is valid too. Let $y \in \{y \in X \setminus \neg U : \text{there exists } v \in \neg U \text{ y } v < y\}$. Therefore, (1) $y \in X \setminus \neg U$ and there exists (2) $v \in \neg U$, which verifies (3) $v < y$. Then, from (2), (3) and (MP2), $v \in m_X \neg U$, from which it follows, by (MP9), that (4) $v \in m_X U \subseteq U$. Therefore, from (3), (4) and (MP8), if $y \notin X \setminus U$, we would have that $y \in \neg U$, which would contradict (1); therefore, $y \in M_X U$.
(MP12): Suppose there exist (1) $x \in X \setminus \triangle^* U$ and (2) $t \in \triangle^* U$ such that (3) $t < x$. By (3) and (MP2), $t \notin max\, X$; therefore, $t \notin M_X U$, from which it follows, by (2), that there exist $z \in M_X(U)$ such that (4) $t < z$. Then, by(3), (4) and (MP2), $z = x$. From this $x \in \triangle^* U$ results, which contradicts (1).
(MP13): Suppose there exist (1) $x \in X \setminus \nabla^* U$ and (2) $t \in \nabla^* U$, such that (3) $t < x$. From (2), if $t \in U$, then from (3) and (MP8), as $x \notin U$ we have that $x \in \neg U$; therefore, $x \in \nabla^* U$, which contradicts (1). If $t \in \neg U$, as by (1), $x \notin \neg U$, then from(3) and (MP11), $x \in M_X U$, from which we conclude that $x \in U$; therefore, $x \in \nabla^* U$, which contradicts (1).
(MP14) is immediate from (MP8); (MP15) is a consequence of (MP8), (MP9) and (MP13); in addition, (MP16) and (MP17) are immediate consequences of (B).

□

Lemma 3.6. *Let X and X' be M_3−spaces and let $h : X \longrightarrow X'$ be a function. Then, the following conditions are equivalent:*
(i) *h is an isomorphism in $\mathfrak{M}_3$,*
(ii) *h is a homeomorphism and an order-isomorphism which satisfies the following conditions:*
 (a) *$\triangle^* h^{-1}(V) = h^{-1}(\triangle^* V)$, for every $V \in D(X')$,*

(b) $\neg h^{-1}(V) = h^{-1}(\neg V)$, *for every* $V \in D(X')$.

Proof. (i) $\Rightarrow$ (ii)

Let $h : X \longrightarrow X'$ be an isomorphism in $\mathfrak{M}_3$, then h is a morphism in the category $\mathfrak{M}_3$ and there exists $g : X' \longrightarrow X$, morphism in $\mathfrak{M}_3$, such that $h \circ g = 1_{X'}$ and $g \circ h = 1_X$. As a result, h and g are continuous and increasing functions which verify the following additional conditions:

(a1) $(M_X h^{-1}(V)] = h^{-1}((M_{X'}V])$, for every $V \in D(X')$,

(a2) $[m_X h^{-1}(V)) \setminus M_X h^{-1}(V) = h^{-1}([m_{X'}V) \setminus M_{X'}V)$, for every $V \in D(X')$,

(a3) $(M_{X'}g^{-1}(U)] = g^{-1}((M_X U])$, for every $U \in D(X)$,

(a4) $[m_{X'}g^{-1}(U)) \setminus M_{X'}g^{-1}(U) = g^{-1}([m_X U) \setminus M_X U)$, for every $U \in D(X)$.

Therefore, h is an isomorphism in the category $\boldsymbol{\mathcal{P}}$; consequently, h is an order isomorphism and a homeomorphism. On the other hand, from conditions (a1), (a2) and Lemma 3.5, (a) and (b) are satisfied.

(ii) $\Rightarrow$ (i)

As h is a homeomorphism and an order-isomorphism, then h is an isomorphism in the category $\boldsymbol{\mathcal{P}}$; therefore, there exists a $P-$function $g : X' \longrightarrow X$ such that $h \circ g = 1_{X'}$ and $g \circ h = 1_X$. Besides, h satisfies (a) and (b); as a consequence, (i) and (ii) in Definition 3.2 are verified, for which h is a morphism in the category $\mathfrak{M}_3$. It only remains to prove that g satisfies these conditions too, that is to say: (c) $(M_{X'}g^{-1}(V)] = g^{-1}((M_X V])$ and (d) $[m_{X'}g^{-1}(V)) \setminus M_{X'}g^{-1}(V) = g^{-1}([m_X V) \setminus M_X V)$ for each $V \in D(X)$. Consider that if $V \in D(X)$, then $g^{-1}(V) = h(V)$.

(c) $(M_{X'}h(V)] = h((M_X V])$: Let $x \in h((M_X V])$, then there exists $z \in (M_X V]$ such that $h(z) = x$. Therefore, there exists $v \in V$ such that $v \in max\, X \cap V$ and $z \leq v$. As h is increasing, $x = h(z) \leq h(v)$ and $h(v) \in h(V)$; in addition, h being an order isomorphism, $h(v) \in max\, X'$ is verified; consequently, $h(v) \in max\, X' \cap h(V)$ and $x \leq h(v)$, which implies that $x \in (M_{X'}h(V)]$. Conversely, if $y \in (M_{X'}h(V)]$, there exists $t \in maxX^* \cap h(V)$ such that $y \leq t$, and as h is an order isomorphism, $t = h(v)$ with $v \in V \cap max\, X$. On the other hand, h being onto, then $y = h(z)$ for some $z \in X$; consequently, $h(z) \leq h(v)$, from which $z \leq v$ with $v \in V \cap max\, X$. Thus, $z \in (M_X V]$ and, as a consequence, $y \in h((M_X V])$.

(d) $h([m_X V) \setminus M_X V) = [m_{X'}h(V)) \setminus M_{X'}h(V)$: Let $x \in h([m_X V) \setminus M_X V)$, then there exists $t \in [m_X V) \setminus M_X V$ such that $x = h(t)$. Therefore, there exists $v \in min\, X \cap V$ such that $v \leq t$ and (1) $t \notin max\, X \cap V$. As h is an order isomorphism, $h(v) \leq h(t) = x$ and $h(v) \in minX' \cap h(V)$ are verified, for which $x \in [m_{X'}h(V))$. If $x \in max\, X' \cap h(V)$, then there exists $u \in V$ such that $x = h(u)$, and as h is injective and $h(t) = x$, we have that $t = u$, from which $t \in maxX \cap V$, which contradicts (1). Thus, we have that $x \in [m_{X'}h(V)) \setminus M_X h(v)$. Conversely, let $x \in [m_{X'}h(V)) \setminus M_{X'}h(V)$; therefore, there exists $t \in m_{X'}h(V)$ such that $t \leq x$ and (1) $x \notin max\, X' \cap h(V)$. Therefore, $t \in minX' \cap h(V)$ and $t \leq x$, from which there exists $u \in V$ such that $t = h(u) \in min\, X'$ with $u \in V$ y $t = h(u) \leq x$. As h is onto, $x = h(y)$; therefore, $h(u) \leq h(y)$, whence $u \leq y$ with $u \in V \cap min\, X$, which implies that $y \in [m_X V)$. If $y \in max\, X \cap V$, then $h(y) \in max\, X' \cap h(V)$, which would imply that $x \in max\, X' \cap h(V)$, which would contradict (1). As a consequence,$y \in m_X V) \setminus M_X V$, and thus we have that $x \in h([m_X V) \setminus M_X V)$.

□

3.3. Dual M_3−lattice of an M_3−space.

Proposition 3.7. *If X is an M_3−space, then $\langle D(X), \cap, \cup, \triangle^*, \neg, \emptyset, X\rangle$ is a bounded M_3−lattice, in which the operations $\triangle^*$ and $\neg$ are those indicated in* (D) *and* (N) *in* Lemma 3.5*, respectively.*

Proof. By (MP5) the operations $\triangle^*$ and $\neg$ are well defined. The proofs to M_3−lattice axioms are obtained from properties (MP6) through (MP17) in Lemma 3.5, as follows:

(M1) $\triangle^*(U \cap \neg U) = \emptyset$: By (MP10), $\triangle^*(U \cap \neg U) = \triangle^*(m_X U)$ and, as by (D), $\triangle^*(m_X U) = (M_X(m_X U)]$ and $M_X(m_X U) = \emptyset$, we conclude that $\triangle^*(U \cap \neg U) = \emptyset$.

(M2) $\neg\neg U = U$: By (MP8), (MP9) and (MP11), we have that $\neg\neg U = m_X U \cup M_X U$, from which it follows, by (MP2), that $\neg\neg U = U$.

(M3) $\triangle^* U \cup \neg\nabla^* U = U$: From (MP6) and (MP15), it follows that $\triangle^* U \cup \neg\nabla^* U \subseteq U$. Conversely, let $x \in U$, then by (MP2), $x \in m_X U$ or $x \in M_X U$. If $x \in m_X U$, by (MP15), $x \in \neg\nabla^* U$, and if $x \in M_X U$, by (D), we have that $x \in \triangle^* U$.

(M4) $\neg\triangle^* U \cup \triangle^* U = \triangle^* U$: From properties (MP8) and (MP12), $\neg\triangle^* U = m_X \triangle^* U$ results; therefore, $\neg\triangle^* U \cup \triangle^* U = \triangle^* U$.

(M5) $\triangle^*\nabla^* U = \nabla^* U$: By (MP5) and (MP6), $\triangle^*\nabla^* U \subseteq \nabla^* U$.
In order to prove the other inclusion, consider that $U \subseteq \triangle^*\nabla^* U$. Indeed, if $x \in M_X U$, from (MP16) and (D), $x \in \triangle^*\nabla^* U$. If (1) $x \in m_X U$, then by (MP2), there exists (2) $t \in max\, X$ such that (3) $x < t$. If $t \in U$, then by (2), $t \in M_X U$; therefore, from (MP16), $t \in M_X \nabla^* U$, from which by (3) and (D), we have that $x \in \triangle^*\nabla^* U$. If $t \notin U$, then $t \notin M_X U$, from which by (1), (2) and (3), $t \in \neg U$ result; as a consequence, by (2), we have (4) $t \in M_X \neg U$. As $M_X \neg U \subseteq M_X \nabla^* U$, then from (3), (4) and (D), $x \in \triangle^*\nabla^* U$.
In an analogous way, $\neg U \subseteq \triangle^*\nabla^* U$ is proved, from which we conclude that $\triangle^*\nabla^* U = \nabla^* U$.

(M6) $\triangle^*(U \cup V) = \triangle^* U \cup \triangle^* V$: It is immediate from the definition of $\triangle^*$, taking into account $M_X(U \cup V) = M_X U \cup M_X V$.

(M7) $\nabla^*(U \cap V) = \nabla^* U \cap \nabla^* V$: By (MP17), $\nabla^*(U \cap V) \subseteq \nabla^* U \cap \nabla^* V$ is verified.
For the converse, if $x \in \nabla^* U \cap \nabla^* V$, then taking into account the definition of ∇^*, it can happen that $x \in U \cap V$, or $x \in U \cap \neg V$, or $x \in V \cap \neg U$ or $x \in \neg U \cap \neg V$. If $x \in U \cap V$, $x \in \nabla^*(U \cap V)$ is immediate. If $x \in U \cap \neg V$, then by (MP8), (1) $x \in U \cap m_X V$ or (2) $x \in U \cap \{x \in X \setminus V :$ there exists$t \in V$ and $t < x\}$. From (1) it follows that $x \in U \cap V$; therefore, $x \in \nabla^*(U \cap V)$. From (2), (3) $x \in U \cap (X \setminus V)$ is immediate, and there exists (4) $t \in V$ such that (5) $t < x$. Therefore, from (3), (4) and (5), taking into account that U is decreasing, there exists $t \in U \cap V$ such that $t < x$, and as $x \notin U \cap V$, then by (MP8), $x \in \neg(U \cap V)$, from which we conclude that $x \in \nabla^*(U \cap V)$.
If $x \in V \cap \neg U$, by analogous reasoning we have that $x \in \nabla^*(U \cap V)$. Finally, if $x \in \neg U \cap \neg V$, then by (MP8), if $x \in m_X U \cap m_X V$, then $x \in U \cap V$, and, therefore, $x \in \nabla^*(U \cap V)$; if $x \in X \setminus U \cup V$ and there exist $u \in U$, $v \in V$, such that $u < x$ and $v < x$, then by (MP2), $u = v$, whence $x \in \neg(U \cap V)$ results and, therefore, $x \in \nabla^*(U \cap V)$.

□

3.4. M_3-space associated to a bounded M_3-lattice.

Proposition 3.8. *If $\langle L, \wedge, \vee, \triangle, \sim, 0, 1\rangle$ is an M_3-lattice, then $\mathcal{I}_p(L)$, the Priestley space of L, is an M_3-space and $\sigma_L : L \longrightarrow D(\mathcal{I}_p(L))$, defined as in* (A1)*, is an* $\mathbf{M_3}-$*isomorphism.*

Proof. According to what was seen in Section 2, we know that $\sigma_L : L \longrightarrow D(\mathcal{I}_p(L))$, defined as in (A1), is an isomorphism of bounded lattices. This allows us to state that if $U \in D(\mathcal{I}_p(L))$, then $U = \sigma_L(a)$, for some $a \in L$. In order to prove conditions (MP3) and (MP4) in Definition 3.1, it is sufficient to prove, taking into account Corollary 3.4, that $\triangle^*\sigma_L(a) = \sigma_L(\triangle a)$ and $\neg\sigma_L(a) = \sigma_L(\sim a)$, which will allow us to state that $\sigma_L : L \longrightarrow D(\mathcal{I}_p(L))$, is an $\mathbf{M_3}-$isomorphism.

(i) $\triangle^*\sigma_L(a) = \sigma_L(\triangle a)$: By (I7), we obtain (1) $M_{\mathcal{I}_p(L)}\ \sigma_L(a) = \{I \in \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L) : a \notin I\}$; as a consequence, if $R \in \triangle^*\sigma_L(a)$, then there exists $J \in \{I \in \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L) : a \notin I\}$ such that (2) $R \subseteq J$. If $\triangle a \in R$, then by (2), $\triangle a \in J$, and, consequently, $a \in J_\triangle$. Besides, as by (I7) and (I8), we have that $J_\triangle = J$; therefore, $a \in J$, which contradicts the hypothesis. From this it follows that $\triangle^*\sigma_L(a) \subseteq \sigma_L(\triangle a)$.
Conversely, let $R \in \sigma_L(\triangle a)$, then, by (M8), $a \notin R$. If $R \in \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L)$, then by (1), it is easy to see that $R \in \triangle^*\sigma_L(a)$. If $R \in \mathcal{NI}_p(L)$, by (I4), $R_\triangle \in \mathcal{I}_p(L) \setminus \mathcal{NI}_p(L)$, $R \subset R_\triangle$ and $a \notin R_\triangle$; therefore, $R_\triangle \in M_{\mathcal{I}_p(L)}\ \sigma_L(a)$ and, in this way, $R \in \triangle^*\sigma_L(a)$.

(ii) $\neg\sigma_L(a) = \sigma_L(\sim a)$: From (I6), we have that $m_{\mathcal{I}_p(L)}\ \sigma_L(a) = \{I \in \mathcal{NI}_p(L) : a \notin I\}$; therefore, by (MP8), we get (3) $\neg\sigma_L(a) = \{I \in \mathcal{NI}_p(L) : a \notin I\} \cup \{I \in \mathcal{I}_p(L) \setminus \sigma_L(a) :$ there exists $R \in \sigma_L(a)$ y $R \subset I\}$. If $I \in \{I \in \mathcal{NI}_p(L) : a \notin I\}$, by (M2), it is immediate that $\sim a \notin I$; therefore, $I \in \sigma_L(\sim a)$. If $I \in \mathcal{I}_p(L) \setminus \sigma_L(a)$ and there exists $R \in \sigma_L(a)$ such that $R \subset I$, then $R = I_\nabla$ by Theorem 3.3; as a consequence, $a \notin I_\nabla$, that is to say $\nabla a \notin I$, and as $a \in I$, $\sim a \notin I$ is verified; therefore, $I \in \sigma_L(\sim a)$, from which it follows that $\neg\sigma_L(a) \subseteq \sigma_L(\sim a)$.
Conversely, if (4) $I \in \sigma_L(\sim a)$ is such that $I \in \mathcal{NI}_p(L)$, we have that $a \notin I$ and, therefore, $I \in \neg\sigma_L(a)$. If $I \notin \mathcal{NI}_p(L)$, $a \notin I_\nabla$; as a consequence, $I_\nabla \in \sigma_L(a)$ and, by (I5), $I_\nabla \subset I$. It remains to prove, taking into account (3), that $I \in \mathcal{I}_p(L) \setminus \sigma_L(a)$. Suppose that $I \in \sigma_L(a)$; therefore, $a \notin I$. By (M1) and (M25), $\triangle a \wedge \triangle \sim a \in I$, then I being a prime ideal, $\triangle a \in I$ or $\triangle \sim a \in I$; however, as $I \notin \mathcal{NI}_p(L)$, by (I7) and (I8), $I = I_\triangle$; therefore, $\triangle a \notin I$, from which $\triangle \sim a \in I$ results; thus $\sim a \in I$, which contradicts (4). Therefore, $I \in \mathcal{I}_p(L) \setminus \sigma_L(a)$.

□

3.5. Duality between $\boldsymbol{\mathcal{M}_3}$ and $\boldsymbol{\mathfrak{M}_3}$.

Propositions 3.7 and 3.8 allow us to say that we have established a correspondence between the objects of the categories $\boldsymbol{\mathcal{M}_3}$ and $\boldsymbol{\mathfrak{M}_3}$. In order to prove that these categories are naturally equivalent and to define the corresponding functors, we will show that there exists a correspondence between the morphisms of such categories.

Lemma 3.9. *Let $h : X \longrightarrow X'$ be an M_3-function (bijective), then $\Psi(h) : D(X') \longrightarrow D(X)$, defined by $\Psi(h)(V) = h^{-1}(V)$ for each $V \in D(X')$, is an* $\mathbf{M_3}-$*homomorphism (isomorphism).*

Proof. It immediately results since h is an M_3-function. □

Proposition 3.10. *Let X be an M_3−space; then $\epsilon_X : X \longrightarrow \mathcal{I}_p(D(X))$, defined by $\epsilon_X(x) = \{V \in D(X) : x \notin V\}$ is an isomorphism in the category $\mathfrak{M}_3$.*

Proof. Given that ϵ_X is a homeomorphism and an order isomorphism, it only rests to prove conditions (a) and (b), item (ii) in Lemma 3.6.

(a) $\triangle^*\epsilon_X^{-1}(V) = \epsilon_X^{-1}(\triangle^*V)$: Let $x \in \triangle^*\epsilon_X^{-1}(V)$, then there exists $y \in max\, X \cap \epsilon_X^{-1}(V)$ such that $x \leq y$; as ϵ_X is an order isomorphism, $y \in max\, X$, $\epsilon_X(x) \leq \epsilon_X(y)$ and $\epsilon_X(y) \in max\, \mathcal{I}_p(D(X))$ are verified. Therefore, there exists $\epsilon_X(y) \in max\, \mathcal{I}_p(D(X)) \cap V$ such that $\epsilon_X(x) \leq \epsilon_X(y)$; as a consequence, $\epsilon_X(x) \in (max\, \mathcal{I}_p(D(X)) \cap V]$ and, therefore, $x \in \epsilon_X^{-1}(\triangle^*V)$.
Conversely, if $x \in \epsilon_X^{-1}(\triangle^*V)$, then $\epsilon_X(x) \in \triangle^*V$ and, by (D), there exists $t \in max\, \mathcal{I}_p(D(X)) \cap V$ such that $\epsilon_X(x) \leq t$; therefore, $x \leq \epsilon_X^{-1}(t)$ and $\epsilon_X^{-1}(t) \in max\, X \cap \epsilon_X^{-1}(V)$, from which $x \in \triangle^*\epsilon_X^{-1}(V)$ results.

(b) $\neg\epsilon_X^{-1}(V) = \epsilon_X^{-1}(\neg V)$: First, let us take into account that, by (MP8) in Lemma 3.5, the following is verified: $\neg\epsilon_X^{-1}(V) = (min\, X \cap \epsilon_X^{-1}(V)) \cup \{x \in X \setminus \epsilon_X^{-1}(V) :$ there exists $t \in \epsilon_X^{-1}(V)$ and $t < x\}$. Next, we will prove that $\neg\epsilon_X^{-1}(V) \subseteq \epsilon_X^{-1}(\neg V)$.
Let $x \in \neg\epsilon_X^{-1}(V)$. If $x \in min\, X \cap \epsilon_X^{-1}(V)$ we have that $\epsilon_X(x) \in min\, \mathcal{I}_p(D(X)) \cap V$ and, therefore, $x \in \epsilon_X^{-1}(\neg V)$. If $y \in \{x \in X \setminus \epsilon_X^{-1}(V)$: there exists $t \in \epsilon_X^{-1}(V)$ y $t < x\}$, then there exists $t \in \epsilon_X^{-1}(V)$ such that $t < y$ and $y \notin \epsilon_X^{-1}(V)$; therefore, $\epsilon_X(t) \subset \epsilon_X(y)$, $\epsilon_X(t) \in V$ and $\epsilon_X(y) \notin V$. Therefore, $\epsilon_X(y) \in \{I \in \mathcal{I}_p(D(X)) \setminus V :$ there exists $R \in V$ y $R \subset I\}$; thus, $y \in \epsilon_X^{-1}(\neg V)$.
Conversely, let $x \in \epsilon_X^{-1}(\neg V)$, then $\epsilon_X(x) \in \neg V$. By (MP8), if $\epsilon_X(x) \in min\, \mathcal{I}_p(D(X)) \cap V$, this is verified: $x \in min\, X \cap \epsilon_X^{-1}(V)$ and, therefore, $x \in \neg\epsilon_X^{-1}(V)$. On the other hand, if $\epsilon_X(x) \notin V$ and there exists $R \in V$ such that $R \subset \epsilon_X(x)$, then $\epsilon_X^{-1}(R) \in \epsilon_X^{-1}(V)$ and $\epsilon_X^{-1}(R) < x$, from which it follows that, by (MP8), $x \in \neg\epsilon_X^{-1}(V)$.

□

Proposition 3.11. *Let L and L' be M_3−lattices and $h : L \longrightarrow L'$, an* **$\mathbf{M_3}$**−*homomorphism; then the application $\Phi(h) : \mathcal{I}_p(L') \longrightarrow \mathcal{I}_p(L)$, defined by $\Phi(h)(I') = h^{-1}(I')$ for each $I' \in \mathcal{I}_p(L')$, is an M_3−function.*

Proof. It results from the fact that $\Phi(h)$ is a P−function ([9]) and Proposition 3.8, because if $V \in D(\mathcal{I}_p(L))$, then $V = \sigma_L(a)$ for some $a \in L$ and $\Phi(h)^{-1}(\sigma_L(a)) = \sigma_L(h(a))$. As a consequence, $\Phi(h)^{-1}(\triangle^*\sigma_L(a)) = \triangle^*\Phi(h)^{-1}(\sigma_L(a))$, $\Phi(h)^{-1}(\neg\sigma_L(a)) = \neg\Phi(h)^{-1}(\sigma_L(a))$ and, therefore, conditions (i) and (ii), in Definition 3.2, are fulfilled. □

From Proposition 3.7 and Lemma 3.9, Ψ is a contravariant-functor of $\mathfrak{M}_3$ in $\boldsymbol{\mathcal{M}_3}$. On the other hand, from Propositions 3.8 and 3.11, we can easily conclude that Φ is a contravariant functor of $\boldsymbol{\mathcal{M}_3}$ in $\mathfrak{M}_3$. These results and Proposition 3.10 allow us to state the following theorem.

Theorem 3.12. *Functors $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are naturally equivalent to the identity functors on $\boldsymbol{\mathcal{M}_3}$ and $\mathfrak{M}_3$, respectively, and these two categories are naturally equivalent.*

4. $\mathbf{M_3}$−congruences and subdirectly irreducible algebras

In this section by means of this duality obtained we describe the congruences and the subdirectly irreducible M_3−lattices.

4.1. Characterization of the $\mathbf{M_3}$−congruences lattice.

One of the important facts of Priestley Duality is that, if L is a bounded distributive lattice, there exists a bijective correspondence between the congruences of L and the closed subsets of $\mathcal{I}_p(L)$; more precisely, H. A. Priestley ([8], [9], [10]) proved the following result.

Theorem 4.1. *Let L be a bounded distributive lattice. If Y is a closed subset of $\mathcal{I}_p(L)$, then*

(A3) $\Theta(Y) = \{(a,b) \in L \times L : \sigma_L(a) \cap Y = \sigma_L(b) \cap Y\}$

is a congruence on L. Conversely, if θ is a congruence of L and $q : L \longrightarrow L/\theta$ is the canonical epimorphism, then

(A4) $Y = \{q^{-1}(I) : I \in \mathcal{I}_p(L/\theta)\}$

is a closed subset of $\mathcal{I}_p(L)$ such that $\Theta(Y) = \theta$ and the correspondence $Y \longrightarrow \Theta(Y)$ establishes an isomorphism between $C(\mathcal{I}_p(L))$, the lattice of the bounded subsets of L and the dual of the lattice $Con(L)$ of the congruences of L .

Another result which will be useful later and which may be consulted in H. A. Priestley's above-mentioned works is the following:

Proposition 4.2. *The set Y indicated in* (A4) *verifies that, if $I \in \mathcal{I}_p(L) \setminus Y$, then there exist $a, b \in L$, such that $(a,b) \in \theta$, $a \in I$ y $b \notin I$.*

Now, we will give a generalization of Theorem 4.1 for M_3−lattices, for which we will obtain a characterization of the lattice of the congruences of an M_3−lattice in terms of certain closed subsets of its associated M_3−space. First we will give some auxiliary results.

Definition 4.1. Let X be an M_3−space. We will say that a subset Y of X is $\triangle$-involutive if $\triangle^* Y = Y$.

Lemma 4.3. *Let X be an M_3−space. Then every maximal chain in X is $\triangle$-involutive.*

Proof. If C is a maximal chain in X, then by (MP2), C is a chain which has exactly two elements. Let $x \in C$. If $x \in max\, X$, then it is immediate that $x \in \triangle^* C$. If $x \in min\, X \cap C$, then there exists $z \in max\, X \cap C$ such that $x < z$, which implies that $x \in \triangle^* C$. Conversely, if $x \in \triangle^* C$, then there exists (1) $t \in max\, X \cap C$ such that $x \leq t$. If $x = t$, it is immediate that $x \in C$. If $x < t$, then we have (2) $x \in C_t$, the chain of two elements which contains t. By (1), (2) and (MP2), we conclude that that $C_t = C$ and, therefore, $x \in C$. □

Theorem 4.4. *Let X be an M_3−space and Y, a non-empty subset of X. Then, the following conditions are equivalent:*

(i) *Y is $\triangle$-involutive,*
(ii) *Y is increasing and decreasing,*
(iii) *Y is a cardinal sum of a family of chains, each of which has exactly two elements.*

Proof. (i) $\Rightarrow$ (ii): From the definition of $\triangle^*Y$, it is immediate that, if Y is $\triangle$-involutive, then Y is decreasing. On the other hand, if $x \in Y$, $y \in X$ are such that (1) $x \leq y$, then, as Y is $\triangle$-involutive, there exists (2) $t \in M_X Y$ such that (3) $x \leq t$. Therefore, if $x = y$, then from (1) it is is immediate that $y \in Y$ and, if $x < y$, then from (1), (3) and (MP2), $t = y$ results and, therefore, from (2), $y \in Y$, from which we can conclude that Y is increasing.
(ii) $\Rightarrow$ (iii): Let $x \in Y$ and let C_x be the chain of two elements that contains x. Since Y is increasing and decreasing, it is immediate that $C_x \subseteq Y$, from which $Y = \bigcup\limits_{x\in Y} C_x$ follows and, as a consequence, by (MP2), Y is the cardinal sum of the chains C_x.
(iii) $\Rightarrow$ (i): Let $Y = \bigcup\limits_{x\in Y} C_x$, with C_x chains of two-element. By Lemma 4.3, for each $x \in Y$, we have that $C_x = \triangle^* C_x$; as a consequence, $\bigcup\limits_{x\in Y} C_x = \triangle^* \bigcup\limits_{x\in Y} C_x$ results and, therefore, Y is $\triangle$-involutive. □

The closed and $\triangle$-involutive closed subsets of the M_3−space associated to an M_3−lattice are essential for the characterization of the $\mathbf{M_3}$−congruences on these algebras as can be seen below.

Proposition 4.5. *Let $L \in \mathbf{M_3}$, $\mathcal{I}_p(L)$ be the M_3−space associated to L and $C_\triangle(\mathcal{I}_p(L))$, the lattice of the closed and $\triangle$-involutive subsets of $\mathcal{I}_p(L)$. If $Y \in C_\triangle(\mathcal{I}_p(L))$, then $\Theta_{C\triangle}(Y) \in Con_{\mathbf{M_3}}(L)$, where $\Theta_{C\triangle}(Y)$ is defined as in* (A3).

Proof. We know, by "By Theorem 4.1, that $\Theta_{C\triangle}(Y)$ is a lattice congruence"; then it only remains to prove that $\Theta_{C\triangle}(Y)$ is compatible with $\triangle$ and $\sim$. Let $(a, b) \in \Theta_{C\triangle}(Y)$, then (1) $\sigma_L(a) \cap Y = \sigma_L(b) \cap Y$. As Y is $\triangle$-involutive and σ_L is an M_3−isomorphism, for each $x \in L$, $\sigma_L(\triangle x) \cap Y = \triangle^*\sigma_L(x) \cap \triangle^*Y$; in addition, by (M25), $\triangle^*\sigma_L(x) \cap \triangle^*Y = \triangle^*(\sigma_L(x) \cap Y)$. Then, $\sigma_L(\triangle x) \cap Y = \triangle^*(\sigma_L(x) \cap Y)$, for each $x \in L$, which implies, taking into account (1), that $\sigma_L(\triangle a) \cap Y = \sigma_L(\triangle b) \cap Y$; as a consequence of which $\Theta_{C\triangle}(Y)$ is compatible with the operation $\triangle$.

On the other hand, $\sigma_L(\sim a) \cap Y = \neg(\sigma_L(a) \cap Y)$ is verified. Indeed, if $I \in \sigma_L(\sim a) \cap Y$, as σ_L is an M_3−isomorphism, $I \in \neg\sigma_L(a)$ results; therefore, there exists (1) $Q \in m_{\mathcal{I}_p(L)}\sigma_L(a)$ such that (2) $Q \subseteq I$ and (3) $I \notin M_{\mathcal{I}_p(L)}\sigma_L(a)$. By Theorem 4.4, as Y is decreasing, from (1) and (2), we have that $Q \in m_{\mathcal{I}_p(L)}(\sigma_L(a) \cap Y)$; from this it follows, by (3), that $I \in [m_{\mathcal{I}_p(L)}(\sigma_L(a) \cap Y)) \setminus M_{\mathcal{I}_p(L)}(\sigma_L(a) \cap Y)$, which implies that $I \in \neg(\sigma_L(a) \cap Y)$.

In order to prove the other inclusion, suppose now that $I \in \neg(\sigma_L(a) \cap Y)$; this implies, taking into account the definition of $\neg$ en $D(\mathcal{I}_p(L))$, that there exists (4) $R \in m_{\mathcal{I}_p(L)}(\sigma_L(a) \cap Y)$ such that (5) $R \subseteq I$ and (6) $I \notin M_{\mathcal{I}_p(L)}(\sigma_L(a) \cap Y)$. Therefore, from (4) and (5), it is clear that $I \in [m_{\mathcal{I}_p(L)}\sigma_L(a))$ and, as by Theorem 4.4, Y is decreasing, we have (7) $I \in Y$. If $I \in M_{\mathcal{I}_p(L)}\sigma_L(a)$, from (7) it would follow that $I \in M_{\mathcal{I}_p(L)}(\sigma_L(a) \cap Y)$, which would contradict (6). Therefore, $I \in [m_{\mathcal{I}_p(L)}\sigma_L(a)) \setminus M_{\mathcal{I}_p(L)}\sigma_L(a)$ and, as a consequence, $I \in \neg\sigma_L(a) \cap Y$.

Thus we have proved that $\Theta_{C\triangle}(Y)$ is compatible with operations $\triangle$ and $\sim$; as a consequence, $\Theta_{C\triangle}(Y) \in Con_{\mathbf{M_3}}(L)$. □

Proposition 4.6. *Let $L \in \mathbf{M_3}$, $\mathcal{I}_p(L)$ be the M_3−space associated to L and $C_\triangle(\mathcal{I}_p(L))$ the lattice of closed and $\triangle$-involutive subsets of $\mathcal{I}_p(L)$. If $\theta \in Con_{\mathbf{M_3}}(L)$, then there exists $Y \in C_\triangle(\mathcal{I}_p(L))$ such that $\theta = \Theta_{C\triangle}(Y)$, where $\Theta_{C\triangle}(Y)$ is defined as in* (A3).

Proof. Let $\theta \in Con_{\mathbf{M_3}}(L)$ and $q : L \longrightarrow L/\theta$ the canonical epimorphism. As $Con_{\mathbf{M_3}}(L)$ is a sublattice of $Con(L)$, then by "By Theorem 4.1, $Y = \{q^{-1}(Q) : Q \in \mathcal{I}_p(L/\theta)\}$ is a closed set of $\mathcal{I}_p(L)$"and $\theta = \Theta_{C\triangle}(Y)$. In order to prove that Y is $\triangle$-involutive, by Theorem 4.4, it is sufficient to prove that Y is increasing and decreasing.

(a) Y is decreasing: Suppose that Y is not decreasing; then there exist $I \in \mathcal{I}_p(L)$ and (1) $Q \in Y$ such that (2) $I \subset Q$ and (3) $I \notin Y$. From (2) and (MP2), $Q \in max\,\mathcal{I}_p(L)$ and, by (I5) and (I7), we have (4) $Q_\nabla = I$; besides, by (I6), (5) $I \in \mathcal{N}\mathcal{I}_p(L)$ results. On the other hand, from (3) and Proposition 4.2, we can state that there exist $a, b \in L$ such that (6) $(a, b) \in \theta$ with (7) $a \in I$ and (8) $b \notin I$. Therefore, from (5) and (7), (9) $\sim a \in I$ results and, by (4) and (8), we have $b \notin Q$ or $b \notin \sim Q$. If $b \notin Q$, then by (2) and (6), $a \notin I$, which contradicts (7). Similarly, if $b \notin\sim Q$, then $\sim a \notin I$, which contradicts (9).

(b) Y is increasing. Suppose that there exist $I \in \mathcal{I}_p(L)$ and (1) $Q \in Y$, such that (2) $Q \subset I$ and (3) $I \notin Y$. From (3) and "By Proposition 4.2, there exist $a, b \in L$ such that (4) $(a, b) \in \theta$, (5) $a \in I$ and (6) $b \notin I$". Therefore, from (2) and (6), $b \notin Q$ results and, as $\theta = \Theta_{C\triangle}(Y)$, from (1) and (4) it follows that (7) $a \notin Q$. By (2), (MP2) and (6), we have that $I \in M_{\mathcal{I}_p(L)}\sigma_L(b)$, and, therefore, $Q \in \triangle^*\sigma_L(b)$, which implies that $\triangle b \notin Q$, σ_L being an M_3-isomorphism. As θ is an $\mathbf{M_3}-$congruence, from (4), $(\triangle a, \triangle b) \in \theta$ is verified and, therefore, (9) $\triangle a \notin Q$. Besides, by (M1) and (M25), we have $0 = \triangle a \wedge \triangle \sim a \in Q$ and, as Q is a prime ideal, from (9), $\triangle \sim a \in Q$ or, what is equivalent, (10) $\sim a \in \triangle^{-1}(Q)$. On the other hand, from (2) and (MP2), $I \in max\,\mathcal{I}_p(L)$ and $Q \in min\,\mathcal{I}_p(L)$; therefore, by Lemmas 3.1 and 3.2, we infer (11) $I = Q_\triangle$ and (12) $Q = I_\nabla$. Therefore, from (10), (11) and (M2), we conclude that $a \in\sim I$. From this assertion, taking into account (5) and (12), we obtain $a \in Q$, which contradicts (7).

□

Theorem 4.7. *Let $L \in \mathbf{M_3}$ and let $\mathcal{I}_p(L)$ be the M_3-space associated to L. Then, the lattice $C_\triangle(\mathcal{I}_p(L))$ of the closed and $\triangle$-involutive subsets of $\mathcal{I}_p(L)$ is isomorphic to the dual of the lattice $Con_{\mathbf{M_3}}(L)$ of the $\mathbf{M_3}-$congruences, and the isomorphism is established by the function $\Theta_{C\triangle}$, defined by the same prescription as that given in* (A3).

Proof. It is a consequence of Propositions 4.5 and 4.6, and the fact that the correspondence $Y \longrightarrow \Theta_{C\triangle}(Y)$ establishes an isomorphism of the lattice $C_\triangle(\mathcal{I}_p(L))$ into the dual lattice $Con_{\mathbf{M_3}}(L)$ of the $\mathbf{M_3}-$congruences of L. □

4.2. Simple and subdirectly irreducible algebras in $\mathbf{M_3}$.

In this section we determine the simple and subdirectly irreducible M_3-lattices by using the characterization of the congruence lattice obtained in Theorem 4.7.

Theorem 4.8. *Let X be an M_3-space and $\Psi(X)$ its associated dual M_3-lattice; then, the conditions given below are equivalent:*

(i) *X is totally ordered,*
(ii) *$\Psi(X)$ is simple,*
(iii) *$\Psi(X)$ is subdirectly irreducible.*

Proof. (i)⇒ (ii): If X is totally ordered, then by (MP2), we have that X is a maximal chain of two elements. Then, by Lemma 4.3, X is $\triangle$-involutive and, as a consequence, it is the only non-empty, closed and $\triangle$-involutive set of X, which implies, by Theorem 4.7, that $\Psi(X)$ is simple.
(ii)⇒ (iii): It is trivial.
(iii)⇒ (i): If $\Psi(X)$ is subdirectly irreducible, then by Theorem 4.7, the family $C_\triangle(X)$ of the closed, $\triangle$-involutive and proper subsets of X, has last element F_0. As F_0 is proper, there is (1) $x \in X \setminus F_0$. If C_x is the chain of two elements which contains x, then by Lemma 4.3, $C_x \in C_\triangle(X)$, from which by (1), $C_x = X$ results, and, therefore, X is totally ordered. □

Corollary 4.9. *Let $L \in \mathbf{M_3}$ have more than one element. Then, the conditions given below are equivalent:*
(i) *L is simple,*
(ii) *L is subdirectly irreducible,*
(iii) *L is isomorphic to* T*, where* T *is given as in* (M33).

Proof. It is the immediate consequence of Theorem 4.8 and Corollary 3.4. □

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