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New approach to the calculation of fractal dimension of the lungs

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Abstract. In the present work, using a new approach, we will calculate the fractal dimension of the lungs, through a precise technique of mathematical simulation. Note that the main objective of this study is to contribute to further confirm the idea of the fractal structure of the lungs; structure permitted by the phenomenon of self-similarity.

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1. Introduction

The lungs are essential organs of respiration. Located in the rib cage, they provide the necessary gas exchange between air and blood. Man has two lungs, a right lung comprises three lobes and the left lung has two lobes and a location on its inner face which corresponds to the place of the heart. The lungs are enveloped by the pleura, thin membrane which prevents the friction against the chest wall. The pleura is constituted by two sheets separated from one another by a small amount of serous fluid that allows the lungs to move during breathing. Below the lungs, there is the diaphragm, which separates the rib cage and the abdomen which is a very important muscle in breathing.

The air that reaches the lungs passes through the pharynx, larynx and trachea extending into the lungs by two bronchi, one right and one left, which in turn are divided into bronchi increasingly small, like branches of a tree.

The smallest bronchi, namely the bronchioles lead to tiny bags that are filled with air, the pulmonary alveoli. The lungs of an adult are provided with an average of 600 millions alveoli, which represent a gas exchange surface of about 140 m^2 [12].

Breathing helps both to bring the blood oxygen which live cells and ridding the blood of carbon dioxide contained therein.

On the inspiration (see Figure 1), the diaphragm contracts and descends, the chest expands, the outside air, full of oxygen enters the lungs and fills the cells (flow). It is through tiny vessels that form the capillary network that will make exchanges between alveoli and blood: the oxygen of the air contained in the alveoli passes into the blood to reach the red blood cells and heart which sends it throughout the body.

Upon expiration, the diaphragm rises, thorax decreases in volume and air, charged with carbon dioxide is expelled to the trachea (ebb). In fact, the carbon dioxide of

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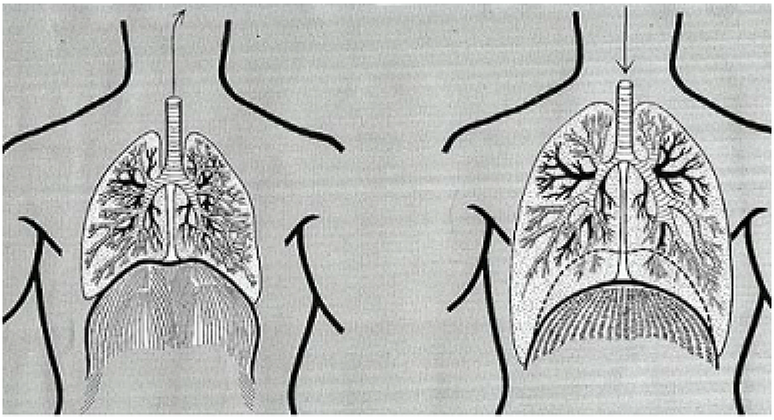

FIGURE 1. Variations in lung volume during inspiration-expiration.

blood takes the opposite route passing from the blood to the alveoli and thence to the outside air.

A particularity in the lungs; the oxygen-rich blood sent from the heart to all the tissues of the body passes through the arteries (the red blood), and the blood responsible for carbon dioxide returns to the heart through the veins (the blue blood). In the lungs it is the opposite: the carbon dioxide-rich blood goes from the heart to the lungs through the pulmonary artery, the oxygen-rich blood from the lungs goes to the heart through the pulmonary vein.

Lung actually allows gas exchange by the combination of several factors such as tissue elasticity, fluid mechanics, the diffusion through the membranes, etc. The complexity of these phenomena entails the coordination of sometimes contradictory effects in appearance [14]. We don't necessarily know all the experimentally measured quantities desired, and we cannot repeat the experiments to infinity to refute or confirm certain assumptions. Hence it would be very wise to model [5]. It is therefore obvious that mathematical modeling of these phenomena would be very useful for an accurate approach. This is exactly the great interest of fractal geometry. Indeed, the Euclidean geometry is powerless in solving such problems; it is applicable only in the case of smooth and regular shapes. Thus, a point has a dimension equal to zero, a line has a dimension one, a plane has a dimension two and a volume has a dimension equal to three. In contrast, fractal geometry, meanwhile, deals with non-integer dimensions [1], [19], for example ranging between one and two and between two and three, etc. The fractal dimension is actually the size of irregular curves [7], [8]. And it is this specificity that opens huge horizons in vast areas and in the medical field in particular.

What means the word "fractal"? The fractals are mathematical objects sufficiently broken and irregular. Another property of the term "fractal" is self-similarity: the fractal objects have the same information at different scales, i.e. similar and identical reproductions at scales smaller and smaller.

The main tool in the study of fractals is the notion of fractal dimension, in its many forms. Given the diversity and complexity of these fractals, there are several definitions of fractal dimensions and which do not always coincide. Among others, the Hausdorff dimension, the box dimension and the dimension of self-similar sets. They are used mainly to measure "the degree of irregularity" or "filling rate" for example, in the plane of a fractal curve. It will stick in the present paper to the self-similarity dimension.

The pulmonary anatomy has a branching tree structure. This tree, shown in Figure 2, has a self-similar phenomenon in large structures like small. The small sizes of trees are similar to those of the larger size. Therefore, the lung has a fractal structure, allowing optimization (compromise) and thus a mathematical modeling of the lungs and the respiratory function [15], through fractal analysis [11].

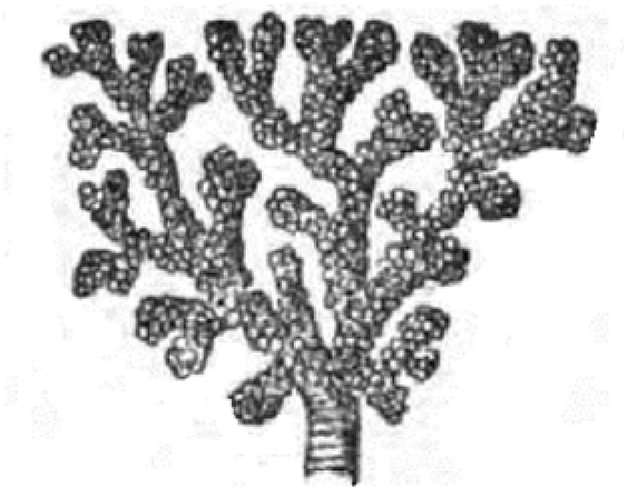

FIGURE 2. The fractal appearance of the bronchial ramifications.

The objective of this work is to determine a modeling technique in order to estimate the fractal dimension of the human lungs. So, after a brief overview on the pulmonary anatomy and respiratory physiology in the first section, we will attempt in Section 2 to show and highlight the fractal structure of the lungs. Thereafter and in Section 3 we will, by a new approach, calculate the fractal dimension of the lungs using a modeling technique based on one of the variants of the Von Koch curve. We will close this paper with the conclusions and discussions.

2. Fractal dimension and fractal structure of lungs

2.1. Fractal dimension. To evaluate the fractal dimension, several methods are proposed in the literature. We will use in this work the self-similarity dimension. This method applies to fractal curves and figures with the property of self-similarity, such that the various component parts are constructed by successive iterations with a same reduction factor q. According to Sapoval [18], the self-similarity dimension corresponds to the logarithm of the number of pieces needed to cover the object, relative to the logarithm of the report of enlargement by aligning the pieces with the initial object.

More generally, according to Gouyet [4] and Falconer [2], this fractal dimension D_F is given by the following relation:

$$D_F = -\frac{\log N}{\log q}, \tag{1}$$

where N is the decomposition factor and q is the single reduction factor from one stage to another (which can be also written as $D_A = \frac{\log N}{\log \frac{1}{q}}$, where $1/q$ is the report of enlargement).

Note that the position of the elements in the set is not involved. Only count their numbers and their relative sizes. Different shapes may have the same dimension.

2.2. Fractal structure of lungs. The role of the bronchial tree is driving the ambient air, rich in oxygen and low in carbon dioxide, to the exchange surface with the blood within the acini. Figure 3 shows the molding of a human lung (according to Weibel [22]). In yellow, there are the airways, in red, the pulmonary veins and in blue, pulmonary arteries. The complexity of the structure is striking. One can observe the tree geometry of the lung, and specifically, that this tree is almost dichotomous. This observation allows us to consider it as a succession of generations, see Figure 4 (according to Weibel [22]). The first generation is the largest branch, the trachea. It has a diameter of about two centimeters. The latter is located at the bottom of the acinus, at the twenty-third generation; it has a diameter on the order of half a millimeter. The number of branches of this tree is roughly 2^{24}, i.e. more than sixteen millions. The bronchi and bronchioles, until the $17th$ generation

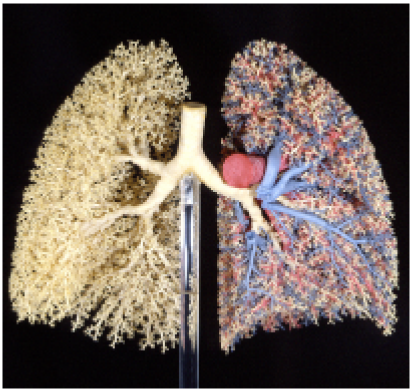

FIGURE 3. Molding of human lung.

of the tree, are structures whose the only role is to ensure the air conduction to the latest generations. Rather asymmetrical at first, especially because of the presence of the heart on the left, dichotomous bifurcations become quickly practically homothetic from one generation to the next [21].

The lower ducts, from the seventeenth generation, correspond to the breathing zone of lung, where gas exchange with blood held. Over there, we find neither cartilage nor smooth muscle. They are grouped into acini in number approximately 30000 by lung, dichotomous sub-trees, to six generations which the alveolar bags cling. These

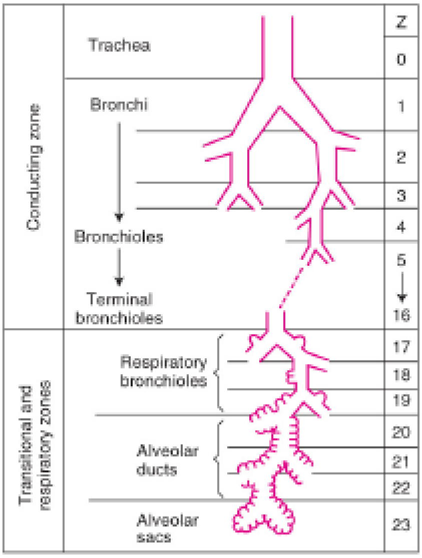

FIGURE 4. The different functional zones of lung airways.

bags are like clusters of alveoli and form the last generation of the lung. In fact, the channel diameter is constant in this region [21].

In the human adult, the gas exchange surface is of 140 square meters, which is the equivalent of half a tennis court. The area occupied by the lungs is small compared to the surface which allows gas exchange. The bronchial tree; which is a fractal structure, allows to increase the gas exchange area in a very significant way. Although the surface of the lungs involved in gas exchange is not infinite, it is considered the lungs and their self-similarity as a natural example of fractal. It enables major gas exchange on a yet reduced lung volume. This impressive gain of surface and space is proof of the interest of a fractal organization adopted by nature. The major advantage of these mathematical structures is that they allow a computer modeling of the lungs, and therefore a real quantification of the exchange zone [14].

3. Numerical simulation and results

3.1. Modeling technique. Lung bronchi are hollow tubes that branch like the branches of a tree, to distribute air evenly to both lungs. The trachea which leads air to the lungs falls within the thorax to be divided into two main bronchi, one for each lung. By phenomenon of self-similarity, bronchi then divide about 23 times (see Figure 4) to get the air to the alveoli [13]. The lung is a real fractal structure. We can mathematically model this fractal structure of the lung, using one of the variants of the Von Koch curve [2], [10]. The construction of this curve is based on the basic principle illustrated in Figure 5. Let $[AB]$ be a line segment of unit length. The transformation consists in removing the segment $[CD]$ and replacing it by the other tow sides of the isosceles triangle based on the removed segment (see Figure 5). Obtaining the points C, E, D from the points A and B is based on the following

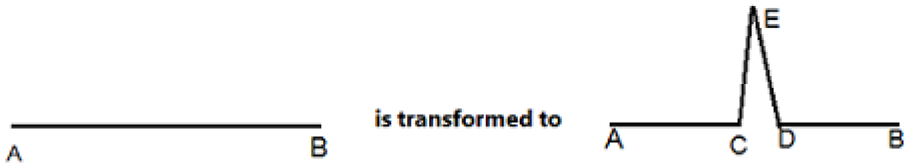

FIGURE 5. Principal and technique of modeling.

formulas:

$$AC = \frac{5}{11}AB \quad \text{and} \quad AD = \frac{6}{5}AC, \tag{2}$$

such that $AC = CE = ED = DB$ and that $(CD, CE) = \alpha$.
Express the two equalities of equation (2) in cartesian coordinates :

$$\begin{aligned} x_C &= \frac{5}{11}(x_B - x_A) + x_A \quad \text{and} \quad y_C = \frac{5}{11}(y_B - y_A) + y_A, \\ x_D &= \frac{6}{5}(x_C - x_A) + x_A \quad \text{and} \quad y_D = \frac{6}{5}(y_C - y_A) + y_A. \end{aligned}$$

Now, to be able to trace this fractal curve, we need to determine the coordinates of the point E. For this, we express CE in terms of CD. In fact, we have

$$CD = AD - AC = \frac{1}{5}AC.$$

However, in length $AC = CE$, which allows to assert that $CE = 5CD$. It follows that CE is the image of $5CD$ by rotation $\Re(C, \alpha)$. Therefore, the coordinates of point E are:

$$\begin{aligned} x_E &= 5\left[(x_D - x_C)\cos(\alpha) - (y_D - y_C)\sin(\alpha)\right] + x_C \\ y_E &= 5\left[(x_D - x_C)\sin(\alpha) + (y_D - y_C)\cos(\alpha)\right] + y_C. \end{aligned}$$

In the other hand, it is possible to give an approximate value of α.
Indeed $CD = \frac{1}{11}$ then $\frac{1}{2}CD = \frac{1}{22}$. So, we can apply trigonometric formula to get:

$$\cos(\alpha) = \frac{1}{2}\frac{CD}{CE} = \frac{1}{10}.$$

Then, an approximate value in radians of α is $\alpha \approx 1,47 rad$.
Thus, it is possible to carry out the iterations of this fractal representing the structure of the lung (see Figure 6).

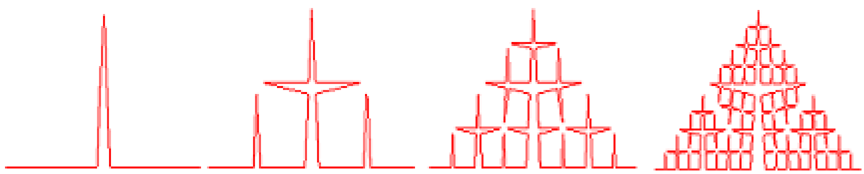

FIGURE 6. Modeling curve: construction steps.

The fourth iteration corresponds to the function of the air sacs composed of lung alveoli for gas exchange.

3.2. Results. The main artery divides into two and then the following arteries divide into two as well. This process is repeated 23 times. In the end, we get about 2^{23} separations of arteries, i.e. 8388608 arterioles, namely the equivalent of sixteen millions of branches.

Alveoli have a diameter $L_{al} = 0.2\ mm$. For the calculations, we use a cube of side $I = L_{al}$ to represent alveoli. On this cube, five faces will be allocated to the exchange surface while the last face cannot be used because it is necessary that the air enters the alveoli. Thus, we can estimate an exchange surface S_{al} by alveoli as

$$S_{al} = 5 \times (0.2)^2 = 2 \times 10^{-3}\ cm^2. \tag{3}$$

From an anatomical point of view, there are 2^{16} acini in lungs, i.e. 65536 acini. Knowing too that the total exchange surface of the lungs is about $140m^2$ [12]. Then, the exchange surface S_{ac} for each acini can be calculated as

$$S_{ac} = \frac{140 \times 10^4}{65536} = 21,36\ cm^2. \tag{4}$$

On the other hand, the pulmonary volume of a man is about 5 liters. It follows that the volume of each acini is

$$V_{ac} = \frac{5 \times 10^3}{65536} = 0,076\ cm^3. \tag{5}$$

Hence, from equation (5), the size L_{ac} of each acini is approximately equal to:

$$L_{ac} = \sqrt[3]{0,076} = 4,23\ mm. \tag{6}$$

To calculate the fractal dimension D_F of the lungs, we proceed as follows:
For each acini, stacking the alveoli on a fractal of dimension D_F, to get a total exchange surface S_{ac}. So, the number N of small cubes stacked on each acini is given by the relation

$$N = \frac{S_{ac}}{S_{al}}. \tag{7}$$

Replacing the values obtained from equations (3) and (4) in equation (7), we deduce that

$$N = 10680. \tag{8}$$

On the other side, from equation (6), the reduction factor q is expressed as

$$q = \frac{L_{al}}{L_{ac}} = \frac{0,2}{4,23}. \tag{9}$$

Then, the factor q is approximately equal to $0,04$. The fractal dimension D_F is expressed as

$$D_F = -\frac{\log N}{\log q}.$$

Consequently, the value of D_F can be calculated as follows:

$$D_F = -\frac{\log(10680)}{\log(0,04)}. \tag{10}$$

The fractal dimension of the lungs is therefore

$$D_F = 2,88. \tag{11}$$

4. Conclusions and discussions

According to the calculations and mathematical simulation used in this work, we arrive at a value of the fractal dimension of lungs of 2.88. This value is indeed a non-integer dimension (non-Euclidean) and therefore confirms the fractal structure of the lungs. The literature cited others values of fractal dimension of lungs, different from ours, but always non-integer.

In [16], Nelson and Manchester found that the fractal dimension D_F of the lungs varies between 2.64 and 2.76, this using the airway lengths as the measuring stick. As for Nelson, West and Goldenberg [17], from experimental data, leaded to values slightly lower. Indeed, they found $D_F = 2.4$, based on power scaling of the airways' length and $D_F = 2.26$ when the basis was the airways'diameter. Afterwards, Weibel [20] achieved a value $D_F = 2.35$, based on scaling of the average airways'diameter.

This heterogeneity of the results is directly related to the different experimental methods used and the choice of mathematical modeling type. But nevertheless, all these values share non-Euclidean property and confirm all "fractality" of the pulmonary system. Note that this fractal property of the lungs occurs very early, even when the formation of the lungs during the fetal stage. This fractal structure gives the lungs very advantageous properties, all working for a fundamental objective: an area of maximum gas exchange in a very small volume.

If have used the geometry of a sphere (Euclidean geometry), for example, to increase the surface, it would increase the radius. However, the lungs are in a closed and limited environment (rib cage). Counting the gas exchange surface of a human, there are approximately 140 square meters. Note that for the case of a sphere, the radius of the lung should measure a gigantic value of 3.3 meters !

The surface of lungs is tiny relative to the gas exchange surface as possible. Consequently, fractals allow living beings and for man in particular to have a very efficient respiratory system, while having a realistic volume.

It would be ideal and very wise to find a unified value of fractal dimension of lungs, which would do a "biological constant." Such an outcome would be very useful in the medical field and especially in the pulmonary diseases include among others asthma, emphysema, respiratory failure, amputation of lung lobes (case of cancers), ... Moreover, such a value would have a major impact in the sporting field, including monitoring and evaluation of performance of athletes high levels.

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