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On the existence of infinitely many solutions of a nonlinear Neumann problem involving the m-Laplace operator

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Abstract. This paper surveys the existence of infinitely many solutions of a nonlinear Neumann problem of the following type:

$$-\mathrm{div}(|\nabla u|^{m-2}\nabla u) + |u|^{m-2}u = f(x,u) \text{ in } \Omega, \quad |\nabla u|^{m-2}\frac{\partial u}{\partial \nu} = g(x,u) \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\partial/\partial\nu$ denotes the outward normal derivative, the functions $f(x,u)$ and $g(x,u)$ are continuous on $\overline{\Omega}\times\mathbb{R}$ and on $\partial\Omega\times\mathbb{R}$, respectively, and odd with respect to u, while the constant m satisfies certain alternative inequalities. More specifically, we demonstrate the existence of a sequence of solutions which diverge to infinity provided that the nonlinear term is locally superlinear and the existence of a sequence of solutions which converge to zero provided that the nonlinear term is locally sublinear.

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1. Introduction

In this paper we will analyze nonlinear elliptic equations with nonlinear Neumann boundary conditions. We start from the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) + |u|^{m-2}u = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{m-2}\frac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, ν stands for the outward unit normal to $\partial\Omega$, $\Delta_m\cdot = \mathrm{div}(|\nabla\cdot|^{m-2}\nabla\cdot)$ denotes the m-Laplace operator, the functions $f(x,u)$ and $g(x,u)$ are continuous on $\overline{\Omega}\times\mathbb{R}$ and on $\partial\Omega\times\mathbb{R}$, respectively, and odd with respect to u, while the constant m satisfies some inequalities which we will provide below. Before that, we introduce a representative example, namely

$$\begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) + |u|^{m-2}u = a(x)|u|^{p-1}u & \text{in } \Omega, \\ |\nabla u|^{m-2}\frac{\partial u}{\partial \nu} = b(x)|u|^{q-1}u & \text{on } \partial\Omega \end{cases} \tag{2}$$

provided that $a \in C(\overline{\Omega})$, $b \in C(\partial\Omega)$, $a(x)$ and $b(x)$ may change their signs, $a(x_1) > 0$ at some $x_1 \in \Omega$, $b(x_2) > 0$ at some $x_2 \in \partial\Omega$ and p, q fulfill either

$$0 < q < m-1 < p < \frac{(m-1)N+m}{N-m} \tag{3}$$

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or

$$0 < p < m - 1 < q < \frac{(m-1)N}{N-m}. \tag{4}$$

Taking $N = 1, 2, ..., m$, the right hand sides of (3) and (4) are replaced by ∞. Therefore, we shall prove (see Example 3.1) the fact that problem (2) admits at least two sequences u_k and v_k of solutions so as

$$\|u_k\|_{W^{1,m}(\Omega)} \to 0, \quad \|u_k\|_{C(\overline{\Omega})} \to 0 \text{ as } k \to \infty,$$

$$\|v_k\|_{W^{1,m}(\Omega)} \to \infty, \quad \|v_k\|_{C(\overline{\Omega})} \to \infty \text{ as } k \to \infty,$$

where $\| \cdot \|_{W^{1,m}(\Omega)}$ represents the $W^{1,m}(\Omega)$ norm, that is,

$$\|u\|_{W^{1,m}(\Omega)} := \left(\int_\Omega (|\nabla u|^m + |u|^m)\, dx \right)^{\frac{1}{m}},$$

and $\| \cdot \|_{C(\overline{\Omega})}$ denotes the maximum norm.

In the first instance, we recall the m-Laplace Emden-Fowler equation

$$-\mathrm{div}(|\nabla u|^{m-2} \nabla u) = |u|^{p-1} u \quad \text{in } \Omega \subseteq \mathbb{R}^N, \tag{5}$$

where $N \geq m$, $0 < m - 1 < p$, approached in [5] where the author analyzed the isolated singularities and the behavior near infinity of nonradial positive solutions when $p < \frac{(m-1)N}{N-m}$, giving a complete classification of local and global radial solutions of any sign, for any p.

When $m = 2$, equation (5) becomes the Emden-Fowler equation

$$-\Delta u = |u|^{p-1} u \quad \text{in } \Omega \subseteq \mathbb{R}^N, \tag{6}$$

which has been intensively studied. If $N \geq 3$, two critical values $\frac{N}{N-2}$ and $\frac{N+2}{N-2}$ arise. It is known that the first studies in *the radial case* are due to Emden; afterwards, in papers [14], [15], [16], Fowler has provided us existence results and a full classification of the global radial solutions in $\mathbb{R}^N$ or $\mathbb{R}^N \setminus \{0\}$. In *the nonradial case* we refer to Lions [26] where the behavior near origin for positive solutions was studied for $p < \frac{N}{N-2}$, while for the case $p = \frac{N}{N-2}$ we find similar approach in Aviles [4]; next, Gidas and Spruck ([19]) pointed out local and global results for $p < \frac{N+2}{N-2}$, and then Caffarelli, Gidas and Spruck [6] obtained similar results when $p = \frac{N+2}{N-2}$.

For the general case when $m > 1$, we refer to Ni and Serrin [28], where was demonstrated existence of the following critical values:

$$q_1 = \frac{(m-1)N}{N-m} \quad \text{and} \quad q_2 = \frac{(m-1)N + m}{N-m},$$

if $N > m$. Then, in [20] Guedda and Véron obtained *radial positive solutions* near origin when $p < q_2$; also, for *the nonradial case* when $p < q_1$, the authors provided some results under conditions of integrability or majorization of u near origin.

Furthermore, in terms of sign changing solutions, the equation (5) (in possibly unbounded domains or in the whole space $\mathbb{R}^N$) was treated on stability of solutions. For instance, Damascelli, Farina, Sciunzi and Valdinoci [11] proved Liouville type theorems for *stable solutions* or for solutions which are stable outside a compact set. The results hold true for $m > 2$ and $m - 1 < p < p_c(N, m)$, where $p_c(N, m)$ is a new critical exponent, which is infinity in low dimension and is always larger than the classical critical one.

Problems like (5) were also studied in the context of the Neumann boundary condition; see, e.g., the very recent paper Gasiński and Papageorgiou [18].

Returning to the Emden-Fowler equation (6) with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$, such that $1 < p < \infty$ if $N = 1, 2$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, we cite Ambrosetti [3] and Rabinowitz [29] where we can see that this problem has a sequence u_k of solutions in $H_0^1(\Omega)$ whose $H_0^1(\Omega)$ norm diverges to infinity as $k \to \infty$. Considering problem (6), Kajikiya and Naimen explained intelligible the aim of paper [23]; the same approach we take into account for understand the purpose of the present paper too.

We shall introduce a *locally sublinear condition* and a *locally superlinear condition*. Kajikiya and Naimen proposed the same conditions in [23], while DeFigueiredo, Gossez and Ubilla put similar conditions in [12] and [13], but anyway their conditions differ from ours, and besides this, they demonstrated only the existence of positive solutions. In the present paper we firstly propose to prove the existence of infinitely many solutions. Thus, we intend to expand the results of Garcia-Azorero, Peral and Rossi [17], and Naimen [27] to nonlinear terms $f(x, u)$ and $g(x, u)$ or, in another train of thoughts, to extend the results in [23] by replacing the Laplace operator with m-Laplace operator. On the other hand, we want to show that the locally sublinear condition implies the existence of solutions converging to zero, while the locally superlinear condition implies the existence of solutions diverging to infinity. For this reason, it is possible to consider the case when $a(x)$ and $b(x)$ change their signs in problem (2). The last goal is to deal the following three cases:

- One of $f(x, u)$ and $g(x, u)$ is locally sublinear and another is locally superlinear.
- $f(x, u)$ is both locally sublinear and locally superlinear.
- $g(x, u)$ is both locally sublinear and locally superlinear.

Therefore, for each of the three cases from above, we shall demonstrate that there exist at least two sequences of solutions so as one sequence converges to zero and another diverges to infinity.

2. The main results

We say that u is *weak solution* of problem (1) if $u \in W^{1,m}(\Omega) \cap C(\overline{\Omega})$ and satisfies

$$\int_\Omega \left(|\nabla u|^{m-2}\nabla u \nabla v + |u|^{m-2}uv\right) dx - \int_\Omega f(x, u)v\, dx - \int_{\partial\Omega} g(x, u)v\, d\sigma = 0, \tag{7}$$

for any $v \in W^{1,m}(\Omega)$, where $d\sigma$ means the surface measure on $\partial\Omega$. Considering the Sobolev embedding, we infer that each $v \in W^{1,m}(\Omega)$ belongs to $L^{\frac{mN}{N-m}}(\Omega)$ and $L^{\frac{m(N-1)}{N-m}}(\partial\Omega)$ when $N > m$ and to $L^r(\Omega) \cap L^r(\partial\Omega)$ for $r < \infty$ and $N = 1, 2, ..., m$. Consequently, relation (7) is well defined. Next, we consider

$$F(x, u) := \int_0^u f(x, t)\, dt \quad \text{and} \quad G(x, u) := \int_0^u g(x, t)\, dt. \tag{8}$$

The following assumption denotes *the locally sublinear condition* on f and g:

Assumption 2.1. *The functions $f(x, s)$ and $g(x, s)$ are continuous on $\overline{\Omega} \times \mathbb{R}$ and on $\partial\Omega \times \mathbb{R}$, respectively, and odd with respect to s. We assume either (**f1**) or (**g1**) from below.*

(f1): *There are an element $x_0 \in \Omega$ and $\delta > 0$ so as $B(x_0, \delta) \subset \Omega$ and*

$$\lim_{s \to 0} \left(\inf_{x \in B(x_0, \delta)} \frac{F(x, s)}{|s|^m} \right) = \infty, \tag{9}$$

where $B(x_0, \delta)$ stands for a ball centered at x_0 and of radius δ.

(g1): *There are an element $x_0 \in \partial\Omega$ and $\delta > 0$ so as*

$$\lim_{s\to 0}\left(\inf\left\{\frac{G(x,s)}{|s|^m};\ x\in B(x_0,\delta)\cap\partial\Omega\right\}\right)=\infty \tag{10}$$

and

$$\liminf_{s\to 0}\left(\inf\left\{\frac{F(x,s)}{|s|^m};\ x\in B(x_0,\delta)\cap\Omega\right\}\right)>-\infty. \tag{11}$$

We call (**f1**) and (**g1**) a *locally sublinear condition.*

Theorem 2.2. *Under Assumption 2.1, there is a sequence u_k of solutions of problem* (1) *so as $u_k \in W^{1,r}(\Omega)$ for any $r < \infty$, $u_k \not\equiv 0$ and $\|u_k\|_{W^{1,r}(\Omega)}$ converges to zero when $k \to \infty$.*

We point out some remarks.

Remark 2.1. Using the Sobolev embedding, since u_k belongs to $W^{1,r}(\Omega)$ for any $r < \infty$, then $u_k \in C^{0,\theta}(\overline{\Omega})$ for every $\theta \in (0,1)$.

Remark 2.2. Theorem 2.2 states that if f or g is sublinear in a neighborhood of a point x_0, there is a sequence of solutions which converges to zero (it happens even if $f(x,u)$ and $g(x,u)$ have any behavior except for a neighborhood of x_0).

Remark 2.3. If $f(x,s)/(|s|^{m-2}s)$ diverges to infinity uniformly on $B(x_0,\delta)$, then for any $L > 0$ there is an $\varepsilon > 0$ so as

$$f(x,s)\ge L|s|^{m-2}s \ \text{ for } \ x\in B(x_0,\delta),\ |s|<\varepsilon.$$

Integrating both sides with respect to s we obtain

$$F(x,s)\ge (L/m)|s|^m \ \text{ for } \ x\in B(x_0,\delta),\ |s|<\varepsilon,$$

that is, (**f1**) holds. This means that assumption (**f1**) is weaker than the condition that the infimum of $f(x,s)/(|s|^{m-2}s)$ in $B(x_0,\delta)$ diverges to infinity as $s \to 0$. Similarly, we can deduce that assumption (10) is weaker than the condition $g(x,s)/(|s|^{m-2}s)$ diverges to infinity.

Assumption 2.3. *There are constants μ, τ, p, q and C so as $0 \le \tau < m < \mu$, $C > 0$, $0 < p < \infty$ for $N = 1,2,...,m$ and $0 < p < ((m-1)N+m)/(N-m)$ for $N > m$, $0 < q < \infty$ for $N = 1,2,...,m$ and $0 < q < (m-1)N/(N-m)$ if $N > m$ and*

$$|f(x,s)|\le C(|s|^p+1), \tag{12}$$
$$\mu F(x,s)-sf(x,s)\le C(|s|^\tau+1), \tag{13}$$

for $s \in \mathbb{R}$ and $x \in \Omega$, where F is given by (8) *and*

$$|g(x,s)|\le C(|s|^q+1), \tag{14}$$
$$\mu G(x,s)-sg(x,s)\le C(|s|^\tau+1), \tag{15}$$

for $s \in \mathbb{R}$ and $x \in \partial\Omega$, where G is given by (8).

We emphasize that under *subcritical conditions* (12) and (14), the Lagrangian functional is well defined in $W^{1,m}(\Omega)$, while assumptions (13) and (15) assure the Palais-Smale condition.

The next assumption means the *locally superlinear condition* on f and g:

Assumption 2.4. *The functions $f(x,s)$ and $g(x,s)$ are continuous on $\overline{\Omega}\times\mathbb{R}$ and on $\partial\Omega\times\mathbb{R}$, respectively, and odd with respect to s. We assume either (**f2**) or (**g2**) below.*

(f2): *There are an element* $x_0 \in \Omega$ *and* $\delta > 0$ *so as* $B(x_0, \delta) \subset \Omega$ *and*

$$\lim_{|s|\to\infty} \left(\inf_{x\in B(x_0,\delta)} \frac{F(x,s)}{|s|^m} \right) = \infty. \tag{16}$$

(g2): *There are an element* $x_0 \in \partial\Omega$ *and* $\delta > 0$ *so as*

$$\lim_{|s|\to\infty} \left(\inf \left\{ \frac{G(x,s)}{|s|^m};\ x \in B(x_0,\delta) \cap \partial\Omega \right\} \right) = \infty \tag{17}$$

and

$$\liminf_{|s|\to\infty} \left(\inf \left\{ \frac{F(x,s)}{|s|^m};\ x \in B(x_0,\delta) \cap \Omega \right\} \right) > -\infty. \tag{18}$$

We call (**f2**) and (**g2**) a *locally superlinear condition.* Similar arguments as in Remark 2.3 yield that (**f2**) is weaker than the condition that $f(x,s)/(|s|^{m-2}s)$ diverges to infinity as $s \to \infty$ uniformly on $B(x_0, \delta)$.

Theorem 2.5. *Under Assumptions 2.3 and 2.4, there is a sequence* v_k *of solutions of problem* (1) *so as* v_k *belongs to* $W^{1,r}(\Omega)$ *for every* $r < \infty$, $\|v_k\|_{W^{1,m}(\Omega)}$ *and* $\|v_k\|_{C(\overline{\Omega})}$ *diverge to infinity as* $k \to \infty$. *Therefore,* $\|v_k\|_{W^{1,r}(\Omega)}$ *also diverges to infinity for every* r *that meets* $m \le r < \infty$.

By combining Theorem 2.2 and Theorem 2.5 we get the following result:

Corollary 2.6. *Under Assumptions 2.1, 2.3 and 2.4, there are at least two sequences* u_k *and* v_k *of solutions of problem* (1) *so as* u_k, $v_k \in W^{1,r}(\Omega)$ *for every* $r < \infty$ *and* $\|u_k\|_{W^{1,r}(\Omega)} \to 0$, $\|v_k\|_{W^{1,r}(\Omega)} \to \infty$ *as* $k \to \infty$ *for every* r *that meets* $m \le r < \infty$.

3. Examples

This section contains several examples of the nonlinear terms $f(x,u)$ and $g(x,u)$. We apply our theorems to them to prove the existence of at least two sequences of solutions.

Example 3.1. Let $f(x,u) = a(x)|u|^{p-1}u$ and $g(x,u) = b(x)|u|^{q-1}u$ so as $a \in C(\overline{\Omega})$, $b \in C(\partial\Omega)$, $a(x_1) > 0$, $b(x_2) > 0$ at some $x_1 \in \Omega$, $x_2 \in \partial\Omega$, and p, q fulfill either (3) or (4). Also, $a(x)$ and $b(x)$ may change their signs. Then there are two sequences u_k and v_k of solutions of problem (1) so as u_k, $v_k \in W^{1,r}(\Omega)$ for every $r < \infty$ and

$$\lim_{k\to\infty} \|u_k\|_{W^{1,r}(\Omega)} = 0, \quad \lim_{k\to\infty} \|v_k\|_{W^{1,r}(\Omega)} = \infty \ \text{ for } m \le r < \infty.$$

To prove this assertion, we assume that (3) is satisfied. Since $a(x_1) > 0$, we pick $\delta > 0$ sufficiently small such that $a(x) > a(x_1)/2$ occurs in $B(x_1, \delta)$. Therefore,

$$\frac{F(x,s)}{|s|^m} = a(x)\frac{|s|^{p-m+1}}{p+1} \ge \frac{a(x_1)|s|^{p-m+1}}{2(p+1)},$$

that is, (**f2**) holds true because $p > m-1$. Next, we want to show that (**g1**) is verified by replacing x_1 by x_2. Indeed, since $b(x_2) > 0$, we pick $\delta > 0$ small enough such that $b(x) > b(x_2)/2$ occurs in $B(x_2, \delta) \cap \partial\Omega$. Consequently,

$$\frac{G(x,s)}{|s|^m} = b(x)\frac{|s|^{q-m+1}}{q+1} \ge \frac{b(x_2)|s|^{q-m+1}}{2(q+1)}.$$

Since $q < m-1$, we derive that (10) is verified. Further, it is obvious that

$$\frac{F(x,s)}{|s|^m} \ge -\|a\|_{L^\infty(\Omega)} \frac{|s|^{p-m+1}}{p+1},$$

which ensures (11) (recall that $\|\cdot\|_{L^\infty(\Omega)}$ denotes the $L^\infty(\Omega)$ norm). We also shall show that Assumption 2.3 takes place. In truth, we have

$$|f(x,s)| = |a(x)||s|^p \le C(|s|^p + 1),$$

$$|g(x,s)| = |b(x)||s|^q \le C(|s|^q + 1),$$

that means, relations (12) and (14) hold. Let $\mu := p+1$. This implies

$$\begin{aligned}\mu F(x,s) - sf(x,s) &= (p+1)\int_0^s f(x,t)\,dt - sf(x,s)\\ &= a(x)\int_0^s (p+1)|t|^{p-1}t\,dt - a(x)|s|^{p+1}\\ &= a(x)\int_0^s \left(|t|^{p+1}\right)'\,dt - a(x)|s|^{p+1}\\ &= 0,\end{aligned}$$

which leads us to relation (13). Finally, $G(x,s)$ can be estimated as

$$\begin{aligned}\mu G(x,s) - sg(x,s) &= (p+1)\int_0^s g(x,t)\,dt - sg(x,s)\\ &= b(x)\frac{p+1}{q+1}\int_0^s (q+1)|t|^{q-1}t\,dt - b(x)|s|^{q+1}\\ &= b(x)\frac{p+1}{q+1}\int_0^s \left(|t|^{q+1}\right)'\,dt - b(x)|s|^{q+1}\\ &= b(x)\frac{p-q}{q+1}|s|^{q+1}\\ &\le \frac{p-q}{q+1}\|b\|_{L^\infty(\partial\Omega)}|s|^{q+1},\end{aligned}$$

which yields relation (15). Hence, all conditions of Assumption 2.3 are fulfilled. We conclude that there are at least two sequences of solutions.

On the other hand, it can be proved that Assumptions 2.1, 2.3 and 2.4 hold also if we consider relation (4) instead of (3).

Example 3.2. Set $f(x,u) := a(x)|u|^{p-1}u + b(x)|u|^{q-1}u$ and $g \equiv 0$. The functions a, $b \in C(\overline{\Omega})$, $a(x_1) > 0$, $b(x_2) > 0$ at some x_1, $x_2 \in \Omega$, p and q verify

$$0 < q < m-1 < p < \frac{(m-1)N+m}{N-m} \quad \text{when } N > m, \tag{19}$$

$$0 < q < m-1 < p < \infty \quad \text{when } N = 1,2,...,m. \tag{20}$$

In this example x_1 and x_2 may be equal. We claim that the same conclusion as in Example 3.1 occurs. Here $f(x,u)$ is *superlinear* near x_1 as $u \to \pm\infty$, *sublinear* near x_2 as $u \to 0$, and fulfills relation (13) by taking $\mu := p+1$. Another example is also $f \equiv 0$ and $g(x,u) := a(x)|u|^{p-1}u + b(x)|u|^{q-1}u$. Here a, $b \in C(\partial\Omega)$, $a(x_1) > 0$, $b(x_2) > 0$ at some x_1, $x_2 \in \partial\Omega$, p and q fulfill (4).

Example 3.3. Set $f(x,u) = a(x)|u|^{p-1}u - b(x)u\log|u|$ and $g \equiv 0$. The functions a, $b \in C(\overline{\Omega})$, $a(x)$ may change its sign, while $b(x) \ge 0$ in Ω, $a(x_1) > 0$, $b(x_2) > 0$, at some x_1, $x_2 \in \Omega$, $1 < p < ((m-1)N+m)/(N-m)$ for $N > m$, and $1 < p < \infty$ for $N = 1,2,...,m$. Then $f(x,u)$ is *superlinear* near x_1 as $u \to \pm\infty$ and *sublinear* near

x_2 as $u \to 0$. For $\mu := p+1$, we obtain

$$\begin{aligned}\mu F(x,s) - sf(x,s) &= (p+1)\int_0^s f(x,t)\,dt - sf(x,s)\\ &= a(x)\int_0^s \left(|t|^{p+1}\right)'\,dt - (p+1)b(x)\int_0^s t\log|t|\,dt\\ &\quad - a(x)|s|^{p+1} + b(x)s^2\log|s|\\ &= -(p+1)b(x)\frac{s^2}{2}\log|s| + (p+1)b(x)\frac{s^2}{4} + b(x)s^2\log|s|\\ &= -b(x)\left(\frac{p-1}{2}\log|s| - \frac{p+1}{4}\right)s^2\\ &\le C,\end{aligned}$$

with $x \in \overline{\Omega}$, $s \in \mathbb{R}$ and $C > 0$ independent of x and s. In other words f fulfills (13). Thus, we get the same conclusion as in Example 3.1. Also, the same assertion as above is valid for $f \equiv 0$ and $g(x,u) := a(x)|u|^{p-1}u - b(x)u\log|u|$.

4. A priori estimates

In this section we establish that a weak solution in $W^{1,m}(\Omega)$ belongs to $W^{1,r}(\Omega)$ for any $r < \infty$ and give $W^{1,r}(\Omega)$ a priori estimates. We consider that $f(x,s)$ and $g(x,s)$ fulfill inequalities (12) and (14), respectively. This means that the functions f and g are subcritical. Thus, for $u \in W^{1,m}(\Omega)$, we have

$$f(x,u) \in L^{\frac{mN}{(m-1)N+m}}(\Omega), \quad g(x,u) \in L^{\frac{m(N-1)}{(m-1)N}}(\partial\Omega),$$

if $N > m$, and

$$f(x,u) \in L^r(\Omega), \quad g(x,u) \in L^r(\partial\Omega) \ \text{ for every } r < \infty,$$

if $N = 1, 2, ..., m$. Now we are in position to assert that all integrals in (7) are finite for every u, $v \in W^{1,m}(\Omega)$. When the functions f and g satisfy (12) and (14), we say that u is a $W^{1,m}(\Omega)$ *solution* if $u \in W^{1,m}(\Omega)$ and also fulfills (7) for every $v \in W^{1,m}(\Omega)$.

The following lemma plays a significant role in our analysis, seeing that will be indirectly used to demonstrate both Theorems 2.2 and 2.5.

Lemma 4.1. *Let $f(x,s)$ and $g(x,s)$ fulfill* (12) *and* (14), *respectively. Then every $W^{1,m}(\Omega)$ solution u belongs both $L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$ and satisfies*

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\partial\Omega)} \le C\left(\|u\|^d_{W^{1,m}(\Omega)} + 1\right), \tag{21}$$

where C, d are positive constants independent of u.

Proof. Let $\lambda > 1$ a parameter which we will choose later. If we multiply the first equation in (1) by $|u|^{\lambda-1}u$, integrate over Ω and use the second equation in (1), we acquire

$$\lambda\int_\Omega |\nabla u|^m|u|^{\lambda-1}\,dx + \int_\Omega |u|^{\lambda+m-1}\,dx = \int_\Omega f(x,u)|u|^{\lambda-1}u\,dx + \int_{\partial\Omega} g(x,u)|u|^{\lambda-1}u\,d\sigma.$$

For simplicity, we denote the $L^p(\Omega)$ norm of u by $\|u\|_{p,\Omega}$ and the $L^q(\partial\Omega)$ norm of u by $\|u\|_{q,\partial\Omega}$. Also, hereafter $C > 0$ denotes various constants independent of u and λ.

We now employ (12) and (14) involving that

$$\lambda \int_\Omega |\nabla u|^m |u|^{\lambda-1}\,dx + \int_\Omega |u|^{\lambda+m-1}\,dx \\ \leq C\left(\|u\|^{\lambda+p}_{\lambda+p,\Omega} + \|u\|^{\lambda}_{\lambda,\Omega} + \|u\|^{\lambda+q}_{\lambda+q,\partial\Omega} + \|u\|^{\lambda}_{\lambda,\partial\Omega}\right). \tag{22}$$

By a simple computation we can show that

$$|\nabla u|^m |u|^{\lambda-1} = \left(\frac{m}{\lambda+m-1}\right)^m \left|\nabla\left(|u|^{(\lambda+m-1)/m}\right)\right|^m.$$

We put $v = |u|^{(\lambda+m-1)/m}$, and thus $|\nabla v|^m = \left|\nabla\left(|u|^{(\lambda+m-1)/m}\right)\right|^m$, to get

$$\lambda \int_\Omega |\nabla u|^m |u|^{\lambda-1}\,dx + \int_\Omega |u|^{\lambda+m-1}\,dx = \lambda\left(\frac{m}{\lambda+m-1}\right)^m \int_\Omega |\nabla v|^m\,dx \\ + \int_\Omega |v|^m\,dx. \tag{23}$$

Combining (22) with (23) we get

$$\lambda\left(\frac{m}{\lambda+m-1}\right)^m \int_\Omega |\nabla v|^m\,dx + \int_\Omega |v|^m\,dx \\ \leq C\left(\|u\|^{\lambda+p}_{\lambda+p,\Omega} + \|u\|^{\lambda}_{\lambda,\Omega} + \|u\|^{\lambda+q}_{\lambda+q,\partial\Omega} + \|u\|^{\lambda}_{\lambda,\partial\Omega}\right).$$

The fact that $\lambda > 1$ provides us $\lambda\left(\frac{m}{\lambda+m-1}\right)^m < 1$. Therefore, by the above inequality we arrive at

$$\|v\|^m_{W^{1,m}(\Omega)} \leq \frac{C}{\lambda}\left(\frac{\lambda+m-1}{m}\right)^m \left(\|u\|^{\lambda+p}_{\lambda+p,\Omega} + \|u\|^{\lambda}_{\lambda,\Omega} + \|u\|^{\lambda+q}_{\lambda+q,\partial\Omega} + \|u\|^{\lambda}_{\lambda,\partial\Omega}\right). \tag{24}$$

Further, we consider $N > m$ and intend to show that

$$p - q = \frac{m}{N-m}. \tag{25}$$

Therefor we take p_1 satisfying the relations

$$p < p_1 \quad \text{and} \quad p + \frac{m}{N-m} < p_1 < \frac{(m-1)N+m}{N-m}.$$

We choose $q_1 := p_1 - m/(N-m)$ and thus $q < q_1 < (m-1)N/(N-m)$. Thereby, the functions $f(x,s)$ and $g(x,s)$ fulfill (12) and (14), respectively, by replacing p and q with p_1 and q_1, respectively. For the sake of simplicity, we rewrite p_1 and q_1 as p and q, respectively. So, indeed, p and q fulfill identity (25). Employing the Sobolev embedding, we derive that

$$\|v\|^m_{mN/(N-m),\Omega} + \|v\|^m_{m(N-1)/(N-m),\partial\Omega} \leq C\|v\|^m_{W^{1,m}(\Omega)}. \tag{26}$$

Using (24) together with (26) and taking $v = |u|^{(\lambda+m-1)/m}$, it follows that

$$\|u\|^{\lambda+m-1}_{\frac{(\lambda+m-1)N}{N-m},\Omega} + \|u\|^{\lambda+m-1}_{\frac{(\lambda+m-1)(N-1)}{N-m},\partial\Omega} \\ \leq \frac{C(\lambda+m-1)^m}{\lambda m^m}\left(\|u\|^{\lambda+p}_{\lambda+p,\Omega} + \|u\|^{\lambda}_{\lambda,\Omega} + \|u\|^{\lambda+q}_{\lambda+q,\partial\Omega} + \|u\|^{\lambda}_{\lambda,\partial\Omega}\right).$$

By virtue of Hölder's inequality we have

$$\|u\|^{\lambda}_{\lambda,\Omega} = \int_\Omega |u|^\lambda\,dx \leq |\Omega|^{\frac{p}{\lambda+p}}\left(\int_\Omega |u|^{\lambda+p}\,dx\right)^{\frac{\lambda}{\lambda+p}} = |\Omega|^{\frac{p}{\lambda+p}}\|u\|^{\lambda}_{\lambda+p,\Omega},$$

where $|\Omega|$ represents the volume of Ω. There is a positive constant C so as $|\Omega|^{p/(\lambda+p)} \leq C$ for any $1 < \lambda < \infty$, and thus $\|u\|_{\lambda,\Omega}^{\lambda} \leq C\|u\|_{\lambda+p,\Omega}^{\lambda}$. Reasoning as above, we can also show that $\|u\|_{\lambda,\partial\Omega}^{\lambda} \leq C\|u\|_{\lambda+q,\partial\Omega}^{\lambda}$. Using once again the relation $\lambda > 1$, we find that $\frac{(\lambda+m-1)^m}{\lambda m^m} < \lambda$. Consequently,

$$\|u\|_{\frac{(\lambda+m-1)N}{N-m},\Omega}^{\lambda+m-1} + \|u\|_{\frac{(\lambda+m-1)(N-1)}{N-m},\partial\Omega}^{\lambda+m-1} \\ \leq \lambda C \left(\|u\|_{\lambda+p,\Omega}^{\lambda+p} + \|u\|_{\lambda+p,\Omega}^{\lambda} + \|u\|_{\lambda+q,\partial\Omega}^{\lambda+q} + \|u\|_{\lambda+q,\partial\Omega}^{\lambda} \right). \tag{27}$$

We now define two positive sequences α_k and β_k as

$$\beta_1 := \frac{m(N-1)}{N-m}, \quad \beta_k := (\beta_{k-1} - q + m - 1)\frac{N-1}{N-m}, \tag{28}$$

$$\alpha_k := (\beta_{k-1} - q + m - 1)\frac{N}{N-m} = \frac{N}{N-1}\beta_k. \tag{29}$$

As a result, β_k is calculated as follows

$$\beta_k = \beta_1 + (\beta_2 - \beta_1)\frac{r^{k-1}-1}{r-1}, \quad r := \frac{N-1}{N-m}. \tag{30}$$

On the basis of $r > 1$, we acquire that β_k and α_k are strictly increasing and diverge to infinity. By making the substitution $\lambda = \beta_k - q$ in (27) and given the identity (25), we find that

$$\|u\|_{\alpha_{k+1},\Omega}^{\beta_k-q+m-1} + \|u\|_{\beta_{k+1},\partial\Omega}^{\beta_k-q+m-1} \\ \leq C\beta_k \left(\|u\|_{\beta_k+\frac{m}{N-m},\Omega}^{\beta_k+\frac{m}{N-m}} + \|u\|_{\beta_k+\frac{m}{N-m},\Omega}^{\beta_k-q} + \|u\|_{\beta_k,\partial\Omega}^{\beta_k} + \|u\|_{\beta_k,\partial\Omega}^{\beta_k-q} \right). \tag{31}$$

Taking into account that β_k is increasing, we have $\beta_k \geq \beta_1 = m(N-1)/(N-m)$. This together with relation (29) lead us to

$$\beta_k + \frac{m}{N-m} \leq \alpha_k \quad \text{for every } k \in \mathbb{N}.$$

Considering the Hölder's inequality, bearing in mind the above inequality, and making the notation

$$\frac{1}{\lambda_k} := 1 - \frac{\beta_k + m/(N-m)}{\alpha_k}, \tag{32}$$

it results that

$$\begin{aligned} \|u\|_{\beta_k+\frac{m}{N-m},\Omega}^{\beta_k+\frac{m}{N-m}} &= \int_\Omega |u|^{\beta_k+\frac{m}{N-m}}\,dx \\ &\leq |\Omega|^{\frac{1}{\lambda_k}} \left(\int_\Omega |u|^{\alpha_k}\,dx \right)^{\frac{\beta_k+\frac{m}{N-m}}{\alpha_k}} \\ &\leq |\Omega|^{\frac{1}{\lambda_k}} \|u\|_{\alpha_k,\Omega}^{\beta_k+\frac{m}{N-m}}. \end{aligned} \tag{33}$$

By relations (29) and (32), taking into consideration that $\beta_k \to \infty$ as $k \to \infty$, we obtain

$$\frac{1}{\lambda_k} = 1 - \frac{\beta_k + m/(N-m)}{(N/(N-1))\beta_k} \to \frac{1}{N}.$$

Consequently, there is a positive constant C independent of k so as $|\Omega|^{1/\lambda_k} \leq C$, and thus (33) becomes

$$\|u\|_{\beta_k+\frac{m}{N-m},\Omega}^{\beta_k+\frac{m}{N-m}} \leq C\|u\|_{\alpha_k,\Omega}^{\beta_k+\frac{m}{N-m}}.$$

In the same fashion, we show that

$$\|u\|^{\beta_k-q}_{\beta_k+\frac{m}{N-m},\Omega} \le C\|u\|^{\beta_k-q}_{\alpha_k,\Omega}.$$

The last two inequalities together with (31) assure us that

$$\begin{aligned}\|u\|^{\beta_k-q+m-1}_{\alpha_{k+1},\Omega}&+\|u\|^{\beta_k-q+m-1}_{\beta_{k+1},\partial\Omega}\\ &\le C\beta_k\left(\|u\|^{\beta_k+\frac{m}{N-m}}_{\alpha_k,\Omega}+\|u\|^{\beta_k-q}_{\alpha_k,\Omega}+\|u\|^{\beta_k}_{\beta_k,\partial\Omega}+\|u\|^{\beta_k-q}_{\beta_k,\partial\Omega}\right).\end{aligned}$$

We further define

$$A_k := \max\left(\|u\|_{\alpha_k,\Omega},\, \|u\|_{\alpha_k,\partial\Omega},\, 1\right),$$

and, thus, due to the last inequality we infer

$$A^{\beta_k-q+m-1}_{k+1} \le C\beta_k A_k^{\beta_k+\frac{m}{N-m}}.$$

Let us take

$$\xi_k := (C\beta_k)^{1/(\beta_k-q+m-1)} \quad \text{and} \quad \zeta_k := \frac{\beta_k+m/(N-m)}{\beta_k-q+m-1},$$

and thereby, the above inequality will be written as $A_{k+1} \le \xi_k A_k^{\zeta_k}$. These lead us to

$$\begin{aligned}A_k \le \xi_{k-1}A_{k-1}^{\zeta_{k-1}} &\le \xi_{k-1}\left(\xi_{k-2}A_{k-2}^{\zeta_{k-2}}\right)^{\zeta_{k-1}}\\ &\le \cdots \le \xi_{k-1}\xi_{k-2}^{\zeta_{k-1}}\xi_{k-3}^{\zeta_{k-1}\zeta_{k-2}}\ldots\xi_1^{\zeta_{k-1}\zeta_{k-2}\cdots\zeta_2}A_1^{\zeta_{k-1}\cdots\zeta_1}.\end{aligned} \tag{34}$$

We propose to argue the relations

$$0 < \prod_{k=1}^{\infty}\zeta_k < \infty, \tag{35}$$

$$0 < \prod_{k=1}^{\infty}\xi_k < \infty. \tag{36}$$

In order to do this, we recall (30) which provides us that there is a positive constant c satisfying the relations

$$r^k \le c(\beta_k - q + m - 1) \quad \text{for every } k \in \mathbb{N}. \tag{37}$$

We point out again relation (25) to deduce the following:

$$\zeta_k = 1 + \frac{p-m+1}{\beta_k-q+m-1}, \quad \sum_{k=1}^{\infty}\frac{p-m+1}{\beta_k-q+m-1} < \infty,$$

which imply (35). We now make the notation $d := \prod_{k=1}^{\infty}\zeta_k$, and hence $1 < d < \infty$. With the aid of $1 < \zeta_k$, it is obvious that

$$\zeta_{k-1}\zeta_{k-2}\ldots\zeta_i \le d \quad \text{for } i \le k-1.$$

Withal, based on $1 < \xi_k$ we also deduce

$$\xi_{k-1}\xi_{k-2}^{\zeta_{k-1}}\xi_{k-3}^{\zeta_{k-1}\zeta_{k-2}}\ldots\xi_1^{\zeta_{k-1}\zeta_{k-2}\cdots\zeta_2} \le (\zeta_1\zeta_2\ldots\zeta_{k-1})^d.$$

We notice that there is a positive constant C so as $\beta_k \le Cr^k$ for all $k \in \mathbb{N}$. Combining this last inequality with (37), we have

$$\log\xi_k = \log\left[(C\beta_k)^{\frac{1}{(\beta_k-q+m-1)}}\right] = \frac{\log(C\beta_k)}{\beta_k-q+m-1} \le \frac{\bar{C}k}{r^k},$$

where $\bar{C} > 0$ is independent of k. Consequently, we deduce the following:

$$\log\left(\prod_{k=1}^{n} \xi_k\right) = \sum_{k=1}^{n} \log \xi_k, \quad \sum_{k=1}^{\infty} \log \xi_k < \infty,$$

and, therefore, we reach (36). Putting together (34), (35) and (36), we obtain the existence of a positive constant C so as $A_k \le C A_1^d$ for all $k \in \mathbb{N}$. Considering the definition of A_k and letting $k \to \infty$, we get

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\partial\Omega)} \le C A_1^d. \tag{38}$$

Moreover, by (28), (29) and the Sobolev embedding, we conclude that

$$\begin{aligned} A_1 &= \max\left(\|u\|_{\alpha_1,\Omega},\, \|u\|_{\beta_1,\partial\Omega},\, 1\right) = \max\left(\|u\|_{mN/(N-m),\Omega},\, \|u\|_{m(N-1)/(N-m),\partial\Omega},\, 1\right) \\ &\le C\left(\|u\|_{W^{1,m}(\Omega)} + 1\right). \end{aligned}$$

This last inequality together with (38) yield (21).

If $N = 1, 2, ..., m$, then similar arguments as in the case $N > m$ lead us to the same conclusion. The proof of Lemma 4.1 is complete. □

Our second lemma is somewhat a m-Laplace version of Proposition 4.1 due to Garcia-Azorero, Peral and Rossi [17].

Lemma 4.2. *Let $N \ge m$. Assume that $f \in L^r(\Omega)$ with $r \in (m-1, N)$, $g \in L^s(\partial\Omega)$ with $s > m-1$, and let $\phi \in W^{1,m}(\Omega)$ be a weak solution of problem (1). Then $\phi \in W^{1,\alpha}(\Omega)$ and there exist $C_r > 0$ and $C_s > 0$ such that*

$$\|\phi\|_{W^{1,\alpha}(\Omega)} \le C_r \|f\|_{L^r(\Omega)} + C_s \|g\|_{L^s(\partial\Omega)} \tag{39}$$

with $\alpha \le \frac{Nr}{N-r}$ and $\alpha \le \frac{Ns}{N-1}$.

Proof. Multiplying (1) by a regular test function $\varphi \in C^1(\Omega)$, integrating both sides over Ω and using the boundary condition, it is obvious that

$$\int_\Omega |\nabla\phi|^{m-2}\nabla\phi\nabla\varphi\, dx + \int_\Omega |\phi|^{m-2}\phi\varphi\, dx \le \int_\Omega |f\varphi|\, dx + \int_{\partial\Omega} |g\varphi|\, d\sigma.$$

By Hölder's inequality we have

$$\int_\Omega |f\varphi|\, dx \le \|f\|_{L^r(\Omega)} \|\varphi\|_{L^{r'}(\Omega)},$$

with $\frac{1}{r} + \frac{1}{r'} = 1$, and by Sobolev embedding we can take $\varphi \in W^{1,\beta'}(\Omega)$ with $\beta' = \frac{Nr'}{N+r'}$. As a consequence, using Proposition 1 in [8], we get

$$\phi \in W^{1,\beta}(\Omega) \quad \text{and} \quad \|\phi\|_{W^{1,\beta}(\Omega)} \le C_r \|f\|_{L^r(\Omega)}, \quad \text{where} \quad \beta = \frac{Nr}{N-r}. \tag{40}$$

Also, Hölder's inequality implies

$$\int_\Omega |g\varphi|\, dx \le \|g\|_{L^s(\partial\Omega)} \|\varphi\|_{L^{s'}(\partial\Omega)},$$

with $\frac{1}{s} + \frac{1}{s'} = 1$. By density we can take $\varphi \in W^{1,\gamma}(\Omega)$ and then, applying the trace theorem, we obtain

$$\varphi|_{\partial\Omega} \in W^{1-\frac{1}{\gamma'},\gamma'}(\partial\Omega) \subset L^{\frac{\gamma'(N-1)}{N-\gamma'}}(\partial\Omega),$$

where $s' = \frac{\gamma'(N-1)}{N-\gamma'}$ or, in another train of thoughts, $\gamma = \frac{Ns}{N-1}$. Thus, by Proposition 1 in [8], we deduce that

$$\phi \in W^{1,\gamma}(\Omega) \quad \text{and} \quad \|\phi\|_{W^{1,\gamma}(\Omega)} \le C_s \|g\|_{L^s(\partial\Omega)}. \tag{41}$$

By (40) and (41), we conclude that $\phi \in W^{1,\alpha}(\Omega)$ and satisfies (39) provided that $\alpha \leq \frac{Nr}{N-r}$, $\alpha \leq \frac{Ns}{N-1}$, $f \in L^r(\Omega)$ and $g \in L^s(\partial\Omega)$. The above estimate still remains valid when $N = 1, 2, ..., m-1$, with eventual slight modifications. □

Remark 4.1. In light of Lemma 4.2, we point out that, if $f \in L^\infty(\Omega)$ and $g \in L^\infty(\partial\Omega)$, then (39) becomes

$$\|u\|_{W^{1,\alpha}(\Omega)} \leq C_\alpha \|f\|_{L^\infty(\Omega)} + C_\alpha \|g\|_{L^\infty(\partial\Omega)} \tag{42}$$

for every $\alpha < \infty$.

We are now in a position to prove the following result.

Proposition 4.3. *Let $f(x,s)$ and $g(x,s)$ satisfy* (12) *and* (14)*, respectively. Then the following two claims hold true.*

(i) *Every $W^{1,m}(\Omega)$ solution u belongs to $W^{1,r}(\Omega)$ for any $r < \infty$ and fulfills*

$$\|u\|_{W^{1,r}(\Omega)} \leq C_r \|u\|^{dp}_{W^{1,m}(\Omega)} + C_r \|u\|^{dq}_{W^{1,m}(\Omega)} + C_r, \tag{43}$$

where the positive constant C_r depends only on r and not on u, but the positive constant d is independent of u and r. Here p and q in (43) *are the same as in Assumption 2.3.*

(ii) *Let u_k be a sequence of $W^{1,m}(\Omega)$ solutions converging to zero in $W^{1,m}(\Omega)$ as $k \to \infty$. Then $\|u_k\|_{W^{1,r}(\Omega)} \to 0$ for every $r < \infty$.*

Proof. (i) Let u be any $W^{1,m}(\Omega)$ solution. Using Lemma 4.1, we know that u also belongs to $L^\infty(\Omega)$ and to $L^\infty(\partial\Omega)$. Based on (12) and (14), we reach

$$\|f(x,u)\|_{L^\infty(\Omega)} \leq C\|u\|^p_{L^\infty(\Omega)} + C \leq C\|u\|^{dp}_{W^{1,m}(\Omega)} + C,$$

$$\|g(x,u)\|_{L^\infty(\partial\Omega)} \leq C\|u\|^q_{L^\infty(\partial\Omega)} + C \leq C\|u\|^{dq}_{W^{1,m}(\Omega)} + C.$$

Substituting the above inequalities in (42), we get exactly (43).

(ii) Let u_k be a sequence of $W^{1,m}(\Omega)$ solutions converging to zero in $W^{1,m}(\Omega)$. For $\alpha < m$, it is obvious that $\|u_k\|_{W^{1,\alpha}(\Omega)} \to 0$. Give $\alpha \in (m, \infty)$ arbitrarily. Set $\beta > \alpha$. We define θ as $\frac{1}{\alpha} = \frac{\theta}{m} + \frac{1-\theta}{\beta}$. By using interpolation inequality we obtain

$$\|u_k\|_{L^\alpha(\Omega)} \leq \|u_k\|^\theta_{L^m(\Omega)} \|u_k\|^{1-\theta}_{L^\beta(\Omega)}, \tag{44}$$

$$\|\nabla u_k\|_{L^\alpha(\Omega)} \leq \|\nabla u_k\|^\theta_{L^m(\Omega)} \|\nabla u_k\|^{1-\theta}_{L^\beta(\Omega)}. \tag{45}$$

By (43) we get that $\|u_k\|_{W^{1,\beta}(\Omega)}$ is bounded as $k \to \infty$, and from assumption we have $\|u_k\|_{W^{1,m}(\Omega)} \to 0$ as $k \to \infty$. Consequently, taking into account (44) and (45), we infer that $\|u_k\|_{W^{1,\alpha}(\Omega)} \to 0$. The proof of Proposition 4.3 is now complete. □

According to Proposition 4.3, by the Sobolev embedding, we derive that every $W^{1,m}(\Omega)$ solution belongs to $C^{0,\theta}(\overline{\Omega})$ for any $\theta \in (0,1)$. Particularly, a $W^{1,m}(\Omega)$ solution belongs to $C(\overline{\Omega})$. We shall show that, for a sequence u_k of $W^{1,m}(\Omega)$ solutions, if $\|u_k\|_{W^{1,m}(\Omega)}$ is divergent, then $\|u_k\|_{C(\overline{\Omega})}$ is also divergent.

Lemma 4.4. *Consider the assumption of Proposition 4.3. Let u_k a sequence of $W^{1,m}(\Omega)$ solutions. If $\|u_k\|_{W^{1,m}(\Omega)} \to \infty$, then $\|u_k\|_{C(\overline{\Omega})} \to \infty$ as $k \to \infty$.*

Proof. It is clear that, multiplying (1) by u and then integrating over Ω, we get

$$\begin{aligned}\int_\Omega |\nabla u|^m \, dx + \int_\Omega |u|^m \, dx &= \int_\Omega f(x,u)u \, dx + \int_{\partial\Omega} g(x,u)u \, d\sigma \\ &\leq C \max\left\{|f(x,s)s| + |g(x,s)s| : |s| \leq \|u\|_{C(\overline{\Omega})}\right\},\end{aligned}$$

for a positive constant C independent of u. This ensure us that, if u is bounded in $C(\overline{\Omega})$, then u is also bounded in $W^{1,m}(\Omega)$. The proof of Lemma 4.4 is complete. $\square$

5. Proof of the main results

This last section will be devoted to the proof of the main theorems by using the variational method with the help of the a priori $W^{1,r}(\Omega)$ estimates provided in Section 4. Our method is based on the symmetric mountain pass lemma. With this in mind, we define the Lagrangian functional $I(u)$ by

$$I(u) := \frac{1}{m}\int_\Omega (|\nabla u|^m + |u|^m)\,dx - \int_\Omega F(x,u)\,dx - \int_\Omega G(x,u)\,d\sigma, \tag{46}$$

where F and G are defined in (8), and by $d\sigma$ we refer to the surface measure on $\partial\Omega$. Standard arguments show that $I \in C^1(W^{1,m}(\Omega),\mathbb{R})$ with the derivative given by

$$\begin{aligned}\langle I'(u), v\rangle = &\int_\Omega \left(|\nabla u|^{m-2}\nabla u \nabla v + |u|^{m-2}uv\right)dx - \int_\Omega f(x,u)v\,dx \\ &- \int_{\partial\Omega} g(x,u)v\,d\sigma,\end{aligned} \tag{47}$$

for any $u, v \in W^{1,m}(\Omega)$. We introduce the *Palais-Smale condition*:

(PS) every sequence u_k in $W^{1,m}(\Omega)$ so as $I(u_k)$ is bounded and $I'(u_k) \to 0$ in $W^{1,m}(\Omega)'$ as $k \to \infty$ possesses a convergent subsequence.

We denoted by $W^{1,m}(\Omega)'$, the dual space of $W^{1,m}(\Omega)$. We verify the Palais-Smale condition by a standard method (see Rabinowitz [29] or Struwe [31], where the Laplace operator is involved).

Lemma 5.1. *Under Assumption 2.3, the functional I fulfills Palais-Smale condition (PS).*

Proof. Let $(u_k) \subset W^{1,m}(\Omega)$ be a sequence such that $|I(u_k)| < M$ for any $k \geq 1$, where $M > 0$ is a constant, and $I'(u_k) \to 0$ as $k \to \infty$. We claim that (u_k) is bounded. Arguing by contradiction, we assume that, passing eventually to a subsequence still denoted by (u_k), $\|u_k\|_{W^{1,m}(\Omega)} \to \infty$ as $k \to \infty$. Then, taking into consideration Assumption 2.3, we obtain

$$\begin{aligned}1 + M + \|u_k\|_{W^{1,m}(\Omega)} &\geq I(u_k) - \frac{1}{\mu}\langle I'(u_k), u_k\rangle \\ &= \frac{\mu - m}{m\mu}\int_\Omega (|\nabla u_k|^m + |u_k|^m)\,dx \\ &\quad - \int_\Omega \left(F(x,u_k) - \frac{1}{\mu} f(x,u_k)u_k\right)dx \\ &\quad - \int_{\partial\Omega} \left(G(x,u_k) - \frac{1}{\mu} g(x,u_k)u_k\right)d\sigma \\ &\geq \frac{\mu - m}{m\mu}\|u_k\|^m_{W^{1,m}(\Omega)} - \frac{C}{\mu}\|u_k\|^\tau_{W^{1,m}(\Omega)} - \frac{C}{\mu}|\Omega|.\end{aligned}$$

Dividing the above relation by $\|u_k\|^m_{W^{1,m}(\Omega)}$ and passing to the limit as $k \to \infty$ we obtain a contradiction.

Thus, (u_k) is bounded in $W^{1,m}(\Omega)$, and, since $W^{1,m}(\Omega)$ is reflexive, there exists an $u_0 \in W^{1,m}(\Omega)$ such that, up to a subsequence, (u_k) converges weakly to u_0 in $W^{1,m}(\Omega)$. Next, we show that (u_k) converges strongly to u_0 in $W^{1,m}(\Omega)$. Bearing in mind Assumption 2.3, we find that the embedding $W^{1,m}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact, for $p \in \left(1, \frac{Nm}{N-m}\right)$. Hence,

$$(u_k) \text{ converges strongly to } u_0 \text{ in } L^{p+1}(\Omega).$$

Given this, by (12) and Hölder's inequality we have

$$\begin{aligned}\left|\int_\Omega f(x,u_k)(u_k-u_0)dx\right| &\le C\int_\Omega |u_k|^p|u_k-u_0|\,dx + C\int_\Omega |u_k-u_0|dx\\ &\le C\|u_k\|_{\frac{p+1}{p},\Omega}\cdot\|u_k-u_0\|_{p+1,\Omega} + C\|u_k-u_0\|_{1,\Omega}\,,\end{aligned}$$

which means that

$$\lim_{k\to\infty}\int_\Omega f(x,u_k)(u_k-u_0)dx = 0. \tag{48}$$

On the other hand, keeping in mind Assumption 2.3 and boundary trace embedding theorem (see, e.g. Adams and Fournier [1] or Ladyzenskaja and Ural'tzeva [25]), we deduce that $W^{1,m}(\Omega) \hookrightarrow L^{q+1}(\partial\Omega)$ is a compact embedding, where $q+1 \in \left(1, \frac{(N-1)m}{N-m}\right)$. Therefore,

$$(u_k) \text{ converges strongly to } u_0 \text{ in } L^{q+1}(\partial\Omega).$$

On account of this fact, by (14) and Hölder's inequality we have

$$\begin{aligned}\left|\int_{\partial\Omega} g(x,u_k)(u_k-u_0)d\sigma\right| &\le C\int_{\partial\Omega} |u_k|^q|u_k-u_0|\,d\sigma + C\int_{\partial\Omega} |u_k-u_0|d\sigma\\ &\le C\|u_k\|_{\frac{q+1}{q},\partial\Omega}\cdot\|u_k-u_0\|_{q+1,\partial\Omega} + C\|u_k-u_0\|_{1,\partial\Omega}\,,\end{aligned}$$

involving that

$$\lim_{k\to\infty}\int_{\partial\Omega} g(x,u_k)(u_k-u_0)d\sigma = 0. \tag{49}$$

Bearing in mind relations (48), (49) and relying on the fact that, by $I'(u_k) \to 0$ as $k \to \infty$,

$$\lim_{k\to\infty}\langle I'(u_k), u_k-u_0\rangle = 0,$$

we arrive at

$$\lim_{k\to\infty}\int_\Omega \left(|\nabla u_k|^{m-2}\nabla u_k\nabla(u_k-u_0) + |u_k|^{m-2}u_k(u_k-u_0)\right)dx = 0. \tag{50}$$

In addition, on the strength of the fact that (u_k) converges weakly to u_0 in $W^{1,m}(\Omega)$ we infer that

$$\lim_{k\to\infty}\int_\Omega \left(|\nabla u_0|^{m-2}\nabla u_0\nabla(u_k-u_0) + |u_0|^{m-2}u_0(u_k-u_0)\right)dx = 0. \tag{51}$$

On the basis of (50) and (51), it happens

$$\begin{aligned}\int_\Omega \Big[&\left(|\nabla u_k|^{m-2}\nabla u_k - |\nabla u_0|^{m-2}\nabla u_0\right)\nabla(u_k-u_0)\\ &+ \left(|u_k|^{m-2}u_k - |u_0|^{m-2}u_0\right)(u_k-u_0)\Big]\,dx \underset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

We apply now the well known inequality (2.2) in Simon [30], namely

$$\left(|\xi|^{m-2}\xi - |\eta|^{m-2}\eta\right)(\xi-\eta) \ge C|\xi-\eta|^m, \quad \xi,\eta\in\mathbb{R}^N,$$

valid for every $m \geq 2$, where C is a positive constant. This together with the above limit imply

$$\lim_{k\to\infty} \|u_k - u_0\|_{W^{1,m}(\Omega)} = 0.$$

This means that $u_k \to u_0$ in $W^{1,m}(\Omega)$. Thus, we have shown that I satisfies the Palais-Smale condition, that is exactly what we had to show. □

Further, in order to deal with symmetric mountain pass lemma, we recall *Krasnoselskii's genus.*

Definition 5.1. Let E be an infinite dimensional Banach space. A subset A of E is said to be *symmetric* if $x \in A$ implies $-x \in A$. For a closed symmetric set A which does not contain the origin, we define a *genus* $\gamma(A)$ of A by the smallest integer k so as there is an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k we set $\gamma(A) = \infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

In the sequel we will establish the properties of the genus that will be used through this work. More information on this subject may be found in many works, such as [2], [7], [9], [10], [24], [29] or [31].

Lemma 5.2. *Let A, B be closed symmetric subsets of E which do not contain the origin.*
(i) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
(ii) *If there is an odd continuous mapping $f \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$.*
(iii) *If U is a symmetric bounded open neighborhood of the origin in $\mathbb{R}^N$, then $\gamma(\partial U) = N$.*
(iv) *Let W be a closed linear subspace of E whose codimension is finite. If $\gamma(A)$ is greater than the codimension of W, then $A \cap W \neq \emptyset$.*

Let Γ_k be the family of closed symmetric subsets A of E so as $0 \notin A$ and $\gamma(A) \geq k$. We introduce the following assumption:

Assumption 5.3. *Let E be an infinite dimensional Banach space, and $I \in C^1(E, \mathbb{R})$ meets:*
(I1) *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ fulfills the Palais-Smale condition (PS).*
(I2) *For any $k \in \mathbb{N}$, there is an $A_k \in \Gamma_k$ so as $\sup_{u\in A_k} I(u) < 0$.*

Under Assumption 5.3, we define c_k and K_c as

$$c_k := \inf_{A\in\Gamma_k} \sup_{u\in A} I(u)\,, \tag{52}$$

$$K_c := \{u \in E : I'(u) = 0,\ I(u) = c\}\,.$$

We now are in a position to give *the symmetric mountain pass lemma* (see, e.g. Ambrosetti and Rabinowitz [3], Clark [9], Kajikiya [21] or Struwe [31]).

Lemma 5.4. *Under Assumption 5.3, any c_k is a critical value of I, $c_k \leq c_{k+1} < 0$ for $k \in \mathbb{N}$ and c_k converges to zero. Furthermore, if $c_k = c_{k+1} = \cdots = c_{k+p} \equiv c$, then $\gamma(K_c) \geq p + 1$.*

The next result is another *critical point theorem* related to *the symmetric mountain pass lemma* (see Kajikiya [21]).

Proposition 5.5. *Under Assumption 5.3, either (i) or (ii) from below holds true.*

(i) *There is a sequence* u_k *in* E *so as* $I'(u_k) = 0$, $I(u_k) < 0$ *and* u_k *converges to zero.*
(ii) *There are two sequences* u_k *and* v_k *in* E *so as* $I'(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $I'(v_k) = 0$, $I(v_k) < 0$, $\lim_{k\to\infty} I(v_k) = 0$, *and* v_k *converges to a non-zero limit.*

In both cases (i) and (ii), there is a sequence $u_k \neq 0$ *of critical points converging to zero.*

Remark 5.1. Whereas our purpose is to prove the existence of a sequence of solutions for (1) that converges to zero in $W^{1,m}(\Omega)$, we can observe that Proposition 5.5 is appropriate to achieve the aim. More precisely, unlike Lemma 5.4 (symmetric mountain pass lemma), which assures the existence of a sequence of *critical values* converging to zero in $\mathbb{R}$, Proposition 5.5 assures the existence of a sequence of *critical points* of I converging to zero in $W^{1,m}(\Omega)$. The Palais-Smale condition (PS) guarantees that a critical point u_k corresponding to c_k converges to zero as $k \to \infty$, whether the problem

$$I(u) = 0 \quad \text{and} \quad I'(u) = 0 \tag{53}$$

possesses only the trivial solution $u = 0$. But we have no certainty that, in the case of our functional I, defined in (46), the equation (53) has only the solution $u = 0$; this is why we can not apply the usual symmetric mountain pass lemma.

Remark 5.2. We point out that, if presume only the Assumption 2.1, our functional I is not well-defined in $W^{1,m}(\Omega)$. For this to happen, we truncate the functions f and g. Therefor, we choose an even function $h \in C_0^\infty(\mathbb{R})$ so as $h(s) = 1$ for $s \leq 1$ and $h(s) = 0$ for $|s| \geq 2$. We define the following:

$$\tilde{f}(x,s) := f(x,s)h(s), \quad \tilde{g}(x,s) := g(x,s)h(s),$$
$$\tilde{F}(x,u) := \int_0^u \tilde{f}(x,s)\,ds, \quad \tilde{G}(x,u) := \int_0^u \tilde{g}(x,s)\,ds.$$

Therefore, $\tilde{f}(x,s)$ and $\tilde{g}(x,s)$ are odd with respect to s, $\tilde{f}$ and $\tilde{F}$ are bounded on $\overline{\Omega} \times \mathbb{R}$, and $\tilde{g}$ and $\tilde{G}$ are bounded on $\partial\Omega \times \mathbb{R}$. So, instead of I, we set

$$\tilde{I}(u) := \frac{1}{m}\int_\Omega (|\nabla u|^m + |u|^m)\,dx - \int_\Omega \tilde{F}(x,u)dx - \int_{\partial\Omega} \tilde{G}(x,u)d\sigma\,.$$

The goal is to prove that $\tilde{I}$ possesses a sequence of critical points $u_k \neq 0$ satisfying $\|u_k\|_{W^{1,m}(\Omega)}$ converges to zero. Then u_k is a solution of the problem

$$\begin{cases} -\operatorname{div}(|\nabla u_k|^{m-2}\nabla u_k) + |u_k|^{m-2}u_k = \tilde{f}(x,u_k) & \text{in } \Omega, \\ |\nabla u_k|^{m-2}\frac{\partial u_k}{\partial \nu} = \tilde{g}(x,u_k) & \text{on } \partial\Omega. \end{cases} \tag{54}$$

Whereas $\tilde{f}$, $\tilde{g}$, $\tilde{F}$ and $\tilde{G}$ are bounded, it is obvious that they fulfill Assumptions 2.1 and 2.3. As a result, Proposition 4.3 ensures that $\|u_k\|_{C(\overline{\Omega})}$ converges to zero. As $\|u_k\|_{C(\overline{\Omega})} < 1$ for k sufficiently large, we have $\tilde{f}(x,u_k) = f(x,u_k)$ and $\tilde{g}(x,u_k) = g(x,u_k)$. This implies the fact that problem (54) is reduced to (1). In this way, we acquire a sequence u_k of solutions of problem (1) satisfying $\|u_k\|_{W^{1,r}(\Omega)}$ converges to zero. We rewrite $\tilde{f}$, $\tilde{g}$ and $\tilde{I}$ as f, g and I, respectively. Accordingly, f, g, F and G are bounded.

We now proceed to the proof of Theorem 2.2.

Proof of Theorem 2.2. As we said, in the proof of Theorem 2.2 we require Proposition 5.5. Therefore, we shall demonstrate that the functional I fulfills (I1) and (I2) in Assumption 5.3. Firstly, let us show that I satisfies (I1). Indeed, as we pointed out in Remark 5.2 from above, F and G are bounded, so we deduce immediately that

$$I(u) \geq \frac{1}{m}\|u\|^m_{W^{1,m}(\Omega)} - C,$$

where C is a positive constant. This means that I is bounded from below. Also, since f, g, F and G are bounded, they fulfill Assumption 2.3 and, thus, via Lemma 5.1, I satisfies Palais-Smale condition (PS). We conclude that (I1) is satisfied.

Next, we verify that I satisfies (I2). To this end, let us first deal with (f1) in Assumption 2.1. We consider $B(x_0, \delta)$ the ball defined in (f1). Let λ_k and ϕ_k be the k-th eigenvalue and eigenfunction, respectively, of the problem

$$\begin{cases} -\mathrm{div}(|\nabla\phi|^{m-2}\nabla\phi) = \lambda|\phi|^{m-2}\phi & \text{in } B(x_0,\delta), \\ \phi = 0 & \text{on } \partial B(x_0,\delta). \end{cases} \tag{55}$$

We extend ϕ_k such that $\phi_k(x) = 0$ in $\Omega \setminus B(x_0,\delta)$ and, thus, $\phi_k \in W^{1,m}_0(\Omega) \cap C(\overline{\Omega})$. Let $k \in \mathbb{N}$. We plan to construct A_k satisfying (I2). Set

$$X := \left\{ \sum_{i=1}^{k} t_i\phi_i : \ t_i \in \mathbb{R} \right\}, \tag{56}$$

the linear space spanned by ϕ_i with $1 \leq i \leq k$. Therefore, X is a linear subspace of $W^{1,m}_0(\Omega) \cap C(\overline{\Omega})$. The fact that X is finite dimensional space, provide us that all norms are equivalent to each other. We infer that there is $\alpha > 0$ so as

$$\|u\|_{W^{1,m}(\Omega)} \leq \alpha\|u\|_{L^m(\Omega)} \ \text{ for } u \in X. \tag{57}$$

Considering (9), we can choose an $\varepsilon > 0$ so small that the following fact to happen:

$$F(x,s) \geq \alpha^m|s|^m \ \text{ for } x \in B(x_0,\delta), \ |s| \leq \varepsilon.$$

On the basis that every $u \in X$ vanishes in $\Omega \setminus B(x_0,\delta)$, we get

$$F(x,u(x)) \geq \alpha^m|u(x)|^m \ \text{ for } x \in \Omega, \ \|u\|_{L^\infty(\Omega)} \leq \varepsilon, \ u \in X. \tag{58}$$

Set

$$A := \left\{ u \in X : \ \|u\|_{L^\infty(\Omega)} = \varepsilon \right\}, \tag{59}$$

a $(k-1)$-dimensional sphere. Via Borsuk-Ulam theorem, the genus of A is k, i.e., $\gamma(A) = k$. Let $u \in A$. This yields $u = 0$ on $\partial\Omega$ and, implicitly, $G(x,u(x))$ vanishes on $\partial\Omega$. Considering $u \in A$, by relations (57) and (58), we have

$$\begin{aligned} I(u) &= \frac{1}{m}\int_\Omega (|\nabla u|^m + |u|^m)\,dx - \int_\Omega F(x,u)dx \\ &\leq \frac{1}{m}\|u\|^m_{W^{1,m}(\Omega)} - \alpha^m \int_\Omega |u|^m dx \\ &\leq \left(\frac{1}{m} - 1\right)\|u\|^m_{W^{1,m}(\Omega)} \\ &< 0. \end{aligned}$$

Since A is compact, we obtain $\sup_{u\in A} I(u) < 0$. So, starting from (f1) in Assumption 2.1, we obtained that I fulfills (I2).

Let us also show that, assuming (g1) in Assumption 2.1, I satisfies (I2). To achieve this goal, we first denote for simplicity $D := B(x_0,\delta) \cap \Omega$ and $S := B(x_0,\delta) \cap \partial\Omega$. Let

k be a positive integer. We choose ψ_i, with $1 \le i \le k$, so as $\psi_i \in C^1(\overline{D})$, $\psi_i(x) = 0$ for $x \in \partial B(x_0, \delta) \cap \Omega$, and $\psi_i|_S$ (i.e., the restrictions of functions ψ_i to S), for $1 \le i \le k$, are linearly independent in $L^m(S)$. Thus, ψ_i are also linearly independent in $C^1(\overline{D})$. We set $\psi_i(x) = 0$ in $\Omega \setminus D$. So, $\psi_i \in W^{1,m}(\Omega) \cap C(\overline{\Omega})$. Let Y be the linear space spanned by ψ_i with $1 \le i \le k$, i.e.,

$$Y := \left\{ \sum_{i=1}^{k} t_i \psi_i : \ t_i \in \mathbb{R} \right\}.$$

Since Y is finite dimensional space, we deduce the existence of an $\alpha > 0$ so as

$$\|u\|_{W^{1,m}(D)} \le \alpha \|u\|_{L^m(S)} \quad \text{for } u \in Y. \tag{60}$$

Recalling that F fulfills (11), we infer that there is a positive constant C' so as

$$F(x,s) \ge -C'|s|^m \quad \text{if } x \in D, \ |s| \le 1,$$

and, consequently,

$$F(x,u(x)) \ge -C'|u(x)|^m \quad \text{if } x \in D, \ \|u\|_\infty \le 1. \tag{61}$$

We take $\beta > 0$ so large that

$$\frac{1}{m} + C' < \left(\frac{\beta}{\alpha}\right)^m. \tag{62}$$

Considering (10), we can choose an $\varepsilon > 0$ so small that

$$G(x,s) \ge \beta^m |s|^m \quad \text{for } x \in S, \ |s| \le \varepsilon.$$

Since each $u \in Y$ vanishes in $\partial\Omega \setminus S$, we obtain

$$G(x,u(x)) \ge \beta^m |u(x)|^m \quad \text{for } x \in \partial\Omega, \ \|u\|_{L^\infty(\Omega)} \le \varepsilon, \ u \in Y. \tag{63}$$

We now define A as in (59), for Y. Thus, for $u \in A$, by (60), (61), (62) and (63) we obtain the following:

$$\begin{aligned} I(u) &= \frac{1}{m}\int_D (|\nabla u|^m + |u|^m)\,dx - \int_D F(x,u)dx - \int_S G(x,u)d\sigma \\ &\le \frac{1}{m}\|u\|^m_{W^{1,m}(D)} + C'\|u\|^m_{L^m(D)} - \beta^m \|u\|^m_{L^m(S)} \\ &\le \left(\frac{1}{m} + C' - \frac{\beta^m}{\alpha^m}\right)\|u\|^m_{W^{1,m}(D)} \\ &< 0. \end{aligned}$$

Since A is compact, we deduce $\sup_{u\in A} I(u) < 0$, that is, I satisfies (I2). We can finally conclude that, via Proposition 5.5, problem (1) possesses a sequence of nontrivial solutions u_k such that $\|u_k\|_{W^{1,m}(\Omega)}$ converges to zero. In addition, via Proposition 4.3, $\|u_k\|_{W^{1,r}(\Omega)}$ converges to zero. The proof of Theorem 2.2 is complete. □

In the following, we are going to demonstrate some auxiliary lemmas that will lead us step by step to the conclusion of Theorem 2.5. Therefore, we assume Assumptions 2.3 and 2.4 and thus, we have ensured that I is well defined and satisfies the Palais-Smale condition (PS). First, we emphasize that X and Y involved in the proof of Theorem 2.2, will be used here as follows: if (f2) in Assumption 2.4 holds, then we take $Z_k := X$, and if (g2) holds, then we take $Z_k := Y$.

Lemma 5.6. *For any $k \in \mathbb{N}$, there is a positive R_k so as*

$$I(u) < 0 \quad \textit{for } u \in Z_k \textit{ with } \|u\|_{W^{1,m}(\Omega)} \ge R_k. \tag{64}$$

Proof. We start assuming that (f2) holds. Thus, $Z_k := X$ and there exists a positive constant α_k so as

$$\|u\|_{W^{1,m}(\Omega)} \leq \alpha_k \|u\|_{L^m(\Omega)} \quad \text{for } u \in Z_k.$$

Also, by (f2), there is a positive constant C_k so as

$$F(x,s) \geq \alpha_k^m |s|^m - C_k \quad \text{for } s \in \mathbb{R},\ x \in B(x_0,\delta).$$

Since $\operatorname{supp}(u) \subset B(x_0,\delta)$ for any $u \in Z_k$, we obtain that $G(x,u) = 0$ for $x \in \partial\Omega$. Then, by the last two inequalities we get

$$\begin{aligned} I(u) &= \frac{1}{m}\int_{B(x_0,\delta)} \left(|\nabla u|^m + |u|^m\right) dx - \int_{B(x_0,\delta)} F(x,u)dx \\ &\leq \frac{1}{m}\|u\|^m_{W^{1,m}(\Omega)} - \alpha_k^m \|u\|^m_{L^m(\Omega)} + C_k |B(x_0,\delta)| \\ &\leq \left(\frac{1}{m} - 1\right)\|u\|^m_{W^{1,m}(\Omega)} + C_k|B(x_0,\delta)| \\ &< 0, \end{aligned}$$

on condition that $\|u\|_{W^{1,m}(\Omega)} \geq R_k$ with R_k sufficiently large. By $|B(x_0,\delta)|$ we meant the volume of the ball $B(x_0,\delta)$.

Next, suppose that (g2) holds. Thus, $Z_k := Y$ and there is a positive constant α_k so as

$$\|u\|_{W^{1,m}(\Omega)} \leq \alpha_k \|u\|_{L^m(\partial\Omega)} \quad \text{for } u \in Z_k. \tag{65}$$

Also, by (g2), there is a positive constant C' so as

$$F(x,s) \geq -C'|s|^m - C' \quad \text{for } s \in \mathbb{R},\ x \in B(x_0,\delta) \cap \Omega. \tag{66}$$

We now consider a positive M so large that

$$\frac{1}{m} + C' < \frac{M}{\alpha_k^m}. \tag{67}$$

Considering (17), we infer the existence of a positive constant C so as

$$G(x,s) \geq M|s|^m - C \quad \text{for } s \in \mathbb{R},\ x \in B(x_0,\delta) \cap \partial\Omega. \tag{68}$$

Every $u \in Z_k$ satisfies $\operatorname{supp}(u) \subset B(x_0,\delta) \cap \Omega$. We take $D := B(x_0,\delta) \cap \Omega$ and $S := B(x_0,\delta) \cap \partial\Omega$. By (65), (66), (67) and (68) we have

$$\begin{aligned} I(u) &= \frac{1}{m}\int_D \left(|\nabla u|^m + |u|^m\right) dx - \int_D F(x,u)dx - \int_S G(x,u)d\sigma \\ &\leq \frac{1}{m}\|u\|^m_{W^{1,m}(D)} + C'\|u\|^m_{L^m(D)} + C'|D| - M\|u\|^m_{L^m(S)} + C|S| \\ &\leq \left(\frac{1}{m} + C' - \frac{M}{\alpha_k^m}\right)\|u\|^m_{W^{1,m}(\Omega)} + C'|D| + C|S| \\ &< 0, \end{aligned}$$

on condition that $\|u\|_{W^{1,m}(\Omega)} \geq R_k$ with R_k sufficiently large. By $|D|$ and $|S|$ we meant the volume of D and the surface area of S, respectively. Consequently, (64) holds true under both hypotheses (f2) and (g2) in Assumption 2.4. The proof of Lemma 5.6 is complete. □

It can be assumed that R_k in Lemma 5.6 is increasing and diverges to infinity as $k \to \infty$. We now involve another *symmetric mountain pass lemma.* For this, we define the following:

$$D_k := \{u \in Z_k : \ \|u\|_{W^{1,m}(\Omega)} \leq R_k\}, \quad \partial D_k := \{u \in Z_k : \ \|u\|_{W^{1,m}(\Omega)} = R_k\},$$

$$G_k := \{g \in C(D_k, W^{1,m}(\Omega)) : \ g \text{ is odd and } g(u) = u \text{ on } \partial D_k\},$$

$$d_k := \inf_{g \in G_k} \max_{u \in D_k} I(g(u)). \tag{69}$$

Lemma 5.7. *d_k is a critical value.*

The proof of Lemma 5.7 can be found in [29] and [31].

We intend to show that d_k diverges to infinity as $k \to \infty$. Define

$$B(r, W^{1,m}(\Omega)) := \{u \in W^{1,m}(\Omega) : \ \|u\|_{W^{1,m}(\Omega)} < r\},$$
$$\partial B(r, W^{1,m}(\Omega)) := \{u \in W^{1,m}(\Omega) : \ \|u\|_{W^{1,m}(\Omega)} = r\}.$$

Lemma 5.8. *Let k be a positive integer and W a closed linear subspace of $W^{1,m}(\Omega)$ whose codimension is less than k. If $g \in G_k$ and $r \in (0, R_k)$, then*

$$g(D_k) \cap \partial B(r, W^{1,m}(\Omega)) \cap W \neq \emptyset.$$

For the proof of Lemma 5.8, we refer the readers to Kajikiya and Naimen [23] or Rabinowitz [29] (Proposition 9.23).

We also recall the following inequality (see relation (2.25) in [25]):

$$\|w\|^m_{L^r(\partial\Omega)} \leq \varepsilon \|\nabla w\|^m_{L^m(\Omega)} + C_\varepsilon \|w\|^m_{L^m(\Omega)} \quad \text{for } w \in W^{1,m}(\Omega),$$

for small positive ε and $r < m(N-1)/(N-m)$, where C_ε is a positive constant depending on ε and m. In the particular case when $r = m$, a standard computation shows that $C_\varepsilon = C/\varepsilon$, and, thus, the following result holds:

Lemma 5.9. *There are positive constants C, ε_0 so as*

$$\|w\|^m_{L^m(\partial\Omega)} \leq \varepsilon \|\nabla w\|^m_{L^m(\Omega)} + \frac{C}{\varepsilon} \|w\|^m_{L^m(\Omega)},$$

for every $w \in W^{1,m}(\Omega)$ and $\varepsilon \in (0, \varepsilon_0)$.

Further, let μ_k be the k-th eigenvalue of the problem

$$\begin{cases} -\mathrm{div}(|\nabla w|^{m-2} \nabla w) = \mu |w|^{m-2} w & \text{in } \Omega, \\ |\nabla w|^{m-2} \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

For each k we denote by w_k an eigenfunction corresponding to μ_k. Since we deal with a homogeneous problem we can assume that for each k we have $\|w_k\|_{W^{1,m}(\Omega)} = 1$ (w_k is the corresponding L^m-normalized eigenfunction or, in other words, the principal eigenfunction). By definition, w_k fulfills

$$\mu_k \|w_k\|^m_{L^m(\Omega)} = \|\nabla w_k\|^m_{L^m(\Omega)}. \tag{70}$$

We know that $\mu_1 = 0$, w_1 is a constant function and $\mu_k > 0$ for $k \geq 2$. Considering that $\partial\Omega$ is smooth, then any w_k is smooth on $\overline{\Omega}$. We define

$$W_k := \left\{ \sum_{i=k}^{\infty} t_i w_i : \ \sum_{i=k}^{\infty} t_i^m < \infty \right\}, \tag{71}$$

the closed linear space spanned by w_i with $i \geq k$, whose codimension is $k - 1$.

Lemma 5.10. *There is a positive constant C so as*

$$\|w\|_{L^m(\Omega)} \leq \mu_k^{-1/m} \|\nabla w\|_{L^m(\Omega)}, \tag{72}$$

$$\|w\|_{L^m(\partial\Omega)} \leq \mu_k^{-1/m^2} \|\nabla w\|_{L^m(\Omega)}, \tag{73}$$

for $w \in W_k$ with $k \geq 2$.

Proof. We consider $w = \sum_{i=k}^{\infty} t_i w_i \in W_k$. By (70) we have

$$\begin{aligned} \|\nabla w\|_{L^m(\Omega)}^m &= \sum_{i=k}^{\infty} t_i^m \|\nabla w_i\|_{L^m(\Omega)}^m \\ &= \sum_{i=k}^{\infty} t_i^m \mu_i \|w_i\|_{L^m(\Omega)}^m \\ &\geq \mu_k \sum_{i=k}^{\infty} t_i^m \|w_i\|_{L^m(\Omega)}^m \\ &= \mu_k \|w\|_{L^m(\Omega)}^m, \end{aligned}$$

which means that (72) holds true.

Next, Lemma 5.9 together with (72) yield

$$\begin{aligned} \|w\|_{L^m(\partial\Omega)}^m &\leq \varepsilon \|\nabla w\|_{L^m(\Omega)}^m + \frac{C}{\varepsilon} \|w\|_{L^m(\Omega)}^m \\ &\leq \varepsilon \|\nabla w\|_{L^m(\Omega)}^m + \frac{C}{\varepsilon \mu_k} \|\nabla w\|_{L^m(\Omega)}^m, \end{aligned} \tag{74}$$

for $\varepsilon \in (0, \varepsilon_0)$. Considering that $\mu_k \to \infty$ as $k \to \infty$, we take k_0 so large that $\mu_{k_0}^{-1/m} < \varepsilon_0$. We make the substitution $\varepsilon = \mu_k^{-1/m}$ and thus

$$\|w\|_{L^m(\partial\Omega)}^m \leq C \mu_k^{-1/m} \|\nabla w\|_{L^m(\Omega)}^m \quad \text{for } k \geq k_0, \tag{75}$$

obtaining (73) for $k \geq k_0$. We now take $\delta_0 > 0$ so small that $\delta_0 / \mu_k^{1/m} < \varepsilon_0$ for every $k \in [2, k_0]$. We make the substitution $\varepsilon = \delta_0 / \mu_k^{1/m}$ in (74) and get relation (75) for $k \in [2, k_0]$. Now the proof of Lemma 5.10 is complete. □

Going further, we recall a lemma regarding d_k defined in (69) (see, e.g. Lemma 4.9 in [22]).

Lemma 5.11. *If R_k fulfills* (64)*, then d_k is independent of the choice of R_k.*

We shall use Lemmas 5.8, 5.10 and 5.11 to prove that $d_k \to \infty$ as $k \to \infty$. In other words, we propose to show the following lemma:

Lemma 5.12. *d_k diverges to infinity.*

Proof. We consider W_k in (71). Taking into consideration that W_k is a closed linear subspace of $W^{1,m}(\Omega)$, by Lemma 5.8 we derive

$$g(D_k) \cap \partial B(r, W^{1,m}(\Omega)) \cap W_k \neq \emptyset,$$

for $g \in G_k$ and $r \in (0, R_k)$. We obtain from here that

$$\max_{u \in D_k} I(g(u)) \geq \inf \left\{ I(u) : \ u \in \partial B(r, W^{1,m}(\Omega)) \cap W_k \right\},$$

for $g \in G_k$. We take here the infimums of both sides over $g \in G_k$ and get

$$d_k \geq \inf \left\{ I(u) : \ u \in \partial B(r, W^{1,m}(\Omega)) \cap W_k \right\}, \tag{76}$$

for $r \in (0, R_k)$. If we replace p and q in Assumption 2.3 by larger constants, we can say that p, $q > m-1$. Thus, there exists a positive constant C so as

$$|F(x,s)| \leq C(|s|^{p+1}+1) \quad \text{for } x \in \overline{\Omega},\ s \in \mathbb{R},$$
$$|G(x,s)| \leq C(|s|^{q+1}+1) \quad \text{for } x \in \partial\Omega,\ s \in \mathbb{R}.$$

The above two inequalities lead us to

$$\begin{aligned} I(u) &\geq \frac{1}{m}\int_{\Omega}(|\nabla u|^m + |u|^m)\,dx - C\int_{\Omega}\left(|u|^{p+1}+1\right)dx - C\int_{\partial\Omega}\left(|u|^{q+1}+1\right)d\sigma \\ &\geq \frac{1}{m}\|u\|^m_{W^{1,m}(\Omega)} - C\|u\|^{p+1}_{L^{p+1}(\Omega)} - C\|u\|^{q+1}_{L^{q+1}(\partial\Omega)} - C \end{aligned} \tag{77}$$

when $u \in W_k$, with $C > 0$ various constants which do not depend on u and k.

We shall continue the proof considering $N > m$. In this case we have $m < p+1 < mN/(N-m)$ and $m < q+1 < m(N-1)/(N-m)$. We now consider α and β defined by the following identities:

$$\frac{1}{p+1} = \frac{\alpha}{m} + \frac{(1-\alpha)(N-m)}{mN} \quad \text{and} \quad \frac{1}{q+1} = \frac{\beta}{m} + \frac{(1-\beta)(N-m)}{m(N-1)}.$$

We appeal to the Hölder's inequality and the Sobolev's inequality to infer

$$\|u\|_{L^{p+1}(\Omega)} \leq \|u\|^{\alpha}_{L^m(\Omega)}\|u\|^{1-\alpha}_{L^{mN/(N-m)}(\Omega)} \leq C\|u\|^{\alpha}_{L^m(\Omega)}\|u\|^{1-\alpha}_{W^{1,m}(\Omega)}, \tag{78}$$

$$\|u\|_{L^{q+1}(\partial\Omega)} \leq \|u\|^{\beta}_{L^m(\partial\Omega)}\|u\|^{1-\beta}_{L^{m(N-1)/(N-m)}(\partial\Omega)} \leq C\|u\|^{\beta}_{L^m(\partial\Omega)}\|u\|^{1-\beta}_{W^{1,m}(\Omega)}. \tag{79}$$

By employing Lemma 5.10 together with (78) and (79) we arrive at

$$\|u\|_{L^{p+1}(\Omega)} \leq \frac{C}{\mu_k^{\alpha/m}}\|u\|_{W^{1,m}(\Omega)}, \quad \|u\|_{L^{q+1}(\partial\Omega)} \leq \frac{C}{\mu_k^{\beta/m^2}}\|u\|_{W^{1,m}(\Omega)},$$

for $u \in W_k$ with $k \geq 2$. We combine (77) with the above two inequalities involving that

$$I(u) \geq \frac{1}{m}\|u\|^m_{W^{1,m}(\Omega)} - \frac{C}{\mu_k^{\alpha(p+1)/m}}\|u\|^{p+1}_{W^{1,m}(\Omega)} - \frac{C}{\mu_k^{\beta(q+1)/m^2}}\|u\|^{q+1}_{W^{1,m}(\Omega)} - C,$$

for $u \in W_k$. This last inequality together with (76) lead to

$$d_k \geq \frac{1}{m}r^m - \frac{C}{\mu_k^{\alpha(p+1)/m}}r^{p+1} - \frac{C}{\mu_k^{\beta(q+1)/m^2}}r^{q+1} - C, \tag{80}$$

for $r \in (0, R_k)$. We make the notation

$$h_k(r) := \frac{1}{m}r^m - \frac{C}{\mu_k^{\alpha(p+1)/m}}r^{p+1} - \frac{C}{\mu_k^{\beta(q+1)/m^2}}r^{q+1} - C.$$

It is obvious that $h_k(r) < 0$ for r sufficiently large. We replace R_k by a larger constant so as $h_k(r) < 0$ for $R_k < r$. Lemma 5.11 help us to remark that the expanse of R_k leaves d_k invariant. We can assume that $p > q$. In truth, we take p_1 so as $\max(p,q) < p_1 < ((m-1)N+m)/(N-m)$. This means that $f(x,s)$ still fulfills (12) even if we replace p by p_1 and C by a larger constant, and, consequently, we can suppose $p > q$. We define

$$a_k := \frac{(p+1)C}{\mu_k^{\alpha(p+1)/m}}, \quad b_k := \frac{(q+1)C}{\mu_k^{\beta(q+1)/m^2}}.$$

Thus, we have

$$h_k(r) = \frac{1}{m} r^m - a_k \frac{r^{p+1}}{p+1} - b_k \frac{r^{q+1}}{q+1} - C.$$

Since p, $q > m - 1$ in Assumption 2.3, $h_k(r)$ achieves its maximum at a unique point $r_k \in (0, R_k)$. In truth, it is obvious that equation $h_k'(r) = 0$ is equivalent to

$$a_k r^{p-m+1} + b_k r^{q-m+1} = 1. \tag{81}$$

Thus, the unique positive solution r_k of (81) corresponds to the maximum value of $h_k(r)$. Equation (81) is the same with

$$\frac{(p+1)C}{\mu_k^{\alpha(p+1)/m}} r_k^{p-m+1} + \frac{(q+1)C}{\mu_k^{\beta(q+1)/m^2}} r_k^{q-m+1} = 1,$$

meaning that $r_k \to \infty$ as $k \to \infty$. Considering inequality (80), we attain $d_k \geq h_k(r_k)$. Since $p > q > m - 1$, by (81) we get

$$\begin{aligned} h_k(r_k) &= r_k^m \left(\frac{1}{m} - \frac{1}{p+1} a_k r_k^{p-m+1} - \frac{1}{q+1} b_k r_k^{q-m+1} \right) - C \\ &\geq r_k^m \left(\frac{1}{m} - \frac{1}{q+1} a_k r_k^{p-m+1} - \frac{1}{q+1} b_k r_k^{q-m+1} \right) - C \\ &= \left(\frac{1}{m} - \frac{1}{q+1} \right) r_k^m - C \to \infty, \end{aligned}$$

as $k \to \infty$, and, thus, d_k diverges to infinity as $k \to \infty$.

Let now consider $N = 1, 2, ..., m$. We take some constants P and Q satisfying $p < P$ and $q < Q$, respectively. Also, we consider α and β defined as follows:

$$\frac{1}{p+1} = \frac{\alpha}{m} + \frac{1-\alpha}{P+1}, \quad \frac{1}{q+1} = \frac{\beta}{m} + \frac{1-\beta}{Q+1}.$$

We use the Hölder's inequality and the Sobolev's inequality to obtain

$$\|u\|_{L^{p+1}(\Omega)} \leq \|u\|^{\alpha}_{L^m(\Omega)} \|u\|^{1-\alpha}_{L^{P+1}(\Omega)} \leq \|u\|^{\alpha}_{L^m(\Omega)} \|u\|^{1-\alpha}_{W^{1,m}(\Omega)},$$

$$\|u\|_{L^{q+1}(\partial\Omega)} \leq \|u\|^{\beta}_{L^m(\partial\Omega)} \|u\|^{1-\beta}_{L^{Q+1}(\partial\Omega)} \leq \|u\|^{\beta}_{L^m(\partial\Omega)} \|u\|^{1-\beta}_{W^{1,m}(\Omega)},$$

and from now on the proof is similar to that for the case $N > m$. □

Proof of Theorem 2.5. Let v_k be a critical point corresponding to d_k. This means that $I(v_k) = d_k$ and $I'(v_k) = 0$. We want to show that $\|v_k\|_{W^{1,m}(\Omega)} \to \infty$. We argue indirectly. So, suppose that there exists a bounded subsequence of v_k in $W^{1,m}(\Omega)$, labeled again v_k. We then obtain, via Proposition 4.3, that $\|v_k\|_{W^{1,r}(\Omega)}$ is bounded for every $r < \infty$. Moreover, by Sobolev embedding, we have that $\|v_k\|_{C(\overline{\Omega})}$ is bounded. Therefore,

$$d_k = I(v_k) = \frac{1}{m} \|v_k\|^m_{W^{1,m}(\Omega)} - \int_\Omega F(x, v_k) dx - \int_{\partial\Omega} G(x, v_k) d\sigma$$

is bounded, a contradiction. Hence, $\|v_k\|_{W^{1,m}(\Omega)} \to \infty$, and, considering Lemma 4.4, we actually infer that $\|v_k\|_{C(\overline{\Omega})} \to \infty$. The proof of Theorem 2.5 is complete. □

References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces, 2nd edition*, Pure Appl. Math. **140**, Academic Press, New York–London, 2003.

[2] A. Ambrosetti and A. Malchiodi, *Nonlinear analysis and semilinear elliptic problems*, Cambridge University Press, Cambridge, 2007.

[3] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.

[4] P. Aviles, Local behavior of solutions of some elliptic equation, *Comm. Math. Phys.* **108** (1987), 177–192.

[5] M.F. Bidaut-Véron, Local and global behavior of solutions of quasilinear equations of Emden-Fowler type, *Arch. Rational Mech. Anal.* **107**(4) (1989), 293–324.

[6] L.A. Caffarelli, B. Gidas, and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure and Appl. Math.* **42** (1989), 271–297.

[7] A. Castro, Metodos variacionales y analisis functional no linear, In: *X Colóquio Colombiano de Matemáticas*, 1980.

[8] P. Cherrier, Problèmes de Neumann non linéaires sur les variétés Riemanniennes, *J. Funct. Anal.* **57** (1984), 154–206.

[9] D.C. Clark, A variant of the Lusternik-Schnirelman theory, *Indiana Univ. Math. J.* **22** (1972), 65–74.

[10] D.G. Costa, Tópicos em análise não-linear e aplicações ás equações diferenciais, In: *Escola Latino-Americana de Matemática*, 1986.

[11] L. Damascelli, A. Farina, B. Sciunzi, and E. Valdinoci, Liouville results for m-Laplace equations of Lane-Emden-Fowler type, *Ann. Inst. H. Poincaré Anal. Non Linaire* **26** (2009), 1099–1119.

[12] D.G. DeFigueiredo, J.-P. Gossez, and P. Ubilla, Local superlinearity and sublinearity for indefinite semilinear elliptic problems, *J. Funct. Anal.* **199** (2003), 452–467.

[13] D.G. DeFigueiredo, J.-P. Gossez, and P. Ubilla, Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity, *J. Eur. Math. Soc.* **8** (2006), 269–288.

[14] R.H. Fowler, The form near infinity of real continuous solutions of a certain differential equation of second order, *Quart. J. Math. Oxford Ser.* **45** (1914), 289–350.

[15] R.H. Fowler, The solutions of Emden's and similar differential equations, *Monthly Notices Roy. Astr. Soc.* **91** (1920), 63–91.

[16] R.H. Fowler, Further studies on Emden's and similar differential equations, *Quart. J. Math. Oxford Ser.* **2** (1931), 259–288.

[17] J. Garcia-Azorero, I. Peral and J. D. Rossi, A convex-concave problem with a nonlinear boundary condition, *J. Differential Equations* **198** (2004), 91–128.

[18] L. Gasiński and N.S. Papageorgiou, Existence and uniqueness of positive solutions for the Neumann p-Laplacian, *Positivity* **17**(2) 2013, 309–332.

[19] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure and Appl. Math.* **34** (1980), 525–598.

[20] M. Guedda and L. Véron, Local and global properties of solutions of quasilinear equations, *J. Diff. Eq.* **76** (1988), 159–189.

[21] R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, *J. Funct. Anal.* **225** (2005), 352–370.

[22] R. Kajikiya, Superlinear elliptic equations with singular coefficients on the boundary, *Nonlinear Analysis T.M.A.* **73** (2010), 2117–2131.

[23] R. Kajikiya and D. Naimen, Two sequences of solutions for indefinite superlinear-sublinear elliptic equations with nonlinear boundary conditions, *Commun. Pure Appl. Anal.* **13** (2014), no. 4, 1593–1612.

[24] M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, MacMillan, New York, 1964.

[25] O.A. Ladyzenskaja and N.N. Ural'tzeva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[26] P.L. Lions, Isolated singularities in semilinear problems, *J. Diff. Eq.* **38** (1980), 441–450.

[27] D. Naimen, Existence of infinitely many solutions for nonlinear Neumann problems with indefinite coefficients, *Electron. J. Diff. Equ.* **181** (2014), 1–12.

[28] W.M. Ni and J. Serrin, Nonexistence theorems for singular solutions of quasilinear partial differential equations, *Comm. Pure and Appl. Math.* **38** (1986), 379–399.
[29] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics **65**, American Mathematical Siciety, Providence, RI, 1986.
[30] J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^n$. In: Ph. Bénilan and J. Robert (eds.) *Journeées d'Analyse Non Linéaire*, Lecture Notes in Mathematics **665**, Springer-Verlag, Berlin, 1978, 205–227.
[31] M. Struwe, *Variational Methods (Second Edition)*, Springer-Verlag, Berlin, 1996.

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