Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(2), 2017, Pages 228–237
ISSN: 1223-6934

Obstinate and maximal prefilters in EQ-algebras

M. Behzadi and L. Torkzadeh

Abstract. In this paper the notion of an obstinate prefilter(filter) in an EQ-algebra ξ is introduced and a characterization of it is obtained by some theorems. Then the notion of maximal prefilter is defined and is characterized under some conditions. Finally, the relations among obstinate, prime, maximal, implicative and positive implicative prefilters are studied.

2010 Mathematics Subject Classification. Primary 60J05; Secondary 60J20.
Key words and phrases. EQ-algebra, obstinate, (prime, maximal, implicative and positive implicative) prefilter.

1. Introduction

A special algebra called EQ-algebra has been recently introduced by Vilém Novák and B. De Baets [12]. Its original motivation comes from fuzzy type theory, in which the main connective is fuzzy equality. An EQ-algebra consists of three binaries (meet, multiplication and a fuzzy equality) and a top element and a binary operation implication is drived from fuzzy equality. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes commutative residuated lattice. These algebras are intended to develop an algebraic structure of truth values for fuzzy type theory. EQ-algebras are interesting and important for studying and researching and residuated lattices [5] and BL-algebras [2,6,14] are particular cases of EQ-algebras. In fact, EQ-algebras generalize non-commutative resiuated lattices [3]. The prefilter theory plays a fundamental role in the general development of EQ-algebras. From a logical point of view, various filters correspond to various sets of provable formulas. Some types of filters on residuated lattice based on logical algebras have been widely studied [7,8,15,16] and some important results have been obtained. The notion of obstinate filter in residuated lattice is introduced in [1]. For EQ-algebras, the notions of prefilters (which coincide with filters in residuated lattices) and prime prefilters were proposed and some of their properties were obtained [3]. Few results for other special prefilters of EQ-algebras have been obtained in [9]. In this paper, we define prefilters in EQ-algebra and characterize them by some theorems. We have shown that if F is an obstinate prefilter of an EQ-algebra E, then E/F is a chain. We hope that these prefilters open a new door into the theory of prefilters in EQ-algebras. This paper is organized as follows: in section 2, the basic definitions, properties and special types of EQ-algebras are reviewed. In section 3, an obstinate prefilter of an EQ-algebra is defined and characterized. In section 4,

Received October 20, 2015.

by defining the notion of maximal prefilters, some characteristics of them are presented. Finally, we study the relation among obstinate, prime, maximal, implicative and positive implicative prefilters.

2. Preliminaries

In this section, we present some definitions and results about EQ-algebras that will be used in the sequel.

Definition 2.1. [3] An *EQ-algebra* is an algebra $\xi = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ which satisfies the following :
(E_1) $(E, \wedge, 1)$ is a $\wedge$-semilattice with a top element 1. We set $a \leq b$ if and only if $a \wedge b = a$,
(E_2) $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone in arguments w.r.t $a \leq b$,
(E_3) $a \sim a = 1$,
(E_4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq (c \sim (d \wedge b))$,
(E_5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$,
(E_6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$,
(E_7) $a \otimes b \leq a \sim b$, for all $a, b, c \in E$.

We denote $\widetilde{a} := a \sim 1$ and $a \to b := (a \wedge b) \sim a$, for all $a, b \in E$.

Theorem 2.1. [3, 12, 13] *Let $\xi = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. For all $a, b, c \in E$ we have*
(e_1) $a \sim b = b \sim a$,
(e_2) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$,
(e_3) $a \sim d \leq (a \wedge b) \sim (d \wedge b)$,
(e_4) $(a \sim d) \otimes ((a \wedge b) \sim c) \leq ((d \wedge b) \sim c)$,
(e_5) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$,
(e_6) $a \otimes b \leq a \wedge b \leq a, b$,
(e_7) $b \leq \widetilde{b} \leq a \to b$,
(e_8) $a \sim b \leq (a \to b) \wedge (b \to a)$,
(e_9) $a \leq b$ implies $a \to b = 1$, $b \to a = a \sim b$, $\widetilde{a} \leq \widetilde{b}$,
$c \to a \leq c \to b$ and $b \to c \leq a \to c$,
(e_{10}) If $a \leq b \leq c$, then $a \sim c \leq a \sim b$ and $a \sim c \leq b \sim c$,
(e_{11}) $a \otimes (a \sim b) \leq \widetilde{b}$,
(e_{12}) $a \sim d \leq (b \to a) \sim (b \to d)$,
(e_{13}) $a \to d \leq (b \to a) \to (b \to d)$,
(e_{14}) $b \to a \leq (a \to d) \to (b \to d)$,
(e_{15}) $(a \to b) \otimes (c \to d) \leq (a \wedge c) \to (b \wedge d)$,
(e_{16}) $(a \to c) \otimes (b \to c) \leq (a \wedge b) \to c$,
(e_{17}) $(c \to a) \otimes (c \to b) \leq c \to (a \wedge b)$,
(e_{18}) $a \to (b \to c) \leq (a \otimes b) \to \widetilde{c}^4$.

Definition 2.2. [12] Let $\xi = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. We say that it is
(i) *spanned*, if it contains a bottom element 0 and $\widetilde{0} = 0$,
(ii) *separated*, if for all $a, b \in E$, $a \sim b = 1$ implies $a = b$,
(iii) *semi-separated*, if for all $a \in E$, $a \sim 1 = 1$ implies $a = 1$.

If an EQ-algebra ξ contains a bottom element 0, we may define the unary operation $\neg$ on E, by $\neg a = a \sim 0$ and $\neg a$ is called a negation of $a \in E$.

Theorem 2.2. [3, 12] *Let $\xi = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra with a bottom element 0. Then for all $a, b, c \in E$, we have*
(i) $\neg 1 = \widetilde{0}, \neg 0 = 1$,
(ii) $0 \to a = 1$ *and* $\neg a = a \to 0$,
(iii) $a \leq b$ *implies* $\neg b \leq \neg a$,
(iv) $\neg \widetilde{0} = \neg\neg 1$,
(v) $a \otimes \neg a \leq \widetilde{0}$,
(vi) $a \to b \leq \neg b \to \neg a$,
(vii) $\neg a \otimes \widetilde{0} \leq \widetilde{a}$,
(viii) $\widetilde{a} \otimes \widetilde{0} \leq \neg a$,
(ix) $a \sim b \leq \neg b \sim \neg a$.

Definition 2.3. [3] A nonempty subset F of an EQ-algebra ξ is called a *prefilter* of E, whenever for all $a, b, c \in E$:
(F_1) $1 \in F$,
(F_2) $a,\ a \to b \in F$ implies $b \in F$.
A prefilter F of ξ is called a filter, if it satisfies the following :
(F_3) $a \to b \in F$ implies $a \otimes c \to b \otimes c \in F$, for any $a, b, c \in E$.

A prefilter (filter) F of an EQ-algebra ξ is called proper, whenever $F \neq E$.

Theorem 2.3. [3] *Let F be a prefilter of an EQ-algebra $\xi = (E, \wedge, \otimes, \sim, 1)$. The following hold, for all $x, y, z, s, t \in E$:*
(i) If $x \in F$ and $x \leq y$, then $y \in F$,
(ii) If $x, x \sim y \in F$, then $y \in F$,
(iii) If $x \sim y \in F$ and $y \sim z \in F$, then $x \sim z \in F$,
(iv) If $x \to y \in F$ and $y \to z \in F$, then $x \to z \in F$,
(v) If $x \sim y \in F$,$s \sim t \in F$, then $(x \wedge s) \sim (y \wedge t) \in F$, $(x \sim s) \sim (y \sim t) \in F$ and $(x \to s) \sim (y \to t) \in F$.

We denote $a \Leftrightarrow b := (a \to b) \wedge (b \to a)$ and $a \Leftrightarrow^\circ b := (a \to b) \otimes (b \to a)$, for all $a, b, c \in E$.

Theorem 2.4. [3] *Let F be a filter of an EQ-algebra $\xi = (E, \wedge, \otimes, \sim, 1)$. Then the following hold :*
(i) $a, b \in F$ implies $a \otimes b \in F$,
(ii) $a \sim b \in F$ iff $a \Leftrightarrow b \in F$ iff $a \to b \in F$ and $b \to a \in F$ iff $a \Leftrightarrow^\circ b$,
(iii) If $a \sim b \in F$, then $(a \otimes c) \sim (b \otimes c) \in F$ and $(c \otimes a) \sim (c \otimes b) \in F$, for all $a, b, c \in E$.

Definition 2.4. [3] A prefilter F of an EQ-algebra ξ is said to be a prime prefilter if for all $a, b \in E$, $a \to b \in F$ or $b \to a \in F$.

For brevity, we need the following notations for all $a, z \in E$ and natural number n:

$$a \to^0 z = z, \quad a \to^1 z = a \to z, \quad a \to^2 z = a \to (a \to z), \quad a \to^n z = a \to (a \to^{n-1} z).$$

Definition 2.5. [13] Let $\emptyset \neq X \subseteq E$. A generated prefilter by X, is the smallest prefilter containing X and denoted by $< X >$. We have

$< X >:= \{a \in E : \exists x_i \in X \text{ and } n \geq 1 \text{ such that } x_1 \to (x_2 \to ...(x_n \to a)...) = 1\}$.
Moreover, for a prefilter F of ξ and $x \in E$,

$$F(x) :=< \{x\} \cup F >= \{a \in E \mid \exists n \geq 1 \text{ such that } x \to^n a \in F\}.$$

Definition 2.6. [9] A prefilter F of an EQ-algebra ξ is called a positive implicative prefilter if it satisfies for any $x, y, z \in E$:
(F_4) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$.

Lemma 2.5. [9] *If F is a positive implicative prefilter of an EQ-algebra ξ, then for all $x \in E$, $F(x) = \{a \in E | x \to a \in F\}$.*

Definition 2.7. [9] A nonempty subset F of E is called an implicative prefilter if it satisfies (F_1) and
(F_5) $z \to ((x \to y) \to x) \in F$ and $z \in F$ imply $x \in F$, for any $x, y, z \in E$.

Theorem 2.6. [9] *Each implicative prefilter of an EQ-algebra ξ is a positive implicative prefilter.*

3. Obstinate prefilters (filters) in EQ-algebras

From now on, unless mentioned otherwise, $\xi = (E, \wedge, \otimes, \sim, 1)$ will be an EQ-algebra, which will be referred to by its support set E.

Definition 3.1. A prefilter F of ξ is called an *obstinate prefilter* of ξ if for all $x, y \in E$,
(F_6) $x, y \notin F$ implies $x \to y \in F$ and $y \to x \in F$.
If F is a filter and satisfies (F_6), then F is called an obstinate filter.

Example 3.1. (i) Let $\xi_1 = (\{0, a, b, c, d, 1\}, \wedge, \otimes, \sim, 1)$ such that $0 < a < b, c < d < 1$. The following binary operations " $\otimes$ " and " $\sim$ " define an EQ-algebra [9]. The implication is also given as follows:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	a	a	b
c	0	0	a	0	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	0	0	0	0	0
a	0	1	d	d	d	d
b	0	d	1	d	d	d
c	0	d	d	1	d	d
d	0	d	d	d	1	1
1	0	d	d	d	1	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	d	1	d	1	1
c	0	d	d	1	1	1
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Then $\{a, b, c, d, 1\}$ is an obstinate prefilter of ξ_1 while $\{1, d\}$ is not an obstinate prefilter, because $0, b \notin \{1, d\}$ and $b \to 0 = \{0\} \notin \{1, d\}$.
(ii) Let $\xi_2 = (\{0, a, b, c, 1\}, \wedge, \otimes, \sim, 1)$, such that $0 < a, b < c < 1$. The following binary operations " $\otimes$ " and " $\sim$ " define an EQ-algebra on ξ_2 and we have the following " $\to$ ":

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\sim$	0	a	b	c	1
0	1	b	a	0	0
a	b	1	1	a	a
b	a	1	1	b	b
c	0	a	b	1	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $\{b, c, 1\}$ and $\{a, c, 1\}$ are obstinate prefilters of ξ_2.

Theorem 3.1. *$\{1\}$ is a prefilter of ξ if and only if ξ is a semi-separated EQ-algebra.*

Proof. Let $\{1\}$ be a prefilter of ξ and $a \sim 1 = 1$ for $a \in E$. We get that $a \sim 1 \in \{1\}$ and so by Theorem 2.3 part (ii), $a = 1$. Therefore, ξ is a semi-separated EQ-algebra. Conversely, let ξ be a semi-separated EQ-algebra and $b, b \to a \in \{1\}$. Then $1 \to a = \{1\}$, we get that $(1 \wedge a) \sim 1 = 1$ and so $a \sim 1 = 1$. Therefore, $a = 1$ and $\{1\}$ is a prefilter of ξ. □

Since every good EQ-algebra is separated, so the above lemma holds for good EQ-algebra.

Lemma 3.2. *Let ξ be a separated EQ-algebra. Then $\{1\}$ is an obstinate prefilter of ξ if and only if E has at most two elements.*

Proof. Let $\{1\}$ be an obstinate prefilter and $x, y \in E - \{1\}$. Then $x \to y \in \{1\}$ and $y \to x \in \{1\}$, and so $(x \wedge y) \sim x = (x \wedge y) \sim y = 1$. Since ξ is a separated EQ-algebra, $x = x \wedge y = y$, thus $x = y$. Therefore, E has at most two elements. The converse is clear. □

Lemma 3.3. *Let F be a prefilter of ξ. Then F is an obstinate prefilter of ξ if and only if $x, y \notin F$ implies $x \sim y \in F$.*

Proof. Let F be an obstinate prefilter of ξ and $x, y \notin F$. Then $x \to y \in F$ and $y \to x \in F$, and so $(x \wedge y) \sim x \in F$ and $(x \wedge y) \sim y \in F$. Therefore, by Theorem 2.3 part (iii) we get that $x \sim y \in F$. Conversely, suppose $x, y \notin F$, then $x \sim y \in F$. Since $x \sim y \leq x \to y, y \to x$, by Theorem 2.3 part (i), we get that $x \to y \in F$ and $y \to x \in F$. Therefore, F is an obstinate prefilter. □

Theorem 3.4. *Let F be a filter of ξ. Then F is an obstinate filter of ξ if and only if $a \Leftrightarrow^{\circ} b \in F$, for all $a, b \in E - F$.*

Proof. Let F be an obstinate filter of ξ and $a, b \in E - F$. Then $a \to b,\ b \to a \in F$. By Theorem 2.4 part(i), we get that $a \Leftrightarrow^{\circ} b = (a \to b) \otimes (b \to a) \in F$. Conversely, let $a, b \in E - F$. Then $a \Leftrightarrow^{\circ} b \in F$. Since $(a \to b) \otimes (b \to a) \leq (a \to b),\ (a \to b)$, by Theorem 2.3 part(i), we have $(a \to b),\ (b \to a) \in F$ and so F is an obstinate filter of ξ. □

Theorem 3.5. *Let bottom element* $0 \in E$ *and* F *be a proper prefilter of* ξ *. Then* F *is an obstinate prefilter of* ξ *if and only if* $x \notin F$ *implies* $\neg x \in F$*, for all* $x \in E$.

Proof. Let F be an obstinate prefilter and $x \notin F$. Then $\neg x = x \to 0 \in F$. Conversely, suppose $x, y \notin F$, then $\neg x, \neg y \in F$. Thus, by Theorem 2.3 part(ii), we conclude that $x \sim y \in F$. Therefore, by Lemma 3.2, F is an obstinate prefilter. □

Corollary 3.6. *Let* ξ *contain a bottom element 0 and* F *be a proper prefilter of* ξ. *Then* F *is an obstinate prefilter of* ξ *if and only if* $x \in F$ *or* $\neg x \in F$*, for all* $x \in E$.

Theorem 3.7. *If* $a \to 0 = 0$*, for all* $a \in E - \{0\}$*, then* $F = E - \{0\}$ *is the only obstinate proper prefilter of* ξ.

Proof. It is clear that by hypothesis, F is a prefilter of ξ . Now let $x, y \notin F = E - \{0\}$. Then, $x = y = 0$ and so $x \to y = y \to x = 0 \to 0 = 1 \in F$. Therefore, F is an obstinate prefilter. Suppose $F = E - \{0\}$ and G are obstinate proper prefilters and $G \neq F$. Then, there is $0 \neq a \in F$ such that $a \notin G$, and so $0 = a \to 0 \in G$ which is a contradiction. □

Theorem 3.8. *(Extension property) Let* F *be an obstinate prefilter of* ξ *and* $F \subseteq G$. *Then* G *is also an obstiante prefilter of* ξ.

Proof. Let F be an obstinate prefilter and $x, y \notin G$. Then, $x, y \notin F$ and so $x \to y \in F$ and $y \to x \in F$. Thus, by hypothesis $x \to y \in G$ and $y \to x \in G$, i.e G is an obstinate prefilter. □

Given a filter F of ξ. The relation on E, $a \approx_F b$ iff $a \sim b \in F$ is a congruence relation. For $a \in E$, we denote its equivalence class w.r.t. $\approx F$ by $[a]_F$ (or [a] for short) and the set of these equivalence classes is denoted by E/F. It is easy to see that $< E/F, \wedge, \otimes, \sim_F, [1] >$ is an EQ-algebra. The ordering in E/F is defined using the derived meet operation in the following way:

$$[a] \leq [b] \text{ iff } [a] \wedge [b] = [a] \text{ iff } a \wedge b \approx_F a \text{ iff } a \wedge b \sim a = a \to b \in F.$$

Theorem 3.9. *Let* F *be an obstinate filter of* ξ. *Then* E/F *is a chain.*

Proof. Let $[a], [b] \in E/F$. If $a \in F$ or $b \in F$, then $a \to b \in F$ or $b \to a \in F$, by Theorem 2.3 part (i) .Then $[a] \leq [b]$ or $[b] \leq [a]$. If $a, b \notin F$, then $a \to b \in F$ and $b \to a \in F$ and so $[a] = [b]$. Therefore, E/F is a chain. □

Let A and B be two EQ-algebras. A function $f : A \to B$ is a homomorphism of EQ-algebras, if it satisfies the following conditions, for every $x, y \in A$:

$$f(1) = 1,$$
$$f(x \otimes y) = f(x) \otimes f(y),$$
$$f(x \sim y) = f(x) \sim f(y),$$
$$f(x \wedge y) = f(x) \wedge f(y).$$

We also define $\ker(f) = \{x \in A : f(x) = 1\}$.
The set of all homomorphisms from A into B is denoted by $Hom(A, B)$.

Theorem 3.10. *Let* $f \in Hom(A, B)$ *and* G *be an obstinate prefilter of* B. *Then,* $f^{-1}(G)$ *is an obstinate prefilter of* A.

Proof. It is clear that $f^{-1}(G)$ is a prefilter of A. Let $x, y \notin f^{-1}(G)$. Then $f(x), f(y) \notin G$, since G is an obstinate prefilter $f(x \to y) = f(x) \to f(y) \in G$ and $f(y \to x) = f(y) \to f(x) \in G$. Thus $x \to y \in f^{-1}(G)$ and $y \to x \in f^{-1}(G)$. Therefore, $f^{-1}(G)$ is an obstinate prefilter of A. □

Proposition 3.11. *Let F be a prefilter of ξ. If $a, b \in F$, then $a \to b, b \to a, a \sim b, a \wedge b \in F$.*

Proof. Let $a, b \in F$. Since $b \leq a \to b$ and $a \leq b \to a$, then by Theorem 2.3 (i), we have $a \wedge b \sim a = a \to b \in F$ and $a \wedge b \sim b = b \to a \in F$. Hence, by Theorem 2.3 part(iii), $a \sim b \in F$. Thus, $a \wedge b \sim a \in F$ and $a \in F$ imply that $a \wedge b \in F$. □

Theorem 3.12. *Let F be an obstinate filter of a spanned EQ-algebra ξ. Then there exists $f \in Hom(E, E)$ such that $ker(f) = F$.*

Proof. Suppose F is an obstinate filter of ξ. f is defined as follows and it is shown that $f \in Hom(E, E)$. It is easy to check that $f(1) = 1$. We consider two arbitrary elements $x, y \in E$ in the following cases:

$$f(x) = \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{if } x \notin F. \end{cases}$$

Case (1): $x, y \in F$.
(1_a) By Theorem 2.3 part(i) we get that $x \otimes y \in F$. Hence, $f(x) = 1 = f(y)$ and $f(x \otimes y) = 1$. Therefore, $f(x \otimes y) = 1 = f(x) \otimes f(y)$.
(1_b) By Theorem 2.3 part(i), we get that $y \leq x \to y$ and so $x \to y \in F$. Thus, $f(x \to y) = 1$ and $f(x) \to f(y) = 1 = 1 \to 1 = 1$. Therefore, $f(x \to y) = f(x) \to f(y)$.
(1_c) By Proposition 3.1, $x \wedge y \in F$, thus $f(x \wedge y) = 1$. We get that $f(x) \wedge f(y) = 1 \wedge 1 = 1$. Therefore, $f(x \wedge y) = 1 = f(x) \wedge f(y)$.
(1_d) By Proposition 3.1, $x \sim y \in F$. Thus, $f(x \sim y) = 1$. So, $f(x) = 1 = f(y)$ and $f(x \sim y) = 1$. Therefore, $f(x \sim y) = 1 = f(x) \sim f(y)$.
Case (2): $x, y \notin F$:
(2_a) By Theorem 2.3 part (i), $x \otimes y \notin F$. So $f(x \otimes y) = 0$. On the other hand, $f(x) \otimes f(y) = 0 \otimes 0 = 0$. It follows that $f(x \otimes y) = f(x) \otimes f(y)$.
(2_b) Since F is an obstinate filter, $x \to y \in F$ and so $f(x \to y) = 1$. On the other hand, $f(x) \to f(y) = 0 \to 0 = 1$. It follows that $f(x \to y) = f(x) \to f(y)$. Similarly, $f(y \to x) = f(y) \to f(x)$.
(2_c) By Theorem 2.3 part (i), $x \wedge y \notin F$, and so $f(x \wedge y) = 0$. On the other hand, $f(x) \wedge f(y) = 0 \wedge 0 = 0$. It follows that $f(x \wedge y) = f(x) \wedge f(y)$.
(2_d) Since F is an obstinate filter, we have $x \sim y \in F$, $f(x) = 0 = f(y)$ and $f(x \sim y) = 1$. Therefore, $f(x \sim y) = 1 = 0 \sim 0 = f(x) \sim f(y)$.
Case (3): $x \notin F$, $y \in F$:
(3_a) We get that $x \otimes y \notin F$. So $f(x \otimes y) = 0$. Therefore, $f(x \otimes y) = 0 = 0 \otimes 1 = f(x) \otimes f(y)$.
(3_b) We have $y \leq x \to y$, hence $x \to y \in F$. Then $f(x \to y) = 1$. On the other hand, $f(x) \to f(y) = 0 \to 1 = 1$. It follows that $f(x \to y) = f(x) \to f(y)$. By F_2 and hypothesis we have, $y \to x \notin F$. Then, $f(y \to x) = 0$. Since ξ is a spanned EQ-algebra, we get that $f(y) \to f(x) = 1 \to 0 = \tilde{0} = 0$. Therefore, $f(y \to x) = 0 = f(y) \to f(x)$.
(3_c) In this case $x \wedge y \notin F$, and so $f(x \wedge y) = 0$. On the other hand, $f(x) \wedge f(y) = 1 \wedge 0 = 0$. It follows that $f(x \wedge y) = f(x) \wedge f(y)$.

(3_d) Since $x \sim y \leq y \to x \notin F$, we get that $f(x \sim y) = 0$. On the other hand, since ξ is a spanned EQ-algebra, $f(x) \sim f(y) = 0 \sim 1 = \tilde{0} = 0$. Therefore, $f(x \sim y) = 0 = f(x) \sim f(y)$.
Case (4): $x \in F$, $y \notin F$: It can be proved similar to case (3).
Summarizing all the above we have proven that $f \in Hom(E, E)$. It is clear that $Ker(f) = f^{-1}(1) = F$. □

4. Maximal prefilters in EQ-algebras

Definition 4.1. A prefilter F of ξ is called a maximal prefilter if it is proper and no proper prefilter of ξ stricly contains F, that is, for each prefilter, $G \neq F$, if $F \subseteq G$, then $G = E$.

Theorem 4.1. *If $0 \in E$ and M is a proper prefilter of ξ, then the following are equivalent:*
(i) M is a maximal prefilter of ξ,
(ii) For any $x \notin M$, there exists $n \geq 1$ such that $x \to^n 0 \in M$.

Proof. $(i) \Rightarrow (ii)$. If $x \notin M$, then $\langle M \bigcup \{x\} \rangle = E$, and so $0 \in \langle M \bigcup \{x\} \rangle$. Thus, there exists $n \geq 1$ such that $x \to^n 0 \in M$.
$(ii) \Rightarrow (i)$. Assume there is a proper prefilter G such that $M \subset G$. Then, there exists $x \in G$ such that $x \notin M$. By hypothesis, there exists $n \geq 1$ such that $x \to^n 0 \in M$. By Definition 2.3 we get that $0 \in G$, which is a contradiction. □

Proposition 4.2. *Let ξ contain a bottom element 0. Then, every obstinate proper prefilter of ξ is a maximal prefilter of ξ.*

Proof. Let F be an obstinate proper prefilter of ξ, G be a prefilter of ξ and $F \subseteq G \subseteq E$. If $F \neq G$, then there is $x \in G$ such that $x \notin F$. So, by Corollary 3.1, $\neg x \in F$ and we get that $\neg x \in G$. By Theorem 2.3 part (ii) we conclude that $0 \in G$. Therefore, $G = E$. □

By the following example we show that the converse of Proposition 4.1 may not be true.

Example 4.1. Let $\xi = (\{0, a, b, c, d, 1\}, \wedge, \otimes, \sim, 1)$ be an EQ-algebra, with $0 < a < b < c < d < 1$. " $\to$ ", " $\otimes$ " and " $\sim$ " defined as the following [9]:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	a	b
c	0	0	0	a	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	c	b	a	0	0
a	c	1	b	a	a	a
b	b	b	1	b	b	b
c	a	a	b	1	c	c
d	0	a	b	c	1	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	1	1	1	1	1	1
b	b	b	1	1	1	1
c	a	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Then, $\{c, d, 1\}$ is a maximal prefilter of ξ while it is not an obstinate prefilter, because $0, b \notin \{c, d, 1\}$ and $b \to 0 = b \notin \{c, d, 1\}$.

Lemma 4.3. *Let F be a maximal and positive implicative prefilter of ξ. Then, F is an obstinate prefilter of ξ.*

Proof. Let $x, y \notin F$. Then $E =< F, y >= \{z \in E \mid y \to z \in F\}$ and so $y \to x \in F$. Similarly, we can obtain $x \to y \in F$. Thus, F is an obstinate prefilter of ξ. □

Lemma 4.4. *Every obstinate prefilter of an EQ-algebra ξ is an implicative prefilter.*

Proof. Let $(x \to y) \to x \in F$. Consider the following cases:
Case (1): If $y \in F$, then $y \leq x \to y$ by Theorem 2.3 part (i) implies $x \to y \in F$. By hypothesis, we obtain $x \in F$.
Case (2): If $x, y \notin F$, since F is an obstinate prefilter, then $x \to y \in F$ and we get that $x \in F$ by hypothesis, which is a contradiction. □

The following example shows that the converse of Lemma 4.2 may not be true.

Example 4.2. Let $\xi = (\{0, a, b, 1\}, \wedge, \otimes, \sim, 1)$ be a chain with Cayley tables as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	1
1	0	a	b	1

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	1	1

Then, ξ is an EQ-algebra [9], and $\{b, 1\}$ is an implicative prefilter, while it is not an obstinate prefilter.

By Theorem 2.5 and Lemma 4.1 we have the following:

Corollary 4.5. *If F is a maximal and implicative prefilter of ξ, then F is an obstinate prefilter of ξ.*

Theorem 4.6. *Every obstinate prefilter F of ξ is a prime prefilter.*

Proof. Let $a, b \in E$, $a \to b \notin F$ and $b \to a \notin F$. Then, $a \leq b \to a$ and $b \leq a \to b$, imply $a, b \notin F$. Since F is an obsinate prefilter, we get that $a \to b \in F$ and $b \to a \in F$, which is a contradiction. □

The converse of the above theorem does not hold in general.

Example 4.3. Consider EQ-algebra in Example 4.1 It is easy to check that $\{1, d\}$ is a prime prefilter of ξ, while it is not an obstinate prefilter, because $a, b \notin \{d, 1\}$ and $b \to a = a \notin \{d, 1\}$.

Corollary 4.7. *Every maximal and implicative prefilter of* ξ *is a prime prefilter of* ξ*.*

The prefilter $F = \{1, d\}$ in Example 4.3 is a prime prefilter while it is not a maximal prefilter.

5. Conclusion and future research

In this paper, we introduced the notions of obstinate prefilters (filters) and maximal prefilters in an EQ-algebra. We established properties of obstinate prefilters and maximal prefilters in an EQ-algebra. We proved some relationships between obstinate prefilters and the other types of prefilters in an EQ-algebra. In future work, we will introduce other types of prefilters and find the relation between them and the prefilters in this paper. Also we will find the relation of obstinate filters with congruences.

References

[1] A. Borumand Saeid, M.Pourkhatoun, Obstinate filters in residuated lattices, *Bull. Math. Soc. Sci. Math. Roumanie* **55**(**103**) (2012), no.4, 413–422.
[2] C.C. Chang, Algebraic analysis of many valued logics, *Trans. Am. Math. Soc.* **88** (1958), 467–490.
[3] M. El-Zekey, V. Novák, R. Mesiar, On good EQ-algebras, *Fuzzy sets and systems* **178** (2011), 1–23.
[4] M. El-Zekey, Repersentable good EQ-algebras, *Soft Computing* **14** (2010), no. 9, 1011–1023.
[5] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, *Residuated Lattices. An Algebraic Glimpse At Substructural Logics*, Elsevier, Amsterdam, 2007.
[6] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer, Dordrecht, 1998.
[7] M. Haveshki, A. Borumand Saeid, E. Eslami, Some types of filters in BL-algebras, *Soft Computing* **10** (2006), 657-664.
[8] Y.B. Jun, Y. Xu, X.H. Zhang, Fuzzy filters of MTL-algebras, *Inform. Sci.* **175** (2005), 120-138.
[9] L. Liu, X. Zhang, Implicative and positive implicative prefilters of EQ-algebras, *Journal of Intelligent and Fuzzy Systems* **26** (2014), 2087–2097.
[10] N. Mohtashamnia, L. Torkzadeh, The lattice of prefilters of an EQ-algebra, to appear.
[11] V. Novák, On Fuzzy Type Theory, *Fuzzy Sets and Systems* **149** (2005), 235–273.
[12] V. Novák, On the syntactico-semantical completeness of rstorder fuzzy logic I, II, *Kybernetika* **26** (1990), no. 47-66, 134–154.
[13] V. Novák, EQ-algebra in progress, in: *Theoretical Advances and Applications of Fuzzy Logic and Soft Computing* (O. Casitillo, et al.,eds.), Advances in Soft Computing 42, Springer, (2007), 876–884.
[14] V. Novák, EQ-algebra-based fuzzy type theory and its extensions, *Logic Journal of the IGPL* **19** (2011), 512–54.
[15] V. Novák, B. De Baets, EQ-algebra, *Fuzzy sets and systems* **160** (2009), 2956–2978.
[16] E. Turunen, *Mathematics Behind Fuzzy Logic*, Physica-Verlag, Heidelberg.
[17] J. Zhan, Y. Xu, Some types of generalized fuzzy filters of BL-algebras, *Computers and Mathematics with Applications* **56** (2008), 1604–1616.
[18] B. Van Gasse, G. Deschrijver, C. Cornelis, E. Kerre, Filters of residuated lattices and triangle algebras, *Inform. Sci.* **180** (2010), 3006-3020.

(M. Behzadi, L. Torkzadeh) Department of Mathematics, Kerman Branch, Islamic Azad University, Kerman, Iran
E-mail address: m_behzadi12@yahoo.com, ltorkzadeh@yahoo.com