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# Filter theory on good hyper $EQ$-algebras

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Abstract. A special hyper algebra has been recently introduced in [1]. Its original motivation comes from $EQ$-algebra. In this paper, we continue the study of hyper $EQ$-algebra in special case named good hyper $EQ$-algebra. We introduce different kinds of (pre)filters and investigate some results about them and relation between them in good hyper $EQ$-algebra. Then we gain conditions which lead to have a quotient hyper $EQ$-algebra.

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## 1. Introduction

$EQ$-algebra introduced by Vilém Novák in [5]. These algebras are intended to become algebras of truth values for a higher-order fuzzy logic (a fuzzy type theory, $FTT$). The concept of hyper structure (called also multialgebra) was introduced by Marty [3] at first. That was 8th Congress of Scandinavian mathematician 1934. Till now, the hyper structures are studied from the theoretical point of view, for their applications to many subject of pure and applied mathematics. A special hyper algebra has been recently introduced in [1], named hyper $EQ$-algebra. In this paper we continue the study of hyper $EQ$-algebra in special case named good hyper $EQ$-algebra and introduce the concept of different kinds of (pre)filters and investigate some results about them and study relation between them. We show that can have good hyper $EQ$-algebra in type godel by positive implicative filter and have hyper $\ell EQ$-algebra by fantastic filter.

## 2. Preliminaries

**Definition 2.1.** [5] *An* $EQ$-algebra *is an algebra* $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ *of type* $(2, 2, 2, 0)$ *such that, for all* $x, y, z, t \in E$:
$(E1)$ $\langle E, \wedge, 1\rangle$ *is a commutative idempotent monoid (i.e.* $\wedge$*-semilattice with top element 1);*
$(E2)$ $\langle E, \otimes, 1\rangle$ *is a commutative monoid and* $\otimes$ *is isotone w.r.t.* " $\leq$ "
*(where* $x \leq y$ *is defined as* $x \wedge y = x$*);*
$(E3)$ $x \sim x = 1$; *( reflexivity axiom )*
$(E4)$ $((x \wedge y) \sim z) \otimes (t \sim x) \leq z \sim (t \wedge y)$; *( substitution axiom )*
$(E5)$ $(x \sim y) \otimes (z \sim t) \leq (x \sim z) \sim (y \sim t)$; *( congruence axiom )*
$(E6)$ $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$; *( monotonicity axiom )*

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$(E7)$ $(x \wedge y) \sim x \leq (x \wedge y \wedge z) \sim (x \wedge z)$; *( monotonicity axiom )*
$(E8)$ $x \otimes y \leq x \sim y$. *( boundedness axiom )*

The hyperstructure theory was introduced by Marty [3], at the 8th Congress of Scandinavian Mathematicians. In his definition, a function $\circ : A \times A \to P^*(A)$, of the set $A \times A$ into the set of all non-empty subsets of $A$, is called a binary hyperoperation, and the pair $(A, \circ)$ is called a hypergroupoid. If $\circ$ is associative, then $A$ is called a semihypergroup, and it is said to be commutative if $\circ$ is commutative. Also, an element $1 \in A$ is called an identity element if $x \in 1 \circ x$, for all $x \in A$.
Note that if $A, B \subseteq H$, then

(i) $x \circ B = \bigcup\limits_{b \in B} (x \circ b)$ , $B \circ x = \bigcup\limits_{b \in B} (b \circ x)$,
(ii) $A \circ B = \bigcup\limits_{a \in A} ( \bigcup\limits_{b \in B} a \circ b)$,

**Definition 2.2.** [1] *A* hyper *EQ*-algebra $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ *is a non-empty set $H$ with a binary operations $\wedge$ and two binary hyper operations $\otimes$, $\sim$ and top element "1" satisfying the following conditions, for all $x, y, z, t \in H$:*
$(HEQ1)$ $\langle H, \wedge, 1\rangle$ *is a commutative idempotent monoid with top element* "1",
$(HEQ2)$ $\langle H, \otimes, 1\rangle$ *is a commutative semihypergroup with "1" as an identity and $\otimes$ is isotone w.r.t. $\leq$, i.e. if $x \leq y$, then $x \otimes z \ll y \otimes z$( where $x \leq y$ if and only if $x \wedge y = x$ ),*
$(HEQ3)$ $((x \wedge y) \sim z) \otimes (t \sim x) \ll z \sim (t \wedge y)$,
$(HEQ4)$ $(x \sim y) \otimes (z \sim t) \ll (x \sim z) \sim (y \sim t)$,
$(HEQ5)$ $(x \wedge y \wedge z) \sim x \ll (x \wedge y) \sim x$,
$(HEQ6)$ $(x \wedge y) \sim x \ll (x \wedge y \wedge z) \sim (x \wedge z)$,
$(HEQ7)$ $x \otimes y \ll x \sim y$.
*where $A \ll B$, means that, for all $a \in A$ there exists $b \in B$ such that $a \leq b$.*

**Proposition 2.1.** [1] *Let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra such that $x \to y = (x \wedge y) \sim x$ and $\bar{x} = x \sim 1$. Then the following conditions hold, for all $x, y, z, t \in H$ and $A, B, C \subseteq H$:*
*(i)* $x \ll x \otimes 1$, $A \ll A \otimes 1$;
*(ii)* $1 \in x \sim x$ , $1 \ll x \to x$ *and* $1 \in A \sim A$;
*(iii)* *if* $A \ll B$ *and* $B \ll C$, *then* $A \ll C$;
*(iv)* $x \sim y \ll x \to y$;
*(v)* *if* $A \ll B$, *then* $A \otimes C \ll B \otimes C$;
*(vi)* $(x \sim y) \otimes (y \sim z) \ll x \sim z$ *and* $(x \to y) \otimes (y \to z) \ll x \to z$;
*(vii)* $x \otimes (x \sim y) \ll \bar{y}$;
*(viii)* *if* $x \leq y$, *then* $\bar{x} \ll \bar{y}$, $x \sim y = y \to x$ , $z \to x \ll z \to y$ , $y \to z \ll x \to z$;
*(ix)* *if* $A \ll B$, *then* $C \to A \ll C \to B$;
*(x)* $y, \bar{y} \ll x \to y$;
*(xi)* $x \to y = x \to (x \wedge y)$;
*(xii)* $x \to y \ll (z \to x) \to (z \to y)$;
*(xiii)* $x \sim y \ll (x \sim z) \sim (y \sim z)$ *and* $x \sim y \ll (z \sim x) \sim (z \sim y)$.

**Definition 2.3.** [1] *Let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra. Then $\mathcal{H}$ is called,*
*(i)* separated *if $1 \in x \sim y$, then $x = y$, for all $x, y \in H$, ( in other words $1 \in x \sim y$ if and only if $x = y$ ),*
*(ii)* good *if $x \sim 1 = x = 1 \sim x$, for all $x \in H$.*

**Proposition 2.2.** [1] *Every good hyper EQ-algebra is separated.*

**Definition 2.4.** [1] *Let $\mathcal{H}$ be a hyper EQ-algebra and $F$ be a subset of $H$ such that $1 \in F$. Then $F$ is called a*
$(i)$ prefilter *of $\mathcal{H}$, if $x \to y \subseteq F$ and $x \in F$, imply $y \in F$ and $(x \otimes y) \subseteq F$, for all $x, y \in F$;*
$(ii)$ filter *of $\mathcal{H}$, if $F$ is a prefilter and $x \to y \subseteq F$, imply $(x \otimes z) \to (y \otimes z) \subseteq F$, for all $x, y, z \in H$.*

**Definition 2.5.** [1] *Let $\mathcal{H}$ be a hyper EQ-algebra and $D$ be a non-empty subset of $H$. Then $D$ is said to be $S_{\to}reflexive$( $S_{\sim}reflexive$) if $(x \to y) \cap D \neq \emptyset((x \sim y) \cap D \neq \emptyset)$, then $x \to y \subseteq D(x \sim y \subseteq D)$, for all $x, y \in H$.*

**Remark 2.1.** [1] *Let $D$ be $S_{\sim}reflexive$ and $(x \to y) \cap D \neq \emptyset$, for $x, y \in D$. Then $((x \wedge y) \sim x) \cap D \neq \emptyset$. Since $D$ is $S_{\sim}reflexive$, then $((x \wedge y) \sim x) \subseteq D$ or $(x \to y) \subseteq D$. Therefore, $D$ is $S_{\to}reflexive$.*

**Lemma 2.1.** [1] *Let $F$ be an $S_{\sim}reflexive$ (pre)filter of $\mathcal{H}$. Then the following conditions hold, for all $x, y, z \in H$ and $A, B \subseteq H$:*
$(i)$ *if $x \in F$ and $x \leq y$, then $y \in F$, and if $A \subseteq F$ and $A \ll B$, then $B \cap F \neq \emptyset$;*
$(ii)$ *if $(x \to y) \subseteq F$ and $(y \to z) \subseteq F$, then $(x \to z) \cap F \neq \emptyset$.*

For $S_{\sim}reflexive$ filter $F$,we define $x \equiv_F y$ if and only if $x \sim y \cap F \neq \emptyset$. Thus we have:

**Theorem 2.1.** [1] *Let $F$ be an $S_{\sim}reflexive$ filter of $\mathcal{H}$. Then $\frac{\mathcal{H}}{\equiv_F} = (\frac{H}{\equiv_F}, \bar{\wedge}, \bar{\otimes}, \bar{\sim}, [1])$ is a hyper EQ-algebra which is separated.*

**Theorem 2.2.** *Let $F$ be an $S_{\sim}reflexive$ filter of $\mathcal{H}$. Then $\frac{\mathcal{H}}{\equiv_F} = (\frac{H}{\equiv_F}, \bar{\wedge}, \bar{\otimes}, \bar{\sim}, [1])$ is a hyper EQ-algebra which is separated. Moreover, if $\mathcal{H}$ is good, then $\frac{\mathcal{H}}{\equiv_F}$ is good, too and $[1]_{\equiv_F} = F$.*

*Proof.* By Theorem 2.2, the proof is clear. □

## 3. (Positive) implicative Filters

In this section we introduce concept of positive implicative (pre)filter and implicative (pre)filter in hyper $EQ$-algebras. Then we investigate some results about them and study relations between them.

**Definition 3.1.** *Let $\mathcal{H}$ be a hyper EQ-algebra and $F$ be a subset of $H$ such that $1 \in F$. Then $F$ is called:*
$(i)$ *a* positive implicative prefilter *of $\mathcal{H}$, if $z \to (y \to x) \subseteq F$ and $z \to y \subseteq F$, imply $z \to x \subseteq F$, for all $x, y, z \in H$ and $x \otimes y \subseteq F$, for all $x, y \in F$.*
$(ii)$ *a* positive implicative filter *of $\mathcal{H}$, if $F$ is a positive implicative prefilter and $x \to y \subseteq F$, imply $(x \otimes z) \to (y \otimes z) \subseteq F$, for all $x, y, z \in H$.*
$(iii)$ *an* implicative prefilter *of $\mathcal{H}$, if $z \to ((x \to y) \to x) \subseteq F$ and $z \in F$, imply $x \in F$, for all $x, y, z \in H$ and $x \otimes y \subseteq F$, for all $x, y \in F$.*
$(iv)$ *an* implicative filter *of $\mathcal{H}$, if $F$ is an implicative prefilter and $x \to y \subseteq F$, imply $(x \otimes z) \to (y \otimes z) \subseteq F$, for all $x, y, z \in H$.*

**Example 3.1.** *(i) Let $\mathcal{H}$ be a separated hyper EQ-algebra such that $x \to y$ and $x \otimes y$ be singleton. Then by [[6], Lemma 15], the concept of (positive)implicative (pre)filter in hyper EQ-algebra and EQ-algebra are coincide.*
*(ii) Let $(H = \{0, a, b, 1\}, \leq)$ be a poset, such that $0 \leq a \leq b \leq 1$. Define $\wedge, \otimes$ and $\sim$ on $H$ as follows:*

$$x \wedge y = x \otimes y = \min\{x, y\} \quad ,$$

| $\sim$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | {1} | {0} | {0} | {0} |
| a | {0} | {1} | {b,1} | {a,1} |
| b | {0} | {b,1} | {1} | {b,1} |
| 1 | {0} | {a,1} | {b,1} | {1} |

*It is easy to see that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra and $F = \{a, b, 1\}$ is a (positive) implicative (pre)filter.*
*(iii)Let $(H = \{0, a, b, 1\}, \leq)$ be a poset, such that $0 \leq a \leq b \leq 1$. Define $\wedge, \otimes$ and $\sim$ on $H$ as follows:*

| $\otimes$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | {0} | {0} | {0} | {0} |
| a | {0} | {0,a} | {0,a} | {0,a} |
| b | {0} | {0,a} | {b} | {b} |
| 1 | {0} | {0,a} | {b} | {1} |

| $\sim$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | {1} | {a,b,1} | {a,1} | {0,1} |
| a | {a,b,1} | {a,1} | {a,b,1} | {a,1} |
| b | {a,1} | {a,b,1} | {b,1} | {b,1} |
| 1 | {0,1} | {a,1} | {b,1} | {1} |

$$x \wedge y = \min\{x, y\}.$$

*It is easy to see that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra. Then $F = \{b, 1\}$ is positive implicative (pre)filter but it is not an implicative (pre)filter.*
*(iv) Let $(H = \{0, a, b, c, 1\}, \leq)$ be a poset, such that $0 \leq a \leq b \leq c \leq 1$. Define $\wedge, \otimes$ and $\sim$ on $H$ as follows:*

| $\sim$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | {1} | {a, b, c, 1} | H | H | H |
| a | {a, b, c, 1} | {1} | {a, b, c, 1} | {a, b, c, 1} | {a, b, c, 1} |
| b | H | {a, b, c, 1} | {1} | {b, c, 1} | {b, c, 1} |
| c | H | {a, b, c, 1} | {b, c, 1} | {1} | {c, 1} |
| 1 | H | {a, b, c, 1} | {b, c, 1} | {c, 1} | {1} |

$$x \wedge y = x \otimes y = \min\{x, y\}.$$

*It is easy to see that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra and $F_1 = \{b, c, 1\}$ is an implicative prefilter of $\mathcal{H}$. Moreover, since $a \to (a \to 0) = \{1\} \subseteq F_1$ and $a \to a = \{1\} \subseteq F_1$ but $a \to 0 \nsubseteq F_1$, then $F_1$ is not a positive implicative prefilter. Also $F_2 = \{a, 1\}$ is not a (positive)implicative (pre)filter.*

**Proposition 3.1.** *Let $\mathcal{H}$ be a good hyper EQ-algebra. Then*
*(i) each positive implicative (pre)filter is a (pre)filter.*
*(ii) each $S_\sim reflexive$ implicative (pre)filter is a (pre)filter.*

*Proof.* (*i*): Let $F$ be a positive implicative (pre)filter, $x \to y \subseteq F$ and $x \in F$, for some $x, y \in H$. Then by goodness $1 \to (x \to y) = x \to y \subseteq F$ and $1 \to x = \{x\} \subseteq F$. Thus by Definition 3.1, $1 \to y \subseteq F$ or $y \in F$. Hence $F$ is a (pre)filter.
(*ii*): Let $F$ be an $S_\sim reflexive$ implicative (pre)filter, $x \to y \subseteq F$ and $x \in F$, for some $x, y \in H$. Then by goodness $x \to y \subseteq x \to ((y \to 1) \to y)$. Thus

$x \to ((y \to 1) \to y) \cap F \neq \emptyset$ or $x \to ((y \to 1) \to y) \subseteq F$. Since $x \in F$, by Definition 3.1, $y \in F$. Hence $F$ is a (pre)filter. □

It is obviously converse of the Proposition 3.1 is not true.

**Example 3.2.** *In Example 3.1(ii), $F = \{1\}$ is a filter but it is not an implicative filter and in Example 3.1(iv), $F = \{a, 1\}$ is a filter but it is not a positive implicative filter.*

**Theorem 3.1.** *Let $\mathcal{H}$ be a hyper EQ-algebra such that $z \to (y \to x) \ll (z \to y) \to (z \to x)$, for all $x, y, z \in H$. Then every $S_{\sim}reflexive$ (pre)filter $F$ of $H$ is a positive implicative (pre)filter.*

*Proof.* Let $z \to (y \to x) \subseteq F$ and $z \to y \subseteq F$, for some $x, y, z \in H$. Then by assumption and Lemma 2.1(i), $(z \to y) \to (z \to x) \cap F \neq \emptyset$ or $(z \to y) \to (z \to x) \subseteq F$. Hence $z \to x \cap F \neq \emptyset$ or $z \to x \subseteq F$ and so $F$ ia a positive implicative (pre)filter. □

**Definition 3.2.** *Let $\mathcal{H}$ be a hyper EQ-algebra. Then we say that $\mathcal{H}$ is satisfies in*
$(i)$ exchange principle *condition or $(EP)$ condition if $A \to (B \to C) \ll B \to (A \to C)$, for all $A, B, C \subseteq H$;*
$(ii)$ residuated condition *, when $A \otimes B \ll C$ if and only if $A \ll B \to C$, for all $A, B, C \subseteq H$.*

**Example 3.3.** $(i)$ *In Examples 3.1(iii) and (iv), $\mathcal{H}$ satisfies in the $(EP)$ condition.*
$(ii)$ *Let $(H = \{0, a, b, c, 1\}, \leq)$ be a poset such that $0 \leq a \leq b \leq 1$, $0 \leq a \leq c \leq 1$ and $b, c$ are incomparable. Moreover, $\wedge, \otimes$ and $\sim$ are defined on $H$ as follows:*

| $\wedge$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | a | a | a |
| b | 0 | a | b | a | b |
| c | 0 | a | a | c | c |
| 1 | 0 | a | b | c | 1 |

| $\otimes$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | {0} | {0} | {0} | {0} | {0} |
| a | {0} | {0} | {0} | {0} | {0, a} |
| b | {0} | {0} | {0} | {0} | {0, b} |
| c | {0} | {0} | {0} | {0} | {0, c} |
| 1 | {0} | {0, a} | {0, b} | {0, c} | {1} |

| $\sim$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | {1} | {a, b, c} | {a, b, c} | {a, b, c} | { 0} |
| a | {a, b, c} | {1} | {a, b, c} | {a, b, c} | {a} |
| b | {a, b, c} | {a, b, c} | {1} | {b, c} | {b} |
| c | {a, b, c} | {a, b, c} | {b, c} | {1} | {c} |
| 1 | {0} | {a} | {b} | {c} | {1} |

*Then $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra and satisfies in the residuated condition.*

**Lemma 3.1.** *Let $\mathcal{H}$ be a good hyper EQ-algebra. Then the following conditions hold, for all $x, y, z \in H$ and $A, B \subseteq H$:*
$(i)$ $x \ll (x \sim y) \sim y$;
$(ii)$ $x \ll (x \to y) \to y$, $A \ll (A \to B) \to B$;
$(iii)$ $x \otimes (x \sim y) \ll y$;
$(iv)$ $x \otimes (x \to y) \ll x \wedge y$, $x \otimes (x \to A) \ll A$;
$(v)$ $x \otimes y \ll x \wedge y$, *especially* $x \otimes 1 \ll x$ ;
$(vii)$ $x \to (y \to z) \ll y \to (x \to z)$;

*(vi)* *if* $x \ll y \to z$, *then* $x \otimes y \ll z$ *and if* $A \ll y \to z$, *then* $A \otimes y \ll z$;
*(vii)* $x \otimes (y \sim z) \ll (x \sim y) \sim z$;
*If* $H$ *contains bottom element "*0*", then*
*(viii)* $x \ll \neg\neg x$.

*Proof.* $(i)$: By Proposition 2.1(i), (ii), (v), $(HEQ4)$ and goodness,

$$x \ll x \otimes 1 \ll (x \sim 1) \otimes (y \sim y) \ll (x \sim y) \sim (1 \sim y) = (x \sim y) \sim y.$$

$(ii)$: By (i) and Proposition 2.1(iii), (iv) and (ix),

$$x \ll (x \sim (x \wedge y)) \sim (x \wedge y) = (x \to y) \sim (x \wedge y) \ll (x \to y) \to (x \wedge y) \ll (x \to y) \to y.$$

The proof of the rest is clear.
$(iii)$: By Proposition 2.1(vii), the proof is straightforward.
$(iv)$: By (iii), $x \otimes (x \to y) = x \otimes (x \sim x \wedge y) \ll x \wedge y$.
$(v)$: By (iv), Proposition 2.1(x) and (v), $x \otimes y \ll x \otimes (x \to y) \ll x \wedge y$.
$(vi)$: Let $x \ll y \to z$, for some $x, y, z \in H$. Then by Proposition 2.1(v), $x \otimes y \ll (y \to z) \otimes y = y \otimes (y \to z) \ll y \wedge z \leq z$. Hence by Proposition 2.1(iii), $x \otimes y \ll z$.
$(vii)$: By $(HEQ4)$, $(x \sim t) \otimes (y \sim z) \ll (x \sim y) \sim (t \sim z)$. Set $t = 1$, then by goodness (vii) is hold.
$(viii)$: By (i), $x \ll (x \sim 0) \sim 0 = \neg\neg x$. □

**Proposition 3.2.** *Let* $\mathcal{H}$ *be a good hyper EQ-algebra and satisfies in the residuated condition. Then the following statements are hold, for all* $x, y, z \in H$ *and* $A, B, C \subseteq H$;
*(i)* $x \to (y \to z) \ll (x \otimes y) \to z$ *and* $(x \otimes y) \to z \ll x \to (y \to z)$;
*(ii)* $x \ll y \to (x \otimes y)$;
*(iii)* $x \ll y \to z$ *if and only if* $y \ll x \to z$ *and* $A \ll B \to C$ *if and only if* $B \ll A \to C$.

*Proof.* $(i)$: By Lemma 3.1(iv), $x \otimes (x \to (y \to z)) \ll y \to z$ and so by residuated condition, we can prove that $x \otimes (x \to (y \to z)) \otimes y \ll z$ or $(x \otimes y) \otimes (x \to (y \to z)) \ll z$ and so $x \to (y \to z) \ll (x \otimes y) \to z$. The proof of other case is the same.
$(ii)$: Since $x \otimes y \ll x \otimes y$, we obtain $x \ll y \to (x \otimes y)$.
$(iii)$: By definition of residuated condition the proof is straightforward. □

**Theorem 3.2.** *Let* $\mathcal{H}$ *be a good hyper EQ-algebra and satisfies in the* $(EP)$ *condition. If* $F$ *is an* $S_{\sim}reflexive$ *filter of* $H$, *then the following statements are equivalent, for all* $x, y, z \in H$:
*(i)* $F$ *is a positive implicative filter;*
*(ii)* *If* $y \to (y \to x) \subseteq F$, *then* $y \to x \subseteq F$;
*(iii)* *If* $(z \to (y \to x)) \subseteq F$, *then* $(z \to y) \to (z \to x) \subseteq F$;
*(iv)* *If* $(z \to (y \to (y \to x))) \subseteq F$ *and* $z \in F$, *then* $y \to x \subseteq F$;
*(v)* $(x \wedge (x \to y)) \to y \subseteq F$;
*Moreover, if* $\mathcal{H}$ *is satisfied in the residuated condition, then the above conditions are equivalent by;*
*(vi)* $x \wedge y \to x \otimes y \subseteq F$.

*Proof.* $(i) \Rightarrow (ii)$: Let $F$ be a positive implicative filter and $y \to (y \to x) \subseteq F$, for $x, y \in H$. Since $1 \in y \to y$ and $F$ is an $S_{\sim}reflexive$ filter, we get $y \to y \subseteq F$. Thus by (i), $y \to x \subseteq F$.
$(ii) \Rightarrow (iii)$: Let $z \to (y \to x) \subseteq F$, for $x, y, z \in H$. Then by Proposition 2.1(xii)

and $(EP)$ condition, $y \to x \ll (z \to y) \to (z \to x) \ll z \to ((z \to y) \to x)$. Thus by Proposition 2.1(ix), $z \to (y \to x) \ll z \to (z \to ((z \to y) \to x))$. Hence by Lemma 2.1(i), $z \to (z \to ((z \to y) \to x)) \cap F \neq \emptyset$ and so $z \to (z \to ((z \to y) \to x)) \subseteq F$. Then by (ii), we can see that $z \to ((z \to y) \to x) \subseteq F$. By $(EP)$ condition, $z \to ((z \to y) \to x) \ll (z \to y) \to (z \to x)$. Then by Lemma 2.1(i), $(z \to y) \to (z \to x) \cap F \neq \emptyset$ and so $(z \to y) \to (z \to x) \subseteq F$.

$(iii) \Rightarrow (iv)$: Let $z \to (y \to (y \to x)) \subseteq F$ and $z \in F$, for $x, y, z \in H$. Since $F$ is a $S_{\sim}reflexive$ filter, $y \to (y \to x) \subseteq F$ and so by (iii), $(y \to y) \to (y \to x) \subseteq F$. Thus by $1 \to (y \to x) \subseteq (y \to y) \to (y \to x)$ and goodness, $y \to x \subseteq F$.

$(iv) \Rightarrow (i)$: Let $z \to (y \to x) \subseteq F$ and $z \to y \subseteq F$, for some $x, y, z \in H$. Then by $(EP)$ condition and Proposition 2.1(xii), $z \to (y \to x) \ll y \to (z \to x) \ll (z \to y) \to (z \to (z \to x))$. Thus $(z \to y) \to (z \to (z \to x)) \cap F \neq \emptyset$ or $(z \to y) \to (z \to (z \to x)) \subseteq F$. Since $z \to y \subseteq F$, by (iv), we can prove that $z \to x \subseteq F$.

$(i) \Rightarrow (v)$: It is clear $x \wedge (x \to y) \ll x, x \to y$, for any $x, y \in H$. Thus $1 \in (x \wedge (x \to y)) \to (x \to y)$ and $1 \in (x \wedge (x \to y)) \to x$. Then $(x \wedge (x \to y)) \to (x \to y) \cap F \neq \emptyset$ and $(x \wedge (x \to y)) \to x \cap F \neq \emptyset$ or $(x \wedge (x \to y)) \to (x \to y) \subseteq F$ and $(x \wedge (x \to y)) \to x \subseteq F$. Hence by (i), we can prove that $(x \wedge (x \to y)) \to y \subseteq F$.

$(v) \Rightarrow (i)$: Let $z \to (y \to x) \subseteq F$ and $z \to y \subseteq F$, for $x, y, z \in H$. Then by $(HEQ6)$, $z \to (y \to x) \ll z \wedge y \to ((y \to x) \wedge y)$. Thus $z \wedge y \to (y \wedge (y \to x)) \cap F \neq \emptyset$ and so $z \wedge y \to (y \wedge (y \to x)) \subseteq F$. Also by Proposition 2.1(xi), $z \to y = z \to (z \wedge y) \subseteq F$. Then by Lemma 2.1(ii), $z \to (y \wedge (y \to x)) \cap F \neq \emptyset$ or $z \to (y \wedge (y \to x)) \subseteq F$. Now, by (v), $(y \wedge (y \to x)) \to x \subseteq F$. Then again by Lemma 2.1(ii), $z \to x \cap F \neq \emptyset$ or $z \to x \subseteq F$.

$(v) \Rightarrow (vi)$: By (v), $(x \wedge (x \to (x \otimes y))) \to (x \otimes y) \subseteq F$, for any $x, y \in H$. Since $\mathcal{H}$ satisfies in the residuated condition, by Proposition 3.2(ii), $y \ll x \to (x \otimes y)$. Thus $x \wedge y \ll x \wedge (x \to (x \otimes y)$ and so $1 \in x \wedge y \to (x \wedge (x \to (x \otimes y)))$. That is $x \wedge y \to (x \wedge (x \to (x \otimes y))) \cap F \neq \emptyset$ or $x \wedge y \to (x \wedge (x \to (x \otimes y))) \subseteq F$. Hence by Lemma 2.1(ii), $x \wedge y \to x \otimes y \cap F \neq \emptyset$ or $x \wedge y \to x \otimes y \subseteq F$.

$(vi) \Rightarrow (ii)$: Let $y \to (y \to x) \subseteq F$. By Proposition 3.2(i), $y \to (y \to x) \ll (y \otimes y) \to x$. Thus $((y \otimes y) \to x) \cap F \neq \emptyset$ and so $(y \otimes y) \to x \subseteq F$. By (vi), $y \to (y \otimes y) \subseteq F$. Then by Lemma 2.1(ii), $y \to x \subseteq F$. □

**Example 3.4.** *Let* $(H = \{0, a, b, 1\}, \leq)$ *be a poset such that* $0 \leq a \leq b \leq 1$. *Define* $\wedge, \otimes$ *and* $\sim$ *on* $H$ *as follows:*

$x \wedge y = \min\{x, y\}$,

| $\sim$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | {1} | {0} | {0} | {0} |
| a | {0} | {1} | {0, a} | {a} |
| b | {0} | {0, a} | {1} | {b} |
| 1 | {0} | {a} | {b} | {1} |

*It is easy to see that* $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ *is a good hyper EQ-algebra and satisfies in the* $(EP)$ *condition and* $F = \{a, b, 1\}$ *is a positive implicative prefilter.*

**Proposition 3.3.** *Let* $\mathcal{H}$ *be a hyper EQ-algebra and satisfies in the residuated condition. Then* $\mathcal{H}$ *satisfies in the* $(EP)$ *condition and* $x \to y \ll (y \to z) \to (x \to z)$, *for all* $x, y, z \in H$.

*Proof.* Let $u \in x \to (y \to z)$, for $x, y, z, u \in H$. Then $u \ll x \to (y \to z)$ and so $u \ll x \to t$, for some $t \in y \to z$. Since $\mathcal{H}$ satisfies in the residuated condition,

$u \otimes x \ll t$. Hence $v \ll t$, for all $v \in u \otimes x$. By $t \in y \to z$, we get $v \ll y \to z$ or $v \otimes y \ll z$. Thus $(u \otimes x) \otimes y \ll z$ or $u \otimes y \ll x \to z$ and so $u \ll y \to (x \to z)$. Hence $x \to (y \to z) \ll y \to (x \to z)$. Then it is easy to see that $A \to (B \to C) \ll B \to (A \to C)$.
Since $y \to z \ll y \to z$, we have $(y \to z) \otimes y \ll z$ or $y \ll (y \to z) \to z$. Now by Proposition 2.1(viii), and by $(EP)$ condition, $x \to y \ll x \to ((y \to z) \to z) \ll (y \to z) \to (x \to z)$. □

**Theorem 3.3.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition. Then every $S_{\sim}reflexive$ implicative (pre)filter is a positive implicative (pre)filter.*

*Proof.* Let $F$ be an implicative (pre)filter and $y \to (y \to x) \subseteq F$, for some $x, y \in H$. Then by Proposition 3.3, $y \to (y \to x) \ll ((y \to x) \to x) \to (y \to x)$. Thus $(((y \to x) \to x) \to (y \to x)) \cap F \neq \emptyset$ or $(((y \to x) \to x) \to (y \to x) \subseteq F$ and by goodness $1 \to (((y \to x) \to x) \to (y \to x)) \subseteq F$. Since $1 \in F$, by Definition 3.1, we can prove that $y \to x \subseteq F$. Therefore $F$ is a positive implicative (pre)filter. □

**Example 3.5.** *in Example 3.1(iii), $\mathcal{H}$ is a good hyper EQ-algebra and satisfies in the residuated condition. We can see that $F = \{b, 1\}$ is a positive implicative filter but not implicative filter.*

**Theorem 3.4.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the $(EP)$ condition, $F$ be a $S_{\sim}reflexive$ filter of $\mathcal{H}$ and $F \subseteq G$. If $F$ is a positive implicative filter, then so is $G$.*

*Proof.* Let $A = y \to (y \to x) \subseteq G$. Then $1 \in A \to (y \to (y \to x))$ and so $A \to (y \to (y \to x)) \cap F \neq \emptyset$ or $A \to (y \to (y \to x)) \subseteq F$. By $(EP)$ condition, $A \to (y \to (y \to x)) \ll y \to (y \to (A \to x))$. Thus $y \to (y \to (A \to x)) \cap F \neq \emptyset$ or $y \to (y \to (A \to x)) \subseteq F$. Thus by Theorem 3.2(ii), $y \to (A \to x) \subseteq F$ and so by $(EP)$ condition, $A \to (y \to x) \cap F \neq \emptyset$ or $A \to (y \to x) \subseteq F \subseteq G$. Since $G$ is a filter and $A \subseteq G$, we obtain $y \to x \subseteq G$. Therefore, $G$ is a positive implicative filter. □

**Lemma 3.2.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition. If $F$ is an $S_{\sim}reflexive$ filter of $\mathcal{H}$, then $\frac{\mathcal{H}}{\equiv_F}$ satisfies in the residuated condition, too.*

*Proof.* Let $[x]\bar{\otimes}[y] \ll [z]$, for some $[x], [y], [z] \in \frac{H}{\equiv_F}$. Then $[t] \leq [z]$, for all $t \in x \otimes y$. Thus $t \to z \subseteq F$, for all $t \in x \otimes y$ and so $(x \otimes y) \to z \subseteq F$. Since $\mathcal{H}$ satisfies in the residuated condition, by Proposition 3.2(i), we can get $x \to (y \to z) \subseteq F$ or $[x] \ll [y]\bar{\to}[z]$. Let $[x] \ll [y] \to [z]$, for some $[x], [y], [z] \in \frac{H}{\equiv_F}$. Then there exists $t \in y \to z$ such that $[x] \leq [t]$ or $x \to t \subseteq F$. Since $F$ is a filter, $x \otimes y \to t \otimes y \subseteq F$. Thus $[x] \otimes [y] \ll [t] \otimes [y]$. By $t \in y \to z$, we have $t \ll y \to z$ or $t \otimes y \ll z$, i.e. $[t] \otimes [y] = \{[u] | u \in t \otimes y\} \ll [z]$. Hence $[x] \otimes [y] \ll [z]$. By Theorem 2.2 and Lemma 3.1(vi), converse is hold. Therefore, $\frac{\mathcal{H}}{\equiv_F}$ satisfies in residuated condition. □

**Theorem 3.5.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition and $x \otimes x$ be singleton, for all $x \in H$. If $F$ be an $S_{\sim}reflexive$ filter of $H$, then $F$ is a positive implicative filter if and only if $\frac{\mathcal{H}}{\equiv_F}$ is a good hyper EQ-algebra such that $[x] \otimes [x] = \{[x]\}$, for all $x \in H$. Moreover if $F$ is a positive implicative filter, then each $S_{\sim}reflexive$ (pre)filter of $\frac{\mathcal{H}}{\equiv_F}$ is a positive implicative (pre)filter.*

*Proof.* Let $F$ be a positive implicative filter. Then by Theorem 2.2, $\frac{\mathcal{H}}{\equiv_F}$ is a good hyper $EQ$-algebra. By Proposition 3.2(ii), $x \ll x \to (x \otimes x)$. Hence $1 \in (x \to (x \to (x \otimes x)))$ or $(x \to (x \to (x \otimes x))) \cap F \neq \emptyset$ or $(x \to (x \to (x \otimes x))) \subseteq F$. Thus by Theorem 3.2(ii), $x \to (x \otimes x) \subseteq F$ and so $[x] \leq [t]$, for all $t \in x \otimes x$. Since by Lemma 3.1(v), $[x]\bar{\otimes}[x] \ll [x]$ we can obtain $[x]\bar{\otimes}[x] = [x]$.
Conversely, let $x \to (x \to y) \subseteq F$, for $x, y \in H$. Then by Proposition 3.2(i), $((x \otimes x) \to y) \cap F \neq \emptyset$ or $(x \otimes x) \to y \subseteq F$. Thus $[x] \otimes [x] \ll [y]$ and so $[x] \leq [y]$ or $x \to y \subseteq F$. Hence $F$ is a positive implicative filter.
Now, let $G$ be an $S_{\sim}reflexive$ (pre)filter and $[x] \to ([x] \to [y]) \subseteq G$. Then by Lemma 3.2 and Proposition 3.2(i), $[x] \to [y] \cap G \neq \emptyset$ or $[x] \to [y] \subseteq G$. Hence $G$ is a positive implicative (pre)filter. □

**Note.** By Theorem 3.5, if we have an $S_{\sim}reflexive$ positive implicative filter of a good hyper $EQ$-algebra, then we can obtain a good hyper $EQ$-algebra in type godel algebra.

**Remark 3.1.** *Let $A, B, C \subseteq H$ and $A \ll B$. Then in general we do not have $B \to C \ll A \to C$.*
*But if $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition this statement is true. Because by Lemma 3.1(ii) and assumption $A \ll B \ll (B \to C) \to C$ and so by Proposition 3.2(iii), $B \to C \ll A \to C$.*

**Theorem 3.6.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the $(EP)$ condition and "0" be a bottom element of $H$. If $F$ be an $S_{\sim}reflexive$ filter of $\mathcal{H}$, then the following statements are equivalent;*
*(i) $F$ is an implicative filter;*
*(ii) if $(x \to y) \to x \subseteq F$, then $x \in F$, for all $x, y \in H$;*
*Moreover, if $\mathcal{H}$ is satisfied in the residuated condition, then the above conditions equivalent by:*
*(iii) $(\neg x \to x) \to x \subseteq F$, for all $x \in H$;*
*(iv) $((x \to y) \to x) \to x \subseteq F$, for all $x, y \in H$;*
*(v) if $x \to (\neg z \to y) \subseteq F$ and $y \to z \subseteq F$, then $x \to z \subseteq F$, for all $x, y, z \in H$;*
*(vi) if $x \to (\neg y \to y) \subseteq F$, then $x \to y \subseteq F$.*

*Proof.* *(i)* ⇒ *(ii)*: By goodness $1 \to ((x \to y) \to x) = (x \to y) \to x \subseteq F$. Then by assumption, $x \in F$.
*(ii)* ⇒ *(i)*: Let $z \to ((x \to y) \to x) \subseteq F$ and $z \in F$, for $x, y, z \in H$. Since $F$ is a filter, $(x \to y) \to x \subseteq F$. Hence by (ii), $x \in F$.
*(ii)* ⇒ *(iii)*: At first we note that, by $\neg x \to x \ll \neg x \to x$ and Proposition 3.2(iii), $\neg x \ll (\neg x \to x) \to x$. Let $A = (\neg x \to x) \to x$. Then by the $(EP)$ condition, Lemma 3.1(ii), (viii) and Proposition 2.1(viii),

$$\begin{aligned}
(\neg x \to x) \to (x \to x) &\ll x \to ((\neg x \to x) \to x) \ll x \to \neg\neg((\neg x \to x) \to x) \\
&= x \to ((((\neg x \to x) \to x) \to 0) \to 0) \\
&\ll (((\neg x \to x) \to x) \to 0) \to (x \to 0) \\
&\ll (((\neg x \to x) \to x) \to 0) \to ((\neg x \to x) \to x) \\
&= (A \to 0) \to A
\end{aligned}$$

Since $1 \in x \to x$, we get $1 \in (\neg x \to x) \to (x \to x)$ and so $(\neg x \to x) \to (x \to x) \cap F \neq \emptyset$ or $(\neg x \to x) \to (x \to x) \subseteq F$. Thus we can obtain, $((A \to 0) \to A) \subseteq F$ and so by

(ii), we can prove that $A \subseteq F$.
$(iii) \Rightarrow (iv)$: By Proposition 2.1(viii), $\neg x \ll x \to y$ and so by Remark 3.1, $(x \to y) \to x \ll \neg x \to x$. Again by Remark 3.1, $(\neg x \to x) \to x \ll ((x \to y) \to x) \to x$. Hence by (iii) and assumption, (iv) is hold.
$(iv) \Rightarrow (ii)$: Let $(x \to y) \to x \subseteq F$. Then by (iv) and assumption we have $x \in F$.
$(v) \Rightarrow (vi)$: Let $x \to (\neg y \to y) \subseteq F$. Then by $1 \in y \to y$ we obtain $y \to y \subseteq F$ and so by (v), $x \to y \subseteq F$.
$(vi) \Rightarrow (v)$: Let $x \to (\neg z \to y) \subseteq F$ and $y \to z \subseteq F$. Then by Proposition 3.2(i), $x \to (\neg z \to y) \ll (x \otimes \neg z) \to y$. Thus $(x \otimes \neg z) \to y \subseteq F$ and so $((x \otimes \neg z) \to y) \otimes (y \to z) \subseteq F$. By Propositions 2.1(vi) and 3.2(i), $((x \otimes \neg z) \to y) \otimes (y \to z) \ll (x \otimes \neg z) \to z \ll x \to (\neg z \to z)$. Hence $x \to (\neg z \to z) \subseteq F$ and by (vi), $x \to z \subseteq F$.
$(vi) \Rightarrow (iii)$: Since $1 \in (\neg x \to x) \to (\neg x \to x)$ we get $(\neg x \to x) \to (\neg x \to x) \subseteq F$ and so by (vi) we can prove $(\neg x \to x) \to x \subseteq F$.
$(i) \Rightarrow (vi)$: Let $x \to (\neg y \to y) \subseteq F$. Since by Proposition 2.1(x), $y \ll x \to y$ we can have by Remark 3.1, $\neg(x \to y) \ll \neg y$ or $x \to (\neg y \to y) \ll \neg y \to (x \to y) \ll (\neg(x \to y) \to (x \to y))$. Thus by goodness and assumption $1 \to (((x \to y) \to 0) \to (x \to y)) = (\neg(x \to y) \to (x \to y)) \subseteq F$. Now, since $1 \in F$, by Definition 3.1$(iii)$, we can prove that $x \to y \subseteq F$. □

**Proposition 3.4.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition, $F$ be an $S_{\sim}reflexive$ filter of $\mathcal{H}$ and $F \subseteq G$. If $F$ is an implicative filter, then so is $G$.*

*Proof.* Let $A = x \to (\neg y \to y) \subseteq G$, for $x, y \in H$. Then by $(EP)$ condition, Propositions 2.1(viii) and 3.2(i), $A \to A = A \to (x \to (\neg y \to y)) \ll A \to ((x \otimes \neg y) \to y) \ll (x \otimes \neg y) \to (A \to y)$. Since $1 \in A \to A$ we obtain $1 \in (x \otimes \neg y) \to (A \to y)$ and so $(x \otimes \neg y) \to (A \to y) \subseteq F$. On the other hand by Remark 3.1 and Proposition 2.1(x), $(A \to y) \to 0 \ll \neg y$ and so $x \otimes (\neg(A \to y)) \ll x \otimes \neg y$ or $(x \otimes \neg y) \to (A \to y) \ll (x \otimes (\neg(A \to y))) \to (A \to y) \ll x \to ((\neg(A \to y) \to (A \to y))$. Since $F$ is an $S_{\sim}reflexive$ implicative filter, then $x \to (A \to y) \subseteq F$. Hence by $(EP)$ condition $A \to (x \to y) \subseteq F \subseteq G$ and by $A \subseteq G$ we get $x \to y \subseteq G$. Therefore, $G$ is an implicative filter. □

**Proposition 3.5.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition and $F$ be a positive implicative filter of $\mathcal{H}$. Then $F$ is an implicative filter if and only if $(x \to y) \to y \subseteq F$ implies $(y \to x) \to x \subseteq F$, for all $x, y \in H$.*

*Proof.* Let $F$ be an implicative filter and $(x \to y) \to y \subseteq F$, for $x, y \in H$. Then by Propositions 2.1(viii), (x), 3.2 and Remark 3.1, $((y \to x) \to x) \to y \ll x \to y$. Thus by Proposition 3.3, we have:

$$\begin{aligned} (x \to y) \to y &\ll (y \to x) \to ((x \to y) \to x) \\ &\ll (x \to y) \to ((y \to x) \to x) \\ &\ll (((y \to x) \to x) \to y) \to ((y \to x) \to x) \end{aligned}$$

Hence $(((y \to x) \to x) \to y) \to ((y \to x) \to x) \cap F \neq \emptyset$ or $(((y \to x) \to x) \to y) \to ((y \to x) \to x) \subseteq F$. Thus by Theorem 3.6(ii), we can prove that $(y \to x) \to x \subseteq F$.
Conversely, let $(x \to y) \to x \subseteq F$. Then by Proposition 3.3, $(x \to y) \to x \ll (x \to y) \to ((x \to y) \to y)$. Hence $((x \to y) \to ((x \to y) \to y)) \cap F \neq \emptyset$ or $(x \to y) \to ((x \to y) \to y) \subseteq F$. Since $F$ is a positive implicative filter, we have

$(x \to y) \to y \subseteq F$ and so $(y \to x) \to x \subseteq F$. On the other hand, by $y \ll x \to y$ and Remark 3.1 we get $(x \to y) \to x \ll y \to x$. Hence $y \to x \cap F \neq \emptyset$ or $y \to x \subseteq F$. Thus by $(y \to x) \to x \subseteq F$, $y \to x \subseteq F$ and Proposition 3.1, we get $x \in F$. Therefore, $F$ is an implicative filter. □

**Definition 3.3.** *Let $F$ be a subset of $H$ such that $1 \in F$. Then $F$ is called a* deductive system *of $\mathcal{H}$, if $x \to y \subseteq F$ and $x \in F$, imply $y \in F$, for all $x, y \in H$.*

Obviously each (pre)filter is a deductive system but not vice versa.

**Definition 3.4.** *Let $\mathcal{H}$ be a hyper $EQ$-algebra and $A$ be a non-empty subset of $H$. Then the smallest deductive system of $\mathcal{H}$ containing $A$, i.e. the intersection of all deductive system containing $A$, is said the* deductive system generated by $A$ *and denoted by $\langle A\rangle$. We write $\langle a\rangle$ instead of $\langle\{a\}\rangle$, for each $a \in H$.*

For all $x, y \in H$ and $n \in \mathbb{N} \cup \{0\}$, $x^n \to y$ is defined as follows:

$$x^0 \to y = 1 \to y, \qquad x^{n+1} \to y = x \to (x^n \to y)$$

**Theorem 3.7.** *Let $\mathcal{H}$ be a good hyper $EQ$-algebra and satisfies in the $(EP)$ condition and $\emptyset \neq A \subseteq H$. Then*
*$\langle A\rangle \subseteq \{x \in H | 1 \in a_1 \to (a_2 \to (... \to (a_n \to x)...))$ for some $a_1, a_2, ..., a_n \in A$ and $n \in \mathbb{N} \cup \{0\}\}$*
*In particular $\langle a\rangle \subseteq \{x \in H | 1 \in a^n \to x,$ for some $n \in \mathbb{N} \cup \{0\}\}$.*

*Proof.* We denote the right side above by $M$. Since $A$ is a non-empty subset of $\mathcal{H}$, it is clear $1 \in M$. Let $x \in M$ and $x \to y \subseteq M$, for some $x, y \in H$. Then $1 \in a_1 \to (a_2 \to (... \to (a_n \to x)...))$ and $1 \in b_1 \to (b_2 \to (... \to (b_m \to (x \to y))...))$, for some $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in A$ and $m, n \in \mathbb{N} \cup \{0\}$. By Proposition 3.3, $b_1 \to (b_2 \to (... \to (b_m \to (x \to y))...)) \ll x \to (b_1 \to (b_2 \to (... \to (b_m \to y)...)))$. Thus $1 \in x \to (b_1 \to (b_2 \to (... \to (b_m \to y)...)))$ and so by goodness and Proposition **??**, $x \ll b_1 \to (b_2 \to (... \to (b_m \to y)...))$. Hence by Proposition **??**(ix), $a_1 \to (a_2 \to (... \to (a_n \to x)...)) \ll a_1 \to (a_2 \to ...(a_n \to (b_1 \to (b_2 \to (... \to (b_m \to y)...))))...)$. Then $1 \in a_1 \to (a_2 \to ...(a_n \to (b_1 \to (b_2 \to (... \to (b_m \to y)...))))...)$ or $y \in M$. Hence $M$ is a deductive system. Since $1 \in x \to x$, for all $x \in A$, we have $A \subseteq M$ and so $\langle A\rangle \subseteq M$. The proof of the rest is clear. □

**Corollary 3.1.** *Let $F$ be an $S_\sim reflexive$ deductive system and $a \in H$. Then*

$$\langle F, a\rangle = \{x \in H | a^n \to x \subseteq F, \textit{for some } n \in \mathbb{N} \cup \{0\}\}.$$

*Proof.* Set $M$ the right side above. Let $x \in \langle F, a\rangle$. Then $1 \in a_1 \to (a_2 \to (...(a_m \to (a^n \to x))...))$, for some $a_1, a_2, ..., a_m \in F$ and $n, m \in \mathbb{N} \cup \{0\}$. Thus $a_1 \to (a_2 \to (...(a_m \to (a^n \to x))...)) \cap F \neq \emptyset$ or $a_1 \to (a_2 \to (...(a_m \to (a^n \to x))...)) \subseteq F$ and so by assumption $a^n \to x \cap F \neq \emptyset$ or $a^n \to x \subseteq F$. Hence $\langle F, a\rangle \subseteq M$. Conversely, let $x \in M$. Then $a^n \to x \subset F$, for some $n \in \mathbb{N}$. Set $t = a^n \to x$, thus $1 \in t \to (a^n \to x) \ll a^n \to (t \to x) = a \to (a \to (...(t \to x)...))$ and so $1 \in a \to (a \to (...(t \to x)...))$. Hence $x \in \langle F, a\rangle$. Hence $M \subseteq \langle F, a\rangle$ and complete the proof. □

**Theorem 3.8.** *Let $\mathcal{H}$ be a hyper $EQ$-algebra and $F$ be an $S_\sim reflexive$ positive implicative (pre)filter of $H$. Then $\langle F, a\rangle = \{x \in H | a \to x \subseteq F\}$ ( We denote $\{x \in H | a \to x \subseteq F\}$ by $F_a$).*

*Proof.* Let $F$ be a positive implicative (pre)filter. Then by $1 \in a \to 1$, we have $a \to 1 \cap F \neq \emptyset$ or $a \to 1 \subseteq F$, i.e. $1 \in F_a$. Now, if $x \to y \subseteq F_a$ and $x \in F_a$, for some $x, y \in H$, then $a \to (x \to y) \subseteq F$ and $a \to x \subseteq F$ and so by assumption $a \to y \subseteq F$ or $y \in F_a$. By $1 \in a \to a$ we can get $a \in F_a$. Since $t \ll a \to t$, for all $t \in F$, we obtain $t \in F_a$. Thus $F_a$ is a deductive system contains $F$ and $a$. If $B$ is a deductive system such that $F \cup \{a\} \subseteq B$ and $u \in F_a$, then $a \to u \subseteq F \subseteq B$. Hence $u \in B$ or $F_a \subseteq B$. Therefore, $F_a$ is a least deductive system contains $F$ and $a$, i.e. $\langle F, a\rangle = \{x \in H | a \to x \subseteq F\}$. □

**Corollary 3.2.** *Let $F$ be a (pre)filter such that $F_a = \{x \in H | a \to x \subseteq F\}$ be a deductive system, for all $a \in H$. Then $F$ is a positive implicative (pre)filter.*

*Proof.* Let $z \to (y \to x) \subseteq F$ and $z \to y \subseteq F$, for some $x, y, z \in H$. Then $y \in F_z$ and $z \to t \subseteq F$, for all $t \in (y \to x)$. Thus $t \in F_z$, for all $t \in (y \to x)$ and so $y \to x \subseteq F_z$. Since $F_z$ is a deductive system, we get $x \in F_z$ or $z \to x \subseteq F$. Therefore, $F$ is a positive implicative (pre)filter. □

**Example 3.6.** *In Example 3.1(iii), if $F = \{b, 1\}$, then $F$ is a (pre)filter. We have:*
*$F_0 = \{x \in H | 0 \to x \subseteq F\} = H$, which is a deductive system;*
*$F_a = \{x \in H | a \to x \subseteq F\} = \emptyset$, which is a deductive system;*
*$F_b = \{x \in H | b \to x \subseteq F\} = \{b, 1\}$, which is a deductive system;*
*$F_1 = \{x \in H | 1 \to x \subseteq F\} = \{b, 1\}$, which is a deductive system.*
*Therefore, $F$ is a positive implicative (pre)filter.*

**Example 3.7.** *In Example 3.1(iii), $F = \{a, b, 1\}$ is a positive implicative filter and $F_0 = \{0\}$ is not deductive system.*

**Corollary 3.3.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition and $\neg\neg x = x$, for all $x \in H$ and $F$ be an $S_{\sim}reflexive$ (pre)filter of $H$. Then $F$ is an implicative (pre)filter if and only if $F$ is a positive implicative (pre)filter.*

*Proof.* If $F$ is an implicative (pre)filter, then by Theorem 3.3, $F$ is a positive implicative (pre)filter. Now, let $F$ be a positive implicative (pre)filter and $x \to (\neg y \to y) \subseteq F$. By Propositions 3.3 and 2.1(viii), $x \to (\neg y \to y) \ll \neg y \to (x \to y) \ll \neg y \to (\neg y \to \neg x)$ and so we can have $\neg y \to \neg x \subseteq F$. Again by Proposition 3.3 and assumption we have $\neg y \to \neg x \ll x \to y$. Hence we can get $x \to y \subseteq F$. Therefore $F$ is an implicative (pre)filter. □

## 4. Fantastic filters

In this section we introduce concept fantastic (pre)filter and investigate some results about them. Then we study the relation between fantastic (pre)filter and (positive)implicative (pre)filters. In the end we show that we can have hyper $\ell EQ$-algebra by fantastic filter.

**Definition 4.1.** *Let $\mathcal{H}$ be a hyper EQ-algebra and $F$ be a subset of $H$ such that $1 \in F$. Then $F$ is called*
$(i)$ fantastic prefilter *of $\mathcal{H}$, if $z \to (y \to x) \subseteq F$ and $z \in F$, then $((x \to y) \to y) \to x \subseteq F$, for all $x, y, z \in H$ and $x \otimes y \subseteq F$, for all $x, y \in F$.*
$(ii)$ fantastic filter *of $\mathcal{H}$, if $F$ is a fantastic prefilter and $x \to y \subseteq F$, imply $(x \otimes z) \to (y \otimes z) \subseteq F$, for all $x, y, z \in H$.*

**Example 4.1.** *(i) Let $\mathcal{H}$ be a separated hyper EQ-algebra such that $x \to y$ and $x \otimes y$ be singleton. Then by [[6], Lemma 15], the concept of fantastic (pre)filter in hyper EQ-algebras and EQ-algebra are coincide.*
*(ii): In Example 3.1(ii), $F = \{a, b, 1\}$ is fantastic (pre)filter and in Example 3.4, $F = \{a, b, 1\}$ is a fantastic prefilter.*
*(iii): In Example 3.1(iii), since for $b \in F$ and $b \to (0 \to 0) \subseteq F$ we have $((0 \to 0) \to 0) \to 0 \nsubseteq F$, then $F = \{1, b\}$ is not a fantastic prefilter.*

**Proposition 4.1.** *Let $\mathcal{H}$ be a good hyper EQ-algebra. Then each fantastic (pre)filter is a (pre)filter.*

*Proof.* Let $F$ be a fantastic filter, $x \to y \subseteq F$ and $x \in F$, for $x, y \in H$. Since $x \to (1 \to y) \subseteq F$ and $x \in F$, we have $((y \to 1) \to 1) \to y \subseteq F$. By goodness $y \in (y \to 1) \to 1) \to y$. Hence $y \in F$ and so $F$ is a filter. □

**Proposition 4.2.** *Let $\mathcal{H}$ be a hyper EQ-algebra and satisfies in the residuated condition. If $F$ is an $S_\sim reflexive$ fantastic (pre)filter of $\mathcal{H}$, then every (pre)filter $G$ containing $F$ is a fantastic (pre)filter.*

*Proof.* Let $y \to x \subseteq G$, for $x, y \in H$. Then by Proposition 3.3, we have, $(y \to x) \to (y \to x) \ll y \to ((y \to x) \to x)$. Since $1 \in (y \to x) \to (y \to x)$ we get, $y \to ((y \to x) \to x) \cap F \neq \emptyset$ or $y \to ((y \to x) \to x) \subseteq F$. Set $A = (y \to x) \to x$. Thus $y \to A \subseteq F$ and so $((A \to y) \to y) \to A \subseteq F$. By $(EP)$ condition:

$$\begin{aligned}((A \to y) \to y) \to A &= ((A \to y) \to y) \to ((y \to x) \to x)\\ &\ll (y \to x) \to (((A \to y) \to y) \to x)\end{aligned}$$

Thus we can prove $(y \to x) \to (((A \to y) \to y) \to x) \subseteq F \subseteq G$. Since $y \to x \subseteq G$ and $G$ is a (pre)filter, $(((A \to y) \to y) \to x) \subseteq G$. On the other hand by Proposition 3.3:

$$\begin{aligned}1 \in (y \to x) \to 1 &\subseteq (y \to x) \to (x \to x) \ll x \to ((y \to x) \to x)\\ &\ll (((y \to x) \to x) \to y) \to (x \to y)\\ &\ll ((x \to y) \to y) \to ((((y \to x) \to x) \to y) \to y)\\ &\ll (((((y \to x) \to x) \to y) \to y) \to x) \to (((x \to y) \to y) \to x)\\ &= ((A \to y) \to y) \to x) \to ((x \to y) \to y) \to x) = B\end{aligned}$$

Thus $B \cap F \neq \emptyset$ or $B \subseteq F \subseteq G$. Since $(((A \to y) \to y) \to x \subseteq G$, we get $(((x \to y) \to y) \to x) \subseteq G$. Therefore, $G$ is a fantastic (pre)filter. □

**Theorem 4.1.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition and $\{1\}$ be an $S_\sim reflexive$ fantastic (pre)filter of $H$. Then any filter is a fantastic (pre)filter.*

*Proof.* By Proposition 4.2, the proof is clear. □

**Proposition 4.3.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition. If $F$ is an $S_\sim reflexive$ implicative (pre)filter of $H$, then $F$ is a fantastic (pre)filter.*

*Proof.* Let $y \to x \subseteq F$, for $x, y \in H$. Then by Proposition 2.1(x), $x \ll ((x \to y) \to y) \to x$. Now, by Remark 3.1, $(((x \to y) \to y) \to x) \to y \ll x \to y$ and so again by Remark 3.1, $(x \to y) \to (((x \to y) \to y) \to x) \ll ((((x \to y) \to y) \to x) \to y) \to$

$(((x \to y) \to y) \to x)$. Thus by Propositions 2.1(xii), 3.3 and $(EP)$ condition, we have:

$$\begin{aligned} y \to x &\ll ((x \to y) \to y) \to ((x \to y) \to x) \\ &\ll (x \to y) \to (((x \to y) \to y) \to x) \\ &\ll ((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \end{aligned}$$

Since $y \to x \subseteq F$, then we can get

$$[((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)] \subseteq F$$

Since $F$ is a implicative (pre)filter, then $((x \to y) \to y) \to x \subseteq F$. □

In the next example we can see that the converse of the above proposition is not true.

**Example 4.2.** *In Example 3.1(iii), $F = \{1, a, b\}$ is a fantastic (pre)filter but it is not an implicative (pre)filter.*

**Theorem 4.2.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition and $F$ be an $S_\sim reflexive$ filter of $H$. Then $F$ is an implicative filter if and only if $F$ is both positive implicative filter and fantastic filter.*

*Proof.* Let $F$ be an implicative filter. Then by Theorem 3.3 and Proposition 4.3, $F$ is both positive implicative and fantastic filter.
Conversely, let $(x \to y) \to x \subseteq F$, for $x, y \in H$. By Proposition 3.3, $(x \to y) \to x \ll (x \to y) \to ((x \to y) \to y)$. Since $F$ is a positive implicative filter we have, $(x \to y) \to y \cap F \neq \emptyset$ or $(x \to y) \to y \subseteq F$. On the other hand by Proposition 2.1(x) and Remark 3.1, $(x \to y) \to x \ll y \to x$, i.e. we can get $y \to x \subseteq F$ and so $((x \to y) \to y) \to x \subseteq F$. Now, since $F$ is a filter and we obtain $x \in F$. Hence $F$ is an implicative filter. □

**Corollary 4.1.** *Let $F$ be an $S_\sim reflexive$ fantastic filter of $H$. Then $F$ is a positive implicative filter if and only if $F$ is an implicative filter.*

*Proof.* By Theorem 4.2 and Theorem 3.3, the proof is clear. □

**Corollary 4.2.** *Let $F$ be an $S_\sim reflexive$ positive implicative filter of $H$. Then $F$ is an implicative filter if and only if $F$ is a fantastic filter.*

*Proof.* By Theorem 4.2, Proposition 4.3, the proof is clear. □

**Remark 4.1.** *We note that if $\neg\neg x = x$, for all $x \in H$, then $\neg x$ is singleton. Let $u, v \in \neg x$. Then $u \to 0 \subseteq (x \to 0) \to 0 = x$. Thus $u \to 0 = x$. Since $v \in x \to 0 = (u \to 0) \to 0 = u$, we get $u = v$, i.e. $\neg x$ is singleton.*

**Theorem 4.3.** *Let $\mathcal{H}$ be a good hyper EQ-algebra contains bottom element "0" such that $\neg\neg x$ be singleton and $F$ be an $S_\sim reflexive$ fantastic filter of $H$. Then $\frac{\mathcal{H}}{\equiv_F}$ is a good hyper EQ-algebra such that $\neg\neg[x] = [x]$, for all $x \in H$.*

*Proof.* By Theorem 2.2, $\frac{\mathcal{H}}{\equiv_F}$ is a good hyper $EQ$-algebra. Since $1 \in 0 \to x$, for all $x \in H$, then $(0 \to x) \cap F \neq \emptyset$ or $0 \to x \subseteq F$. Thus by Definition 4.1, $((x \to 0) \to 0) \to x \cap F \neq \emptyset$ or $\neg\neg x \to x \subseteq F$ and so $[\neg\neg x] \ll [x]$. By Lemma 3.1(viii), $[x] \ll \neg\neg[x]$ and so by Remark 4.1, $\neg\neg[x] = [x]$. □

**Definition 4.2.** *Let $\mathcal{H}$ be a hyper EQ-algebra. Then $\mathcal{H}$ is called a* lattice hyper EQ-algebra( hyper $\ell$EQ-algebra ) *if, it is a lattice and the following substitution axiom holds, for all $x, y, z, t \in H$.*

$$((x \vee y) \sim z) \otimes (t \sim x) \ll (z \sim (t \vee y))$$

**Example 4.3.** $(i)$ *In Example 3.1(iii), it is easy to see that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper $\ell EQ$-algebra.*
$(ii)$ *Let $H = \{0, a, b, c, 1\}$, which $0 < a < b < 1$, $0 < a < c < 1$ such that $b, c$ be incomparable and $\wedge, \otimes$ and $\sim$ are defined on $H$ as follows:*

| $\wedge$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | a | a | a |
| b | 0 | a | b | a | b |
| c | 0 | a | a | c | c |
| 1 | 0 | a | b | c | 1 |

| $\otimes$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | {0} | {0} | {0} | {0} | {0} |
| a | {0} | {0} | {0} | {0} | {0, a} |
| b | {0} | {0} | {0} | {0} | {0, b} |
| c | {0} | {0} | {0} | {0} | {0, c} |
| 1 | {0} | {0, a} | {0, b} | {0, c} | {1} |

| $\sim$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | {1} | {a, b, c} | {a, b, c} | {a, b, c} | { 0} |
| a | {a, b, c} | {1} | {a, b, c} | {a, b, c} | {a} |
| b | {a, b, c} | {a, b, c} | {1} | {b, c} | {b} |
| c | {a, b, c} | {a, b, c} | {b, c} | {1} | {c} |
| 1 | {0} | {a} | {b} | {c} | {1} |

*Then $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra which is not $\ell EQ$-algebra.*

**Theorem 4.4.** *Let $\mathcal{H}$ be a good hyper EQ-algebra contains bottom element "0" such that $\neg\neg x$ be singleton and $F$ be an $S_{\sim}reflexive$ fantastic filter of $H$. Then $\frac{\mathcal{H}}{\equiv_F}$ a lattice good hyper EQ-algebra.*

*Proof.* Let $[x] \vee [y] = \neg(\neg[x] \wedge \neg[y])$. Since $\neg[x] \wedge \neg[y] \leq \neg[x], \neg[y]$ (by Remark 4.1 and Theorem 4.3, $\neg[x]$ is singleton). Then $[x], [y] \leq \neg(\neg[x] \wedge \neg[y]) = [x] \vee [y]$. Let $[x], [y] \leq [c]$, for some $[c] \in \frac{\mathcal{H}}{\equiv_F}$. Then $\neg[c] \leq \neg[x] \wedge \neg[y]$ or $[x] \vee [y] = \neg(\neg[x] \wedge \neg[y]) \leq [c]$. Hence $[x] \vee [y]$ is the supremum of $[x]$ and $[y]$. We can prove that $\frac{\mathcal{H}}{\equiv_F}$ is a lattice by some modification. □

**Corollary 4.3.** *Let $\mathcal{H}$ be a good hyper EQ-algebra contains bottom element "0" such that $\neg\neg x$ be singleton and $F$ be an $S_{\sim}reflexive$ fantastic filter of $H$.Then $\frac{\mathcal{H}}{\equiv_F}$ is an $\ell EQ$-algebra.*

*Proof.* By Theorem 4.4, $\frac{\mathcal{H}}{\equiv_F}$ is a lattice. By Proposition 2.1(v), (xiii), Theorem 4.3 and $(HEQ3)$, we have:

$$\begin{aligned}(([x] \vee [y]) \sim [z]) \otimes ([t] \sim [x]) &\ll (([x] \vee [y]) \sim [z]) \otimes (\neg[t] \sim \neg[x]) \\ &= (\neg(\neg[x] \wedge \neg[y]) \sim [z]) \otimes (\neg[t] \sim \neg[x]) \\ &\ll \neg(\neg(\neg[x] \wedge \neg[y])) \sim \neg[z]) \otimes (\neg[t] \sim \neg[x]) \\ &= ((\neg[x] \wedge \neg[y]) \sim \neg[z]) \otimes (\neg[t] \sim \neg[x]) \\ &\ll \neg[z] \sim (\neg[t] \wedge \neg[y]) \\ &\ll \neg\neg[z] \sim \neg(\neg[t] \wedge \neg[y]) = [z] \sim ([t] \vee [y])\end{aligned}$$

Therefore, $\frac{\mathcal{H}}{\equiv_F}$ is a hyper $\ell EQ$-algebra. □

**Corollary 4.4.** *Let $\mathcal{H}$ be a good hyper EQ-algebra and satisfies in the residuated condition, contains bottom element "0" such that $\neg\neg x$ is singleton and F be an $S_{\sim}reflexive$ fantastic filter of $H$. Then in $\frac{\mathcal{H}}{\equiv_F}$ implicative filters and positive implicative filters are equivalent.*

*Proof.* By Theorem 4.3 and Corollary 3.3, the proof is clear. □

## 5. Conclusions

In this paper, we verify some results in good hyper $EQ$-algebra. Then we introduce different kinds of (pre)filters and gain some results about them. Also we study relations between these (pre)filters. In each kind of filter we obtain conditions that can help us to have a quotient structure. Definition of new filters and study about them and their quotient algebra will be my next task.

## References


[1] R. A. Borzooei, B. Ganji Saffar, R. Ameri, On Hyper $EQ$-algebras, *Italian Journal of Pure and Applied Mathematics* **13** (2013), 13–28.

[2] M. El-Zekey, Representable good $EQ$-algebra, *Soft Computing* **14** (2010), 1011–1023.

[3] F. Marty, Sur une generalization de la notion de groups, *8th congress math. Scandinaves*, Stockholm (1934), 45–49.

[4] V. Novák, $EQ$-algebra, Primary concepts and properties, *Int. Joint Czech Republic-Japan and Taiwan-Japan Symposium*, Kitakyushu, Japan, August (2006), 219–223.

[5] V. Novák, B. De Baets, $EQ$-algebras, *Fuzzy Sets and Systems* **160** (2009), 2956–2978.

[6] M. EL-Zekey, V. Novák, R. Mesiar, On good $EQ$-algebras, *Fuzzy Sets and Systems* **178** (2011), 1–23.

[7] X.L. Xin, P.F. He, Y.W. Yang, Characterizations of some fuzzy prefilter(filters) in $EQ$-algebra, *The Scientific World Journal* **2014** (2014), 1–12.



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