Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(2), 2017, Pages 214–227
ISSN: 1223-6934

Double left stabilizers in BL-algebras

Somayeh Motamed

Abstract. In this paper we introduce the notions of double left stabilizer of X and double left stabilizer of X with respect to Y, for nonempty subsets X and Y of BL-algebra A and we study some properties of them. After that we state and prove some theorems which determine the relationship between these notions and other types of filters in BL-algebras. Finally we introduce the set $N(F)$, for every filter F of A. Also we prove A is an MV-algebra iff $N(F) = N(A) = \{1\}$ iff $D(X_l) = X_l$ iff $D((X, \{1\})_l) = X_l$, for each nonempty subset X and every proper filter F of A.

2010 Mathematics Subject Classification. 03B47; 03G25; 06D99.
Key words and phrases. (linearly, local) BL-algebra, double left stabilizer, (Boolean, obstinate, primary, prime) filter, MV-algebra.

1. Introduction

BL-algebras (basic logic algebras) are the algebraic structures for Hájek basic logic [6], in order to investigate many valued logic by algebraic means. A BL-algebra is an algebra $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that:

$(BL1)$ $(A, \wedge, \vee, 0, 1)$ is a bounded lattice $L(A)$,

$(BL2)$ $(A, *, 1)$ is a commutative monoid,

$(BL3)$ $*$ and $\rightarrow$ form an adjoint pair i.e, $c \leq a \rightarrow b$ if and only if $a * c \leq b$, for all $a, b, c \in A$,

$(BL4)$ $a \wedge b = a * (a \rightarrow b)$,

$(BL5)$ $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

A BL-algebra becomes an MV-algebra if we adjoin to the axioms the double negation law, $a^{--} = a$. Thus, a BL-algebra is in some intuitive way, a "non-double negation MV-algebra". Our basic tools in the study of a BL-algebra A are *deductive systems*, i.e. subsets $D \subseteq A$ such that $1 \in D$ and if $x, x \rightarrow y \in D$, then $y \in D$, [10]. From logical point of view, deductive systems correspond sets of provable formulas. In MV-algebra theory, deductive systems and ideals are dual notions. There deductive systems are also called filters [6]. In order to avoid confusion, we prefer to talk about filters. Hájek [6] introduced the idea of prime filters in BL-algebras. The concept of implicative, positive implicative and fantastic filter were defined in BL-algebras by Haveshki et al. [7]. Turunen was the first to systematically study filter theory in BL-algebras, e.g., maximal, Boolean and prime filters (see [10], [11]). We defined the notions of normal filters, obstinate filters, set of double complemented elements of a filter, $N(A)$ and radical of a filter in [1], [2], [3] and [9], respectively. After that

Received September 22, 2015.

Haveshki et al. in [8] introduced left stabilizer in BL-algebras. For analyzing the BL-algebras and therefore, BL-logic, we study the BL-algebras and get some results [1],[2],[3],[9]. At this work, we continued our studied in BL-algebras and generalized some notions in this structure and get some connection between BL-algebra and other algebraic structures. Since Haveshki proposed the notion of left stabilizers in BL-algebras, his idea have been applied to various algebraic structures. In this paper, we applied Haveshki's idea in BL-algebras and introduced the notions of double left stabilizer of X and double left stabilizer of X with respect to Y, for nonempty subsets X and Y of BL-algebra A and discussed the relation among them.

2. Preliminaries

Lemma 2.1. *([5],[6],[12]) In any BL-algebra A, the following properties hold for all $x, y, z \in A$:*

(1) $x \leq y$ if and only if $x \to y = 1$,
(2) $x \to (y \to z) = (x * y) \to z = y \to (x \to z)$ and $((x \to y) \to y) \to y = x \to y$,
(3) If $x \leq y$, then $y \to z \leq x \to z$, $z \to x \leq z \to y$, $x * z \leq y * z$ and $y^- \leq x^-$, where $x^- = x \to 0$,
(4) $y \leq (y \to x) \to x$ and $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(5) $x * y \leq x \wedge y$, $x * 0 = 0$ and $x * x^- = 0$,
(6) $1 \to x = x$, $x \to x = 1$, $x \leq y \to x$, $x \to 1 = 1$, $0 \to x = 1$,
(7) $x * y = 0$ iff $x \leq y^-$,
(8) $x \vee y = 1$ implies $x * y = x \wedge y$,
(9) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$,
(10) $x^{--} \leq x^- \to x$ and $(x^{--} \to x)^- = 0$,
(11) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

The order of $a \in A$, $a \neq 1$, in symbols $ord(a)$ is the smallest $n \in N$ such that $a^n = 0$; if no such n exists, then $ord(a) = \infty$.

For any BL-algebra A, $B(A)$ denotes the *Boolean algebra* of all complemented elements in $L(A)$ (hence $B(A) = B(L(A))$).

Proposition 2.2. *([5],[6],[12]) For $e \in A$, the following statements are equivalent:*

(i) $e \in B(A)$,
(ii) $e * e = e$ and $e = e^{--}$,
(iii) $e * e = e$ and $e^- \to e = e$,
(iv) $e \vee e^- = 1$,
(v) $(e \to x) \to e = e$, for every $x \in A$.

Hájek [6] defined a *filter* of a BL-algebra A to be a nonempty subset F of A such that (i) $a, b \in F$ implies $a * b \in F$, and (ii) if $a \in F$, $a \leq b$, then $b \in F$. Turunen [10] defined a deductive system of a BL-algebra A. Note that a subset F of a BL-algebra A is a deductive system of A if and only if F is a filter of A. A proper filter F of A is called a *prime filter* of A if for all $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$. Equivalently, F is a prime filter of A if and only if for all $x, y \in A$, either $x \to y \in F$ or $y \to x \in F$. A proper filter M of A is a *maximal filter* of A if and only if $\forall x \notin M$, $\exists n \in N$ such that $(x^n)^- \in M$, see [10]. Let F be a proper filter of A. The intersection of all maximal filters of A containing F is called the *radical* of F and it is denoted by $Rad(F)$. We proved that $Rad(F) = \{a \in A: (a^n)^- \to a \in F, \text{ for all } n \in N\}$, for any filter F of A, (for details, see e.g. [9]).

Definition 2.1. ([1],[2],[7]) Let $x, y, z \in A$. A nonempty subset F of A is called:
∘ A *Boolean filter* of A, if F is a filter of A and $x \vee x^- \in F$, for all $x \in A$,
∘ A *primary filter* of A, if F is a proper filter of A and $(x * y)^- \in F$ implies $(x^n)^- \in F$ or $(y^n)^- \in F$, for some $n \in N$ and for all $x, y \in A$,
∘ A *fantastic filter* of A, if $1 \in F$ and $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$, for all $x \in A$,
∘ A *normal filter* of A, if F is a filter of A and $z \to ((y \to x) \to x) \in F$ and $z \in F$ imply that $(x \to y) \to y \in F$,
∘ An *obstinate filter* of A, if F is a filter of A and $x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$.

A BL-algebra A is called *local* if it has a unique maximal filter. A is a local BL-algebra if and only if $ord(x) < \infty$ or $ord(x^-) < \infty$, for all $x \in A$, see [11]. A BL-algebra A is called *linearly ordered* if $x \leq y$ or $y \leq x$, for all $x, y \in A$.

Theorem 2.3. ([7],[11],[13]) *Let F be a filter of a BL-algebra A. Then*
(1) $\frac{A}{F}$ is a local BL-algebra if and only if F is a primary filter of A.
(2) $\frac{A}{F}$ is a linearly ordered BL-algebra if and only if F is a prime filter of A.
(3) A is an MV-algebra if and only if $\{1\}$ is a fantastic filter of A.

3. $D(X_l)$ in BL-algebras

From now on, unless mentioned otherwise, $(A, \wedge, \vee, *, \to, 0, 1)$ will be a BL-algebra, which will often be referred by its support set A:

Definition 3.1. Let X be a nonempty subset of A. Then we define

$$D(X_l) = \{a \in A: \ a^{--} \to x = x, \text{ for all } x \in X\}$$

is called the left double stabilizers of X.

Let X be a nonempty subset of A. Then ${}^{\perp}X = \{a \in A: \ a \vee x = 1, \text{ for all } x \in X\}$ and $X_l = \{a \in A: \ a \to x = x, \text{ for all } x \in X\}$ are defined in [11] and [8], respectively. In Theorem 3.3 [8], proved that X_l is a filter of A.

In the following example we show that for some nonempty subset X of A, $X \nsubseteq D(X_l) \nsubseteq X$ and $D(X_l) \nsubseteq X_l$.

Example 3.1. Let $A = \{0, a, b, c, d, e, f, g, 1\}$, where $0 < a < b, d, e, g < 1$, $0 < d < e, g < 1$, $0 < b < e < 1$, $0 < c < d, e, f, g < 1$ and $0 < f < g < 1$. Define $*$ and $\to$ as follows:

$*$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	a	0	a	a	0	a	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	c	c	c	c	c	c
d	0	a	a	c	d	d	c	d	d
e	0	a	b	c	d	e	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	a	a	c	d	d	f	g	g
1	0	a	b	c	d	e	f	g	1

$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	f	1	1	f	1	1	f	1	1
b	f	g	1	f	g	1	f	g	1
c	b	b	b	1	1	1	1	1	1
d	0	b	b	f	1	1	f	1	1
e	0	a	b	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	0	b	b	c	e	e	f	1	1
1	0	a	b	c	d	e	f	g	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra. We take $X = \{b, e, 1\}$. Then $X_l = \{f, g, 1\}$ and $D(X_l) = \{c, d, e, f, g, 1\}$.

Theorem 3.1. *Let X be a nonempty subset of A. Then $a \in D(X_l)$ if and only if $(a^{--})^n \rightarrow x = x$, for all $x \in X$ and $n \in N$.*

Proof. Let $a \in D(X_l)$. Then $a^{--} \rightarrow x = x$, for all $x \in X$. So by Lemma 2.1(2) we get

$$\begin{aligned} x = a^{--} \rightarrow x &= a^{--} \rightarrow (a^{--} \rightarrow x) \\ &= a^{--} \rightarrow (a^{--} \rightarrow (a^{--} \rightarrow x)) \\ &= a^{--} * a^{--} * a^{--} \rightarrow x \\ &\vdots \\ &= a^{--} * a^{--} * \ldots * a^{--} \rightarrow x \\ &= (a^{--})^n \rightarrow x, \text{ for all } n \in N. \end{aligned}$$

The converse is clear. □

Theorem 3.2. *Let X be a nonempty subset of A. Then $D(X_l)$ is a filter of A.*

Proof. We know $1^{--} \rightarrow x = x$, for all $x \in X$, hence $1 \in D(X_l)$. Let $a, a \rightarrow b \in D(X_l)$. Then $a^{--} \rightarrow x = x$ and $(a \rightarrow b)^{--} \rightarrow x = x$ for all $x \in X$. Hence by Lemma 2.1(6), (9), for all $x \in X$, we have

$$\begin{aligned} x &\leq b^{--} \rightarrow x \leq (a^{--} \rightarrow b^{--}) \rightarrow (a^{--} \rightarrow x) \\ &= (a \rightarrow b)^{--} \rightarrow x = x. \end{aligned}$$

So $b^{--} \to x = x$, for all $x \in X$. Therefore $b \in D(X_l)$, i.e. $D(X_l)$ is a filter of A. $\square$

$D_s(X) = \{x \in X : \ x^- = 0\}$, where X is a nonempty subset of A, is called the set of dense elements of a nonempty subset X in a BL-algebra A.

In the following we study properties of double left stabilizer.

Theorem 3.3. *Let X and Y be two nonempty subsets and F be a filter of A. Then the following conditions hold:*

(1) $D(X_l) = \cap_{x\in X} D(\{x\}_l) = \{a \in A : \ (a^{--} \to x) \to x = 1, \text{ for all } x \in X\}$,

(2) if $a \in D(\{x\}_l)$, then $x \wedge a^{--} = x * a^{--}$ and $x \vee a^{--} = (x \to a^{--}) \to a^{--}$,

(3) $X_l \subseteq D(X_l)$ and ${}^{\perp}X \subseteq D(X_l)$,

(4) if $a^- \to a = a$, for all $a \in D(X_l)$, then $D(X_l) = X_l$,

(5) $D(\{0\}_l) = D_s(A)$ and $D(\{1\}_l) = A$,

(6) if $X \subseteq Y$, then $D(Y_l) \subseteq D(X_l)$,

(7) $D((X \cup Y)_l) = D(X_l) \cap D(Y_l) \subseteq D((X \cap Y)_l)$,

(8) $D((\frac{X}{F})_l) = \{\frac{a}{F} \in \frac{A}{F} \ : \ (a^{--} \to x) \to x \in F, \text{ for all } x \in X\}$ such that $F \subseteq X$.

Proof. (1) We have

$$\begin{aligned} a \in D(X_l) \quad &\Leftrightarrow \quad a^{--} \to x = x, \text{ for all } x \in X, \\ &\Leftrightarrow \quad a \in D(\{x\}_l), \text{ for all } x \in X, \\ &\Leftrightarrow \quad a \in \cap_{x\in X} D(\{x\}_l). \end{aligned}$$

Let $a \in D(X_l)$. Then $a^{--} \to x = x$, for all $x \in X$. Hence $(a^{--} \to x) \to x = 1$, for all $x \in X$.

Conversely, let $(a^{--} \to x) \to x = 1$, for all $x \in X$. Then $a^{--} \to x \leq x$, for all $x \in X$. Hence by Lemma 2.1(6), we get that $a^{--} \to x = x$, for all $x \in X$. Therefore $a \in D(X_l)$.

(2) Let $a \in D(\{x\}_l)$. Then $a^{--} \to x = x$, hence we have

$$a^{--} \wedge x = a^{--} * (a^{--} \to x) = a^{--} * x.$$

It is clear that $x \vee a^{--} = (x \to a^{--}) \to a^{--}$.

(3) Let $a \in X_l$. Then $a \to x = x$, for all $x \in X$. We have $a \leq a^{--}$, hence by Lemma 2.1(3) we have $a^{--} \to x \leq a \to x = x$, for all $x \in X$. So $a^{--} \to x = x$, for all $x \in X$, hence $a \in D(X_l)$.

Let $a \in^{\perp} X$. Then $a \vee x = 1$, for all $x \in X$. We have $1 = a \vee x \leq a^{--} \vee x$. Hence $a^{--} \vee x = 1$, for all $x \in X$. So $1 = ((a^{--} \to x) \to x) \wedge ((x \to a^{--}) \to a^{--})$, for all $x \in X$. Hence $(a^{--} \to x) \to x = 1$, for all $x \in X$. Then $a^{--} \to x = x$, for all $x \in X$. Therefore $a \in D(X_l)$.

(4) Let $a^- \to a = a$, for all $a \in D(X_l)$. Take $a \in D(X_l)$ then $a^{--} \to x = x$, for all $x \in X$. By Lemma 2.1(10) we have $a^{--} \leq a^- \to a$. Hence by Lemma 2.1(3), for all $x \in X$, we get that

$$(a^- \to a) \to x \leq a^{--} \to x = x.$$

So $(a^- \to a) \to x = x$, for all $x \in X$. Now by hypothesis $a \to x = x$, for all $x \in X$, i.e. $a \in X_l$. Therefore $D(X_l) \subseteq X_l$. Then by part (3) the proof is complete.

(5) We have

$$D(\{0\}_l) = \{a \in A : \ a^{--} \to 0 = 0\} = \{a \in A : \ a^- = 0\} = D_s(A).$$

$$D(\{1\}_l) = \{a \in A : \ a^{--} \to 1 = 1\} = A.$$

(6) Let $a \in D(Y_l)$. Then $a^{--} \to y = y$, for all $y \in Y$. Hence by hypothesis $a^{--} \to x = x$, for all $x \in X$, i.e. $a \in D(X_l)$.

(7) Let

$$\begin{aligned} z \in D((X \cup Y)_l) &\Rightarrow z^{--} \to a = a, \ \forall\, a \in X \cup Y, \\ &\Rightarrow z^{--} \to a = a, \ \forall\, a \in X \text{ and } z^{--} \to a = a, \ \forall\, a \in Y, \\ &\Rightarrow z \in D(X_l) \cap D(Y_l). \end{aligned}$$

Therefore $D((X \cup Y)_l) \subseteq D(X_l) \cap D(Y_l)$.

Conversely, let

$$\begin{aligned} z \in D(X_l) \cap D(Y_l) &\Rightarrow z^{--} \to x = x, \ \forall\, x \in X \text{ and } z^{--} \to y = y \ \forall\, y \in Y, \\ &\Rightarrow z^{--} \to c = c, \ \forall\, c \in X \cup Y, \\ &\Rightarrow z \in D((X \cup Y)_l). \end{aligned}$$

Hence $D(X_l) \cap D(Y_l) \subseteq D((X \cup Y)_l)$. Therefore $D((X \cup Y)_l) = D(X_l) \cap D(Y_l)$. Now let

$$\begin{aligned} z \in D(X_l) \cap D(Y_l) &\Rightarrow z^{--} \to x = x, \ \forall\, x \in X \text{ and } z^{--} \to y = y, \ \forall\, y \in Y, \\ &\Rightarrow z^{--} \to c = c, \ \forall\, c \in X \cap Y, \\ &\Rightarrow z \in D((X \cap Y)_l). \end{aligned}$$

Hence $D(X_l) \cap D(Y_l) \subseteq D((X \cap Y)_l)$.

(8) By Lemma 2.1(6) we have

$$\begin{aligned} D((\frac{X}{F})_l) &= \{\frac{a}{F} \in \frac{A}{F} \ : \ \frac{a^{--}}{F} \to \frac{x}{F} = \frac{x}{F}, \text{ for all } \frac{x}{F} \in \frac{X}{F}\} \\ &= \{\frac{a}{F} \in \frac{A}{F} \ : \ \frac{(a^{--} \to x) \to x}{F} = \frac{1}{F}, \text{ for all } \frac{x}{F} \in \frac{X}{F}\} \\ &= \{\frac{a}{F} \in \frac{A}{F} \ : \ (a^{--} \to x) \to x \in F, \text{ for all } x \in X\}. \end{aligned}$$

□

In the following examples we show that converse of parts (4) and (6) of the above theorem may not hold.

Example 3.2. (a) Let $A = \{0, a, b, c, d, 1\}$, where $0 < a < c < 1$ and $0 < b < c, d < 1$. Define $*$ and $\to$ as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra. Let $G = \{d, 1\}$. Then $D(G_l) = G_l = \{a, c, 1\}$, while $c^{-} \to c = 1 \neq c$.

(b) Consider the BL-algebra $A = \{0, a, b, c, d, e, f, g, 1\}$ in Example 3.2. We can see that $D(\{c\}_l) = \{d, e, g, 1\} \subseteq \{c, d, e, f, g, 1\} = D(\{b\}_l)$, while $\{b\} \not\subseteq \{c\}$.

By part (3) of the above theorem, we have $X_l \subseteq D(X_l)$. Hence by extension property in [2], [6] and [7], we get the following proposition.

Proposition 3.4. *Let X_l be a maximal, prime, (positive) implicative, fantastic, obstinate filter of A. Then $D(X_l)$ is so (respectively).*

Let F be a filter of A. $D(F) = \{x \in A : x^{--} \in F\}$, see [3].

Theorem 3.5. *Let F be a filter of A and $x \in A$. Then the following conditions are equivalent:*
(1) F is a normal filter of A,
(2) $F = D(F)$,
(3) if $x^{--} \in F$ implies $x \in F$,
(4) F is a fantastic filter of A.

Proof. By Remark 3.23 [3], Theorem 3.25 [3], Lemma 3.8 [4] and Corollary 3.11 [4] the proof is clear. □

Theorem 3.6. *A is an MV-algebra if and only if $D(X_l) = X_l$, for each nonempty subset X of A.*

Proof. Let A be an MV-algebra. Then $x^{--} = x$, for all $x \in A$. Hence $D(X_l) = X_l$, for each nonempty subset X of A.

Conversely, let $D(X_l) = X_l$, for each nonempty subset X of A. It is enough to show that $\{1\}$ is a fantastic filter of A. Let $a^{--} \in \{1\}$. So $a^{--} = 1$. Hence $a \in D(X_l) = X_l$, for each nonempty subset X of A. Take $X = \{a\}$. Thus $a \in \{a\}_l$ and so $a \to a = a$. Therefore $a = 1 \in \{1\}$. Hence By Theorem 3.8, $\{1\}$ is a fantastic filter of A. Then By Theorem 2.4(3), we get A is an MV-algebra. □

Theorem 3.7. *The following statements are equivalent:*
(1) $x^{--} = 1$, for all $0 \neq x \in A$,
(2) $D(X_l) = A - \{0\}$, for every $\{1\} \neq X \subseteq A$,
(3) $ord(x) = \infty$ and $ord(x^-) = 1$, for all $0 \neq x \in A$.

Proof. (1) ⇒ (2) Let $x^{--} = 1$, for all $0 \neq x \in A$. We have $D(X_l) \subseteq A - \{0\}$, for each $\{1\} \neq X \subseteq A$. Now let $a \in A - \{0\}$. Hence by part (1), $a^{--} = 1$. Then $a^{--} \to x = x$, for all $x \in X$, i.e. $a \in D(X_l)$. Therefore $A - \{0\} \subseteq D(X_l)$. Then $D(X_l) = A - \{0\}$, for every $\{1\} \neq X \subseteq A$.

(2) ⇒ (1) Let $D(X_l) = A - \{0\}$, for every $\{1\} \neq X \subseteq A$ and $a \in A - \{0\}$. By part (2), $a \in D(X_l)$, for every $\{1\} \neq X \subseteq A$. Hence $a^{--} \to x = x$, for all $x \in X$ and for every $\{1\} \neq X \subseteq A$. Take $X = \{a\}$. So $a^{--} \to a = a$. By Lemma 2.1(10), we have $(a^{--} \to a)^- = 0$. Hence $a^- = 0$. Therefore $a^{--} = 1$, for all $0 \neq a \in A$.

(2) ⇒ (3) Let $D(X_l) = A - \{0\}$, for every $\{1\} \neq X \subseteq A$ and $x \in A - \{0\}$. Then by ((2) ⇔ (1)) we have $x^{--} = 1$. Hence $x^- = 0$, i.e. $ord(x^-) = 1$. Now we have $x \in D(X_l) = A - \{0\}$. Since $D(X_l)$ is a filter of A then $x^n \in D(X_l) = A - \{0\}$, for all $n \in N$. Therefore $x^n \neq 0$, for all $n \in N$, i.e. $ord(x) = \infty$.

(3) ⇒ (2) Let $ord(x) = \infty$ and $ord(x^-) = 1$ for all $0 \neq x \in A$ and $a \in A - \{0\}$. Then $ord(a^-) = 1$. Hence $a^- = 0$ and so $a^{--} = 1$. We get that $a^{--} \to x = x$, for all $x \in X$. Therefore $a \in D(X_l)$, i.e. $A - \{0\} = D(X_l)$ for every $\{1\} \neq X \subseteq A$. □

By above theorem we have:

Corollary 3.8. *Let $D(X_l) = A - \{0\}$, for every $\{1\} \neq X \subseteq A$. Then A is a local BL-algebra.*

Theorem 3.9. *Let $x \in A$. Then the following statements hold:*

(1) $D(\{x\}_l)$ is a prime filter of A.

(2) $D(\{x\}_l)$ is a primary filter of A.

(3) $\frac{A}{D(\{x\}_l)}$ is a linearly ordered BL-algebra.

(4) $\frac{A}{D(\{x\}_l)}$ is a local BL-algebra.

Proof. (1) Let $a \vee b \in D(\{x\}_l)$ and $a, b \notin D(\{x\}_l)$. Then $a^{--} \to x \neq x$ and $b^{--} \to x \neq x$. Hence

$$x < a^{--} \to x \text{ and } x < b^{--} \to x. \text{ (I)}$$

Since $a \vee b \in D(\{x\}_l)$ by Lemma 2.1(11) we get

$$x = (a \vee b)^{--} \to x = (a^{--} \vee b^{--}) \to x = (a^{--} \to x) \wedge (b^{--} \to x).$$

And so $(a^{--} \to x) \wedge (b^{--} \to x) = x$. By (I) we have $x < (a^{--} \to x) \wedge (b^{--} \to x)$. Therefore $x < x$ which is a contradiction. So $a, b \in D(\{x\}_l)$, i.e. $D(\{x\}_l)$ is a prime filter of A.

(2) Since every prime filter of A is a primary filter of A, then by part (1) the proof is complete.

(3) By part (1) and Theorem 2.4(2), the proof is clear.

(4) By part (2) and Theorem 2.4(1), the proof is clear. □

Proposition 3.10. *Let $f : A \longrightarrow B$ be a BL-homomorphism, $\emptyset \neq X \subseteq A$ and $\emptyset \neq Y \subseteq B$. Then we have*

(1) $f(D(X_l)) \subseteq D((f(X))_l)$,

(2) if f is an injective homomorphism, $f^{-1}(Y) \neq \emptyset$, then

$$f^{-1}(D(Y_l)) \subseteq D((f^{-1}(Y))_l),$$

(3) if f is a surjective homomorphism, then $D((f^{-1}(Y))_l) \subseteq f^{-1}(D(Y_l))$.

Proof. (1) Let $b \in f(D(X_l))$. Then there exists $a \in D(X_l)$ such that $b = f(a)$. Hence $a^{--} \to x = x$, for all $x \in X$ and so $f(a)^{--} \to f(x) = f(x)$, for all $f(x) \in f(X)$. Thus $b^{--} \to f(x) = f(x)$, for all $f(x) \in f(X)$. Therefore, $b \in D((f(X))_l)$, i.e. $f(D(X_l)) \subseteq D((f(X))_l)$.

(2) Let $a \in f^{-1}(D(Y_l))$. Then $f(a) \in D(Y_l)$, i.e. $f(a)^{--} \to y = y$, for all $y \in Y$. Take $b \in f^{-1}(Y)$, so $f(b) \in Y$. Hence we have $f(a)^{--} \to f(b) = f(b)$, for all $b \in f^{-1}(Y)$. Since f is an injective homomorphism, we get that $a^{--} \to b = b$, for all $b \in f^{-1}(Y)$. Therefore $a \in D((f^{-1}(Y))_l)$, i.e. $f^{-1}(D(Y_l)) \subseteq D((f^{-1}(Y))_l)$.

(3) Let $a \in D((f^{-1}(Y))_l)$. Then $a^{--} \to b = b$, for all $b \in f^{-1}(Y)$. So we have $f(a)^{--} \to f(b) = f(b)$, for all $f(b) \in Y$. Thus we obtain $f(a)^{--} \to y = y$, for all $y \in Y$, since f is onto so $f(a) \in D(Y_l)$. Therefore $a \in f^{-1}(D(Y_l))$, hence $D((f^{-1}(Y))_l) \subseteq f^{-1}(D(Y_l))$. □

4. $D((X, Y)_l)$ in BL-algebras

Definition 4.1. Let X and Y be two nonempty subsets of A. Then we define

$$D((X, Y)_l) = \{a \in A : (a^{--} \to x) \to x \in Y, \text{ for all } x \in X\}$$

is called the left double stabilizers of X respect to Y.

Haveshki et al. defined

$$(X,Y)_l = \{a \in A : \ (a \to x) \to x \in Y, \ \text{for all } x \in X\}, \text{see [8]}.$$

Theorem 4.1. *Let F and G be two filters of A. Then $D((F,G)_l)$ is a filter of A.*

Proof. We have $(1^{--} \to x) \to x = 1 \in G$, for all $x \in F$. Hence $1 \in D((F,G)_l)$. Now let $a, a \to b \in D((F,G)_l)$. We must to prove that $b \in D((F,G)_l)$. Hence $(a^{--} \to x) \to x \in G$ and

$$((a \to b)^{--} \to x) \to x \in G, \ \text{for all } x \in F. \ \text{(I)}$$

Let $x \in F$. Since $x \le a^{--} \to x$, then $a^{--} \to x \in F$. So by (I), we get

$$((a \to b)^{--} \to (a^{--} \to x)) \to (a^{--} \to x) \in G. \ \text{(II)}$$

By Lemma 2.1(9), we get $b^{--} \to x \le (a^{--} \to b^{--}) \to (a^{--} \to x)$. Hence By Lemma 2.1(3)

$$((a^{--} \to b^{--}) \to (a^{--} \to x)) \to (a^{--} \to x) \le (b^{--} \to x) \to (a^{--} \to x).$$

So by (II), we have

$$(b^{--} \to x) \to (a^{--} \to x) \in G. \ \text{(III)}$$

We claim that $a^{--} \to x = ((a^{--} \to x) \to x) \to x$. We have

$$(a^{--} \to x) \to (((a^{--} \to x) \to x) \to x) = ((a^{--} \to x) \to x) \to ((a^{--} \to x) \to x) = 1.$$

Then $a^{--} \to x \le ((a^{--} \to x) \to x) \to x$. By parts (9) and (2) of Lemma 2.1 (respectively), we get

$$\begin{aligned} a^{--} \to x &\le ((a^{--} \to x) \to x) \to x \\ &\le ((a^{--} \to ((a^{--} \to x) \to x)) \to (a^{--} \to x) \\ &= ((a^{--} \to x) \to (a^{--} \to x)) \to (a^{--} \to x) = a^{--} \to x. \end{aligned}$$

So

$$a^{--} \to x = ((a^{--} \to x) \to x) \to x. \ \text{(IV)}$$

By Lemma 2.1(2) and by (IV), (III) (respectively), we get

$$\begin{aligned} ((a^{--} \to x) \to x) \to ((b^{--} \to x) \to x) &= (b^{--} \to x) \to (((a^{--} \to x) \to x) \to x) \\ &= (b^{--} \to x) \to (a^{--} \to x) \in G. \end{aligned}$$

We have $(a^{--} \to x) \to x \in G$. Therefore $(b^{--} \to x) \to x \in G$ for all $x \in F$. And so $b \in D((F,G)_l)$, i.e. $D((F,G)_l)$ is a filter of A. □

In the following example, we show that in Theorem 4.2, the condition "F and G be two filters of A" is necessary.

Example 4.1. Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define $*$ and $\to$ as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra. It is clear that $\{0\}$ and $\{b, 1\}$ are not filters and also $D((\{0\}, \{b,1\})_l) = \{b, 1\}$ and $D((\{b,1\}, \{0\})_l) = \phi$ are not filters of A.

Theorem 4.2. *Let X, Y, X_1 and Y_1 be nonempty subsets of A and F, G be two filters of A. Then the following statements hold:*

(1) if $1 \in Y$ then $D(X_l) \subseteq D((X,Y)_l)$ and so $D(X_l) \subseteq D((X,F)_l)$,
(2) if $D((X,Y)_l) = A$, then $X \subseteq Y$,
(3) if $F \subseteq Y$, then $D((F,Y)_l) = A$,
(4) $D((F,F)_l) = A$, hence $D(F_l) \subseteq D((F,F)_l)$,
(5) $D((X,\{1\})_l) = D(X_l)$ and $D((\{1\},G)_l) = A$,
(6) if $1 \in X, Y$, then $D(X_l) \cap D(Y_l) \subseteq D((X,Y)_l) \cap D((Y,X)_l)$,
(7) $D((X, Y \cap Y_1)_l) = D((X,Y)_l) \cap D((X,Y_1)_l)$,
(8) if $X \subseteq X_1$ and $Y \subseteq Y_1$, then $D((X_1,Y)_l) \subseteq D((X,Y_1)_l)$,
(9) $(X,F)_l \subseteq D((X,F)_l)$ and $F \subseteq D((X,F)_l)$,
(10) $D((\frac{F}{G})_l) = \frac{D((F,G)_l)}{G}$ such that $G \subseteq F$.

Proof. (1) Let $a \in D(X_l)$. Then $a^{--} \to x = x$, for all $x \in X$. Hence $(a^{--} \to x) \to x = 1 \in Y$, for all $x \in X$. Therefore $a \in D((X,Y)_l)$.

(2) Let $D((X,Y)_l) = A$. Hence $0 \in D((X,Y)_l)$. So $(0^{--} \to x) \to x \in Y$, for all $x \in X$. Therefore $x \in Y$, for all $x \in X$, i.e. $X \subseteq Y$.

(3) Let $a \in A$ and $x \in F$. We know $x \leq (a^{--} \to x) \to x$. So by filter property we get that $(a^{--} \to x) \to x \in F \subseteq Y$. Hence $a \in D((F,Y)_l)$, i.e. $D((F,Y)_l) = A$.

(4) By part (3), the proof is clear.

(5) Let $a \in D((X,\{1\})_l)$. Then $(a^{--} \to x) \to x \in \{1\}$, for all $x \in X$. Hence $(a^{--} \to x) \to x = 1$, for all $x \in X$. And so by Lemma 2.1, we get that $a^{--} \to x = x$, for all $x \in X$. Thus $a \in D(X_l)$. Therefore $D((X,\{1\})_l) \subseteq D(X_l)$. Then by part (1), the proof is complete.

Now let $a \in A$. We have $(a^{--} \to 1) \to 1 \in G$. Hence $a \in D((\{1\},G)_l)$, i.e. $A \subseteq D((\{1\},G)_l)$. Therefore $A = D((\{1\},G)_l)$.

(6) Let $z \in D(X_l) \cap D(Y_l)$. Hence $z \in D(X_l)$ and $z \in D(Y_l)$, so $z^{--} \to x = x$, for all $x \in X$ and $z^{--} \to y = y$, for all $y \in Y$. Thus $(z^{--} \to x) \to x = 1 \in Y$, for all $x \in X$, and $(z^{--} \to y) \to y = 1 \in X$, for all $y \in Y$. So we have $z \in D((X,Y)_l) \cap D((Y,X)_l)$.

(7) We have

$$\begin{aligned} a \in D((X, Y \cap Y_1)_l) &\Leftrightarrow (a^{--} \to x) \to x \in Y \cap Y_1, \ \forall\, x \in X, \\ &\Leftrightarrow (a^{--} \to x) \to x \in Y \text{ and } (a^{--} \to x) \to x \in Y_1, \ \forall\, x \in X, \\ &\Leftrightarrow a \in D((X,Y)_l) \text{ and } a \in D((X,Y_1)_l), \\ &\Leftrightarrow a \in D((X,Y)_l) \cap D((X,Y_1)_l). \end{aligned}$$

(8) Let $a \in D((X_1,Y)_l)$. Then $(a^{--} \to x_1) \to x_1 \in Y$, for all $x_1 \in X_1$. By hypothesis $Y \subseteq Y_1$, so $(a^{--} \to x_1) \to x_1 \in Y_1$, for all $x_1 \in X_1$. Hence by $X \subseteq X_1$, we get that $(a^{--} \to x) \to x \in Y_1$, for all $x \in X$. Therefore $a \in D((X,Y_1)_l)$.

(9) Let $a \in (X,F)_l$. Then $(a \to x) \to x \in F$, for all $x \in X$. By Lemma 2.1, $a \leq a^{--}$ then $(a \to x) \to x \leq (a^{--} \to x) \to x$. Hence $(a^{--} \to x) \to x \in F$, for all $x \in X$, i.e. $a \in D((X,F)_l)$. Therefore $(X,F)_l \subseteq D((X,F)_l)$.

Now let $a \in F$. Then for all $x \in X$, we have $a \leq a^{--} \leq (a^{--} \to x) \to x$. Hence $(a^{--} \to x) \to x \in F$, for all $x \in X$. Therefore $a \in D((X,F)_l)$, i.e. $F \subseteq D((X,F)_l)$.

(10) We have

$$\begin{aligned} D((\frac{F}{G})_l) &= \{\frac{a}{G} \in \frac{A}{G} \ : \ (\frac{a}{G})^{--} \to \frac{b}{G} = \frac{b}{G}, \ \forall \ \frac{b}{G} \in \frac{F}{G}\} \\ &= \{\frac{a}{G} \in \frac{A}{G} \ : \ \frac{(a^{--} \to b) \to b}{G} = \frac{1}{G}, \ \forall \ \frac{b}{G} \in \frac{F}{G}\} \\ &= \{\frac{a}{G} \in \frac{A}{G} \ : \ (a^{--} \to b) \to b \in G, \ \forall \ b \in F\} \\ &= \{\frac{a}{G} \in \frac{A}{G} \ : \ a \in D((F,G)_l)\} = \frac{D((F,G)_l)}{G}. \end{aligned}$$

□

In the following example, we show that the converse of part (9) of above theorem is not true.

Example 4.2. Consider the BL-algebra $A = \{0, a, b, c, d, e, f, g, 1\}$ in Example 3.2. Take $X = \{b, e, 1\}$ and $F = \{f, g, 1\}$. Clearly $D((X, F)_l) = \{c, d, e, f, g, 1\} \not\subseteq \{f, g, 1\} = (X, F)_l$.

By Theorems 4.4(5) and 3.9, we have

Corollary 4.3. *A is an MV-algebra if and only if $D((X, \{1\})_l) = X_l$, for every nonempty subset X of A.*

Theorem 4.4. *Let F and G be two filters of A. Then the following statements hold:*
(1) if F is an obstinate filter of A then $D((G, F)_l)$ is an obstinate filter of A.
(2) if F is a Boolean filter of A then $D((G, F)_l)$ is a Boolean filter of A.

Proof. (1) By Theorem 4.2, $D((G, F)_l)$ is a filter of A. Let $x, y \notin D((G, F)_l)$. Then there exist $a, b \in G$ such that $(x^{--} \to a) \to a \notin F$ and $(y^{--} \to b) \to b \notin F$. By Lemma 2.1 we have $x^{--} \leq (x^{--} \to a) \to a$ and $y^{--} \leq (y^{--} \to b) \to b$, hence $x^{--}, y^{--} \notin F$. Since F is an obstinate filter we get $x^{--} \to y^{--} \in F$ and $y^{--} \to x^{--} \in F$ and so

$$(x \to y)^{--} \in F \text{ and } (y \to x)^{--} \in F. \text{ (I)}$$

We have $(x \to y)^{--} \leq ((x \to y)^{--} \to z) \to z$ and $(y \to x)^{--} \leq ((y \to x)^{--} \to z) \to z$, for all $z \in G$. Hence by (I), we get $((x \to y)^{--} \to z) \to z \in F$ and $((y \to x)^{--} \to z) \to z \in F$, for all $z \in G$. Therefore $x \to y \in D((G, F)_l)$ and $y \to x \in D((G, F)_l)$, i.e. $D((G, F)_l)$ is an obstinate filter of A.

(2) By Theorem 4.2, $D((G, F)_l)$ is a filter of A. We have

$$\begin{aligned} x \vee x^- &\leq (x \vee x^-)^{--} \\ &\leq ((x \vee x^-)^{--} \to z) \to z), \text{ for all } z \in G. \end{aligned}$$

By hypothesis F is a Boolean filter of A, i.e. $x \vee x^- \in F$, for all $x \in A$. Therefore $((x \vee x^-)^{--} \to z) \to z) \in F$, for all $z \in G$. Hence for every $x \in A$ we have $x \vee x^- \in D((G, F)_l)$. So the proof is complete. □

By Theorem 4.1 [2] and Theorem 2.4, we get the following theorem.

Theorem 4.5. *Let F and G be two filters of A and F be an obstinate filter. Then we have*
(1) $\frac{A}{D((G,F)_l)}$ is a linearly ordered BL-algebra.
(2) $\frac{A}{D((G,F)_l)}$ is a local BL-algebra.

5. $N(F)$ in BL-algebras

Borumand et al. in [3] introduced $N(A) = \{a \in A : a^{--} = 1\}$. Now we define

Definition 5.1. Let F be a filter of A. We define

$$N(F) = \{a \in F : a^{--} = 1\}.$$

Theorem 5.1. *Let F be a filter of A. Then $N(F)$ is a filter of A.*

Proof. It is clear that $1 \in N(F)$. Now let $a, a \to b \in N(F) \subseteq F$. Then $b \in F$. We have $a^{--} = 1$ and $(a \to b)^{--} = 1$. So $1 \to b^{--} = 1$, i.e. $b^{--} = 1$. Then by $b \in F$, we have $b \in N(F)$. Hence the proof is complete. □

Proposition 5.2. *(1) If $N(F)$ is a maximal filter of A then $F = N(F)$.*
(2) If $N(F)$ is an obstinate filter of A then $F = N(F)$.

Proof. (1) By $N(F) \subseteq F$ the proof is clear.
(2) By Theorem 4.1 [2] and by part (1), the proof is clear. □

Example 5.1. Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define $*$ and $\to$ as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra and $F = \{b, 1\}$ is not a maximal filter so by Theorem 4.1 [2], F is not an obstinate filter, while $N(F) = F$.

Theorem 5.3. *Let F and G be two proper filters of A. Then the following conditions hold:*
(1) $B(A) \cap N(F) = N(\{1\}) = \{1\}$ and $N(N(F)) = N(F)$,
(2) $N(F) \subseteq D_s(A) \cap D(F)$ and $N(F) \subseteq \{a \in F : a^- \wedge a = a^-\}$,
(3) if $F \subseteq G$ then $N(F) \subseteq N(G)$,
(4) $N(F \cap G) = N(F) \cap N(G) \subseteq N(< F \cup G >)$,
(5) $Rad(N(F)) \subseteq \{a \in A : a^- * (a^n)^- = 0, \text{ for all } n \in N\}$ and $Rad(N(F)) \subseteq \{a \in A : ord(a^-) < \infty\}$,
(6) if $\frac{y}{N(F)} \in \frac{x}{N(F)}$ then $\frac{y^{--}}{F} \in \frac{x^{--}}{F}$,
(7) if $x^{--} = 1$, for every $x \in A$, then $N(F) = F$,
(8) $N(F) \subseteq N(A) \subseteq D(X_l)$ for every nonempty subset X of A,
(9) if $1 \in Y$ then $N(A) \subseteq D((X, Y)_l)$ for every nonempty subsets X and Y of A.

Proof. (1) Let $x \in B(A) \cap N(F)$. Then $x \in N(F)$, i.e. $x \in F$ and $x^{--} = 1$. And also $x \in B(A)$, i.e. $x^{--} = x$. Thus $x = 1$. Therefore $B(A) \cap N(F) \subseteq \{1\}$ and so $B(A) \cap N(F) = \{1\}$. It is clear that $N(\{1\}) = \{1\}$ and $N(N(F)) = N(F)$.

(2) Let $x \in N(F)$. Then $x \in F$ and $x^{--} = 1$. So $x \in A$ and $x^- = 0$. Therefore $x \in D_s(A)$, i.e. $N(F) \subseteq D_s(A)$. We have $N(F) \subseteq F \subseteq D(F)$. And so $N(F) \subseteq D_s(A) \cap D(F)$.

Let $a \in N(F)$. Then $a \in F$ and $a^{--} = 1$. By Lemma 2.1(10), we have $a^{--} \leq a^- \to a$. Thus $a^- \to a = 1$ and so $a \wedge a^- = a^-$.

(3) Let $x \in N(F)$. Then $x \in F$ and $X^{--} = 1$. So $x \in G$ and $x^{--} = 1$, i.e. $x \in N(G)$.

(4) We have $F \cap G \subseteq F, G$. So by part (3) we get $N(F \cap G) \subseteq N(F) \cap N(G)$. Now let $x \in N(F) \cap N(G)$. Then $x \in N(F)$ and $x \in N(G)$. Thus $x \in F$, $x^{--} = 1$ and $x \in G$, $x^{--} = 1$. We have $x \in F \cap G$ and $x^{--} = 1$, i.e. $x \in N(F \cap G)$. Therefore $N(F) \cap N(G) \subseteq N(F \cap G)$. Hence $N(F) \cap N(G) = N(F \cap G)$.

We know that $F, G \subseteq < F \cup G >$. So by part (3) we get $N(F) \cap N(G) \subseteq N(< F \cup G >)$.

(5) Let $a \in Rad(N(F))$. Then $a^- \to a^n \in N(F)$, for all $n \in N$. Then we have $1 = (a^- \to a^n)^{--} = a^- \to (a^n)^{--}$, for all $n \in N$. So $a^- \leq (a^n)^{--} = (a^n)^- \to 0$, for all $n \in N$. Then by $(BL3)$ we have $a^- * (a^n)^- = 0$, for all $n \in N$.

Now let $a \in Rad(N(F))$. Then $a^- * (a^n)^- = 0$, for all $n \in N$. Take $n = 1$, so $(a^-)^2 = a^- * a^- = 0$. Therefore $ord(a^-) < \infty$.

(6) We have

$$\begin{aligned} \frac{y}{N(F)} \in \frac{x}{N(F)} &\Rightarrow x \to y \in N(F) \text{ and } y \to x \in N(F) \\ &\Rightarrow (x \to y)^{--} = 1,\ (x \to y \in F),\ (y \to x)^{--} = 1,\ (y \to x \in F) \\ &\Rightarrow x^{--} \to y^{--} \in F,\ (x \to y \in F),\ y^{--} \to x^{--} \in F,\ (y \to x \in F) \\ &\Rightarrow \frac{y^{--}}{F} \in \frac{x^{--}}{F}. \end{aligned}$$

(7), (8) The proofs are clear.

(9) By part (8) and Theorem 4.4(1), the proof is clear. □

In the following example we show that the converse of part (3) in the above theorem may not hold.

Example 5.2. Consider the BL-algebra $A = \{0, a, b, c, d, 1\}$ in Example 3.6. We take $F = \{d, 1\}$ and $G = \{c, 1\}$. Clearly $N(F) = \{1\} \subseteq N(G) = \{c, 1\}$, while $F \nsubseteq G$.

Theorem 5.4. *Let F be a proper filter of A. Then $N(F) = N(A) = \{1\}$ if and only if A is an MV-algebra.*

Proof. Let $N(F) = N(A) = \{1\}$. Then $D(\{1\}) = \{a \in A \,:\, a^{--} = 1\} = N(A) = \{1\}$. So $D(\{1\}) = \{1\}$. Hence by Theorem 3.8, $\{1\}$ is a fantastic filter of A. And so by Theorem 2.4(3), A is an MV-algebra.

Conversely, the proof is clear. □

By Theorems 3.9 and 5.7 and Corollary 4.6, we have:

Corollary 5.5. *The following statements are equivalent:*

(1) $D(X_l) = X_l$, for each nonempty subset X of A,

(2) $D((X, \{1\})_l) = X_l$, for each nonempty subset X of A,

(3) $N(F) = N(A) = \{1\}$, for each proper filter F of A,

(4) A is an MV-algebra.

Proposition 5.6. *Let A does not generate by any nilpotent element of A and $N(F) = F$, for each proper filter F of A. Then $a^{--} = 1$ for all $a \in A$.*

Proof. Let $a \in A$. If $A =< a >$, so $0 \in < a >$ i.e $a^n = 0$ for some $n \in N$. Hence a is an nilpotent element of A which generate A, this is a contradiction. Thus we have $< a >\subset A$ and by hypothesis $N(< a >) =< a >$. Therefore $a \in N(< a >)$, i.e. $a^{--} = 1$. □

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(Somayeh Motamed) Department of Mathematics, Bandar Abbas Branch, Islamic Azad University, Bandar Abbas, Iran
E-mail address: somayeh.motamed@iauba.ac.ir, s.motamed63@yahoo.com