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Cubic differential systems with an invariant straight line of maximal multiplicity

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Abstract. In this work the estimation $3n-2 \leq M_a(n) \leq 3n-1$ of maximal algebraic multiplicity $M_a(n)$ of an invariant straight line is obtained for two-dimensional polynomial differential systems of degree $n \geq 2$. In the class of cubic systems ($n=3$) we have $M_a(3)=7$. Moreover, we prove that if an affine real invariant straight line has multiplicity equal to 1 (respectively, 2,3,4,5,6,7), then the maximal multiplicity of the line at infinity is 7 (respectively, 5,5,5,4,1,1). Each of these cubic systems has a single affine invariant straight line, is Darboux integrable and their normal forms are given.

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1. Introduction

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \quad \gcd(P,Q) = 1 \tag{1}$$

and the vector field $\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ associated to system (1).

Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n=3$ then system (1) is called cubic.

A curve $f(x,y)=0$, $f \in \mathbb{C}[x,y]$ (the function $f(x,y) = exp(\frac{g}{h})$, $g,h \in \mathbb{C}[x,y]$) is said to be an *invariant algebraic curve* (*invariant exponential function*) of (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, $deg(K_f) \leq n-1$ such that the identity $\mathbb{X}(f) \equiv f(x,y)K_f(x,y)$ holds. We say that an invariant algebraic curve $f(x,y)=0$ has the *parallel multiplicity* equal to m, if m is the greatest positive integer such that f^{m-1} divides K_f. If $f(x,y)=0$ has the parallel multiplicity equal to $m \geq 2$, then $exp(1/f), ..., exp(1/f^{m-1})$ are invariant exponential functions.

The system (1) is called *Darboux integrable* if there exists a non-constant function of the form $F = f_1^{\lambda_1} \cdots f_s^{\lambda_s}$, where f_j is an invariant algebraic curve or an invariant exponential function and $\lambda_j \in \mathbb{C}$, $j = \overline{1,s}$, such that either F is a first integral or F is an integrating factor for (1).

At present, a great number of works are dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of the estimation of the number of invariant straight lines which can have a polynomial differential system was considered in [1]; the problem of coexistence of the invariant straight lines and limit cycles in {[15] : $n=2$}, {[9], $n=3$}, [8]; the problem of coexistence of the invariant straight lines and singular points of a center type for cubic system in [7], [16]. An interesting relation between the number of

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invariant straight lines and the possible number of directions for them is established in [2].

The classification of all cubic systems with the maximum number of invariant straight lines, including the line at infinity, and taking into account their multiplicities, is given in [10].

It was proved in [1] that the cubic system (1) can have at most eight affine invariant straight lines. The cubic systems with exactly eight and exactly seven distinct affine invariant straight lines have been studied in [10], [11], with invariant straight lines having total parallel multiplicity seven in [17], and with six real invariant straight lines along two (three) directions in [13], [14]. The cubic systems with invariant straight lines (including the line at infinity) of total multiplicity eight and with two (respectively, three, four) distinct infinite singularities were investigated in [3] (respectively, [4], [5]).

In this paper the cubic systems with at most two invariant straight lines, including the line at infinity, of maximal multiplicity are investigated.

The work is organized as follows: In Section 2 the following estimation $3n-2 \leq M_a(n) \leq 3n-1$ of the maximal algebraic multiplicity $M_a(n)$ of an affine invariant straight line is given in the class of polynomial differential systems of degree $n \geq 2$. We formulate the conjecture, that $M_a(n) = 3n-2,\ n \geq 2$. For quadratic ($n=2$), cubic ($n=3$) and quartic ($n=4$) systems the conjecture is true (Theorem 2.1, Conjecture 2.1). The coefficient conditions when the cubic system has an affine real invariant straight line l of algebraic multiplicity $m_a(l) \geq k,\ k=2,\ldots,6$ are given in Section 3. Also, in this section we show that in the class of all cubic differential systems with non-degenerate infinity the maximal algebraic multiplicity of an affine invariant straight line is equal to seven. If the cubic system possesses an affine invariant straight line of multiplicity seven, then the multiplicity of the line at infinity is equal to one (Lemmas 3.1-3.5, Theorem 3.6). In Section 4 the same result on maximal infinitesimal, integrable and geometric multiplicity of an invariant straight line is obtained. In Section 5 we show that the multiplicity of the line at infinity is not greater than seven. If the cubic system has the line at infinity of multiplicity seven, then it can have at most one affine invariant straight and this line has the multiplicity equal to one (Theorem 5.1). In Section 6 it is proved that in the class of non-degenerate cubic differential systems with a real affine invariant straight line of multiplicity six (respectively, five, four, three, two) the maximal multiplicity of the line at infinity is equal to one (respectively, four, five, five, five). The normal forms of the systems which realise these cases are given. Each of these systems posses a single affine invariant straight line and is Darboux integrable (Theorems 6.1-6.5).

2. Estimation of the algebraic multiplicity of an affine invariant straight line for polynomial differential systems

Let $P(x,y) = \sum\limits_{k=0}^{n} P_k(x,y)$ and $Q(x,y) = \sum\limits_{k=0}^{n} Q_k(x,y)$, where $P_j(x,y)$, $Q_j(x,y)$ are homogeneous polynomials in x and y of degree j.

Suppose that

$$deg(\gcd(P,Q)) = 0 \tag{2}$$

and

$$yP_n(x,y) - xQ_n(x,y) \not\equiv 0. \tag{3}$$

The condition (2) means that the right-hand sides of (1) do not have common divisors of degree grater that 0, and the condition (3) means that (1) is not degenerate, i.e. the line at infinity does not consist only of singular points.

Definition 2.1. [6] An invariant algebraic curve f of degree d for the vector field $\mathbb{X}$ has *algebraic multiplicity* m when m is the greatest positive integer such that the m-th power of f divides $E_d(\mathbb{X})$, where

$$E_d(\mathbb{X}) = det \begin{pmatrix} \upsilon_1 & \upsilon_2 & ... & \upsilon_k \\ \mathbb{X}(\upsilon_1) & \mathbb{X}(\upsilon_2) & ... & \mathbb{X}(\upsilon_k) \\ ... & ... & ... & ... \\ \mathbb{X}^{k-1}(\upsilon_1) & \mathbb{X}^{k-1}(\upsilon_2) & ... & \mathbb{X}^{k-1}(\upsilon_k) \end{pmatrix}, \tag{4}$$

and $\upsilon_1, \upsilon_2, ..., \upsilon_k$ is a basis of $\mathbb{C}_d[x, y]$.

If $d = 1$ then $\upsilon_1 = 1$, $\upsilon_2 = x$, $\upsilon_3 = y$ and

$$E_1(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P). \tag{5}$$

The polynomial $E_d(\mathbb{X})$ has in x and y the degree (see [12])

$$\frac{1}{24} d(d+1)(d+2)[8 + 3(d+3)(n-1)]. \tag{6}$$

In the case of cubic systems ($n = 3$) and straight lines ($d = 1$) we have $deg(E_1(\mathbb{X})) = 8$.

Denote by $L(P, Q)$ the set of all affine invariant straight lines of the system $\{(1), (2), (3)\}$; $m_a(l)$ the algebraic multiplicity of the line $l \in L(P, Q)$;

$$M_a(n) = max\{m_a(l) | l \in L(P, Q), \max\{\deg(P), \deg(Q)\} = n\}.$$

It is well known that $M_a(n) \leq 3n - 1$.

Theorem 2.1. *In the class of polynomial differential systems* $\{(1),(2),(3)\}$ *of degree* $n \geq 2$ *we have* $3n - 2 \leq M_a(n) \leq 3n - 1$.

Proof. For system

$$\dot{x} = x^n, \qquad \dot{y} = 1 + nx^{n-1}y. \tag{7}$$

the straight line $x = 0$ is invariant and $E_1(\mathbb{X}) = n(n-1)yx^{3n-2}$. □

The system (7) is Darboux integrable and has the first integral

$$\mathcal{F} = \left(1 + (2n-1)x^{n-1}y\right)/x^{2n-1}.$$

Conjecture 2.1. In the class of polynomial differential systems $\{(1),(2),(3)\}$ of degree $n \geq 2$ we have $M_a(n) = 3n - 2$.

3. Classification of cubic differential systems with an algebraic multiple real affine invariant straight line

We consider the cubic differential system

$$\begin{cases} \dot{x} = P_0 + P_1(x, y) + P_2(x, y) + P_3(x, y) \equiv P(x, y), \\ \dot{y} = Q_0 + Q_1(x, y) + Q_2(x, y) + Q_3(x, y) \equiv Q(x, y), \end{cases} \tag{8}$$

where $P_k = \sum_{j=0}^{k} a_{l(k)+j} x^{k-j} y^j$, $\quad Q_k = \sum_{j=0}^{k} b_{l(k)+j} x^{k-j} y^j$, $\; l(0) = 0, l(1) = 1, l(2) = 3$ and $l(3) = 6$.

Suppose that

$$yP_3(x, y) - xQ_3(x, y) \not\equiv 0, \qquad \gcd(P, Q) = 1, \tag{9}$$

i.e. at infinity the system (8) has at most four distinct singular points and the right-hand sides of (8) do not have the common divisors of degree greatest that 0.

Let the system (8) have a real invariant straight line l. By an affine transformation we can make l to be described by the equation $x = 0$. Then, the system (8) looks as:

$$\begin{array}{l} \dot{x} = x(a_1 + a_3x + a_4y + a_6x^2 + a_7xy + a_8y^2), \\ \dot{y} = b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + b_6x^3 + b_7x^2y + b_8xy^2 + b_9y^3. \end{array} \tag{10}$$

For (10) the determinant $E_1(\mathbb{X})$ is a polynomial in x and y of degree 8. We write it in the form:

$$\begin{array}{rl} E_1(\mathbb{X}) = & x(A_1(y) + A_2(y)x + A_3(y)x^2 + A_4(y)x^3 + A_5(y)x^4 + \\ & +A_6(y)x^5 + A_7(y)x^6 + A_8(y)x^7). \end{array} \tag{11}$$

The algebraic multiplicity $m_a(l)$ of the invariant straight line $x = 0$ of the system (10) is at least two if the identity $A_1(y) \equiv 0$ holds. Then we have $A_1(y) = -A_{11}(y) \cdot A_{12}(y)$, where

$$\begin{array}{rcl} A_{11}(y) & = & b_0 + b_2y + b_5y^2 + b_9y^3, \\ A_{12}(y) & = & a_1^2 + a_4b_0 - a_1b_2 + 2a_1a_4y + 2a_8b_0y - 2a_1b_5y + a_4^2y^2 + 2a_1a_8y^2 + \\ & & +a_8b_2y^2 - a_4b_5y^2 - 3a_1b_9y^2 + 2a_4a_8y^3 - 2a_4b_9y^3 + a_8^2y^4 - a_8b_9y^4. \end{array}$$

From conditions (9) the polynomial $A_{11}(y)$ is not identically equal to zero.

Let $A_{12}(y) \equiv 0$, i.e.

$$\begin{array}{l} a_1^2 + a_4b_0 - a_1b_2 = 0, \ a_4(a_8 - b_9) = 0, \\ a_1a_4 + a_8b_0 - a_1b_5 = 0, \ a_8(a_8 - b_9) = 0, \\ a_4^2 + 2a_1a_8 + a_8b_2 - a_4b_5 - 3a_1b_9 = 0. \end{array} \tag{12}$$

The system of equalities (12) is compatible if and only if at least one of following four sets of conditions holds:

$$a_1 = a_4 = a_8 = 0; \tag{13}$$

$$a_4 = a_8 = b_5 = b_9 = 0, \ b_2 = a_1, \ a_1 \neq 0; \tag{14}$$

$$a_8 = b_9 = 0, \ b_0 = a_1(b_2 - a_1)/a_4, \ b_5 = a_4; \tag{15}$$

$$b_0 = a_1(b_5 - a_4)/a_8, \ b_2 = a_1 + a_4(b_5 - a_4)/a_8, \ b_9 = a_8. \tag{16}$$

In this way we have proved the following lemma.

Lemma 3.1. *For cubic differential system* $\{(10), (9)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is at least two if and only if one of the sets of conditions* (13), (14), (15), (16) *is satisfied.*

The algebraic multiplicity of the invariant straight line $x = 0$ is greater than two if the identity $A_2(y) \equiv 0$ holds. Putting each of the conditions (13)-(16) in the polynomial $A_2(y)$ we have respectively:

$$A_2(y) = -A_{11}(y) \cdot (a_7b_0 - a_3b_2 - 2a_3b_5y - (a_7b_5 + 3a_3b_9)y^2 - 2a_7b_9y^3); \tag{17}$$

$$\begin{array}{l} A_2(y) = -2a_1a_3b_0 - a_7b_0^2 + a_1^2b_1 + a_1b_0b_4 + 2a_1(-a_1a_3 - 2a_7b_0 + a_1b_4 \\ +b_0b_8)y + 3a_1^2(b_8 - a_7)y^2; \end{array} \tag{18}$$

$$\begin{array}{l} A_2(y) = (a_1 + a_4y)(3a_1^2a_3a_4 - a_1^3a_7 + 2a_1a_4^2b_1 - 4a_1a_3a_4b_2 + 2a_1^2a_7b_2 \\ -a_4^2b_1b_2 + a_3a_4b_2^2 - a_1a_7b_2^2 - a_1^2a_4b_4 + a_1a_4b_2b_4 + 2a_1a_4(-a_3a_4 + 2a_1a_7 \\ -2a_7b_2 + a_4b_4 - a_1b_8 + b_2b_8)y - a_4^2(a_3a_4 + a_1a_7 + 2a_7b_2 - a_4b_4 \\ -2a_1b_8 - b_2b_8)y^2 + 2a_4^3(b_8 - a_7)y^3; \end{array} \tag{19}$$

$$\begin{aligned}
A_2(y) = {} & (a_1 + a_4y + a_8y^2)(a_3a_4^3 - a_1a_4^2a_7 + 2a_1a_3a_4a_8 + a_4^2a_8b_1 + a_1a_8^2b_1 \\
& - a_1a_4a_8b_4 - 2a_3a_4^2b_5 + 2a_1a_4a_7b_5 - 2a_1a_3a_8b_5 - a_4a_8b_1b_5 + a_1a_8b_4b_5 \\
& + a_3a_4b_5^2 - a_1a_7b_5^2 + 2a_8(a_3a_4^2 + 2a_1a_4a_7 - a_1a_3a_8 + a_4a_8b_1 + a_1a_8b_4 \\
& - 2a_3a_4b_5 - 2a_1a_7b_5 - a_8b_1b_5 + a_3b_5^2 - a_1a_4b_8 + a_1b_5b_8)y + a_8(3a_4^2a_7 \\
& - 3a_3a_4a_8 - 3a_1a_7a_8 - a_8^2b_1 + 2a_4a_8b_4 - 4a_4a_7b_5 + 2a_3a_8b_5 - a_8b_4b_5 + a_7b_5^2 \\
& - a_4^2b_8 + 3a_1a_8b_8 + a_4b_5b_8)y^2 + a_8^2(b_8 - a_7)(2a_4 + a_8y)y^3).
\end{aligned} \tag{20}$$

Taking into account (9) the identity $A_2(y) \equiv 0$ gives, in each of the cases (17)-(20), the following series of conditions:

(17) $\Rightarrow$

$$a_3 = a_7 = 0,\ a_6 \neq 0; \tag{21}$$

$$a_7 = b_2 = b_5 = b_9 = 0, a_3 \neq 0; \tag{22}$$

$$b_0 = a_3b_2/a_7,\ b_5 = b_9 = 0; \tag{23}$$

(18) $\Rightarrow$

$$b_1 = a_3b_0/a_1,\ b_4 = a_3 + a_7b_0/a_1,\ b_8 = a_7; \tag{24}$$

(19) $\Rightarrow$

$$b_2 = 2a_1,\ b_4 = a_3 + a_1a_7/a_4,\ b_8 = a_7; \tag{25}$$

$$b_1 = a_3(b_2 - a_1)/a_4,\ b_4 = a_3 + a_7(b_2 - a_1)/a_4,\ b_8 = a_7; \tag{26}$$

(20) $\Rightarrow$

$$\begin{aligned}
& a_1 = -(2a_4^2 - 3a_4b_5 + b_5^2)/a_8,\ b_8 = a_7, \\
& b_1 = (2a_4^2a_7 - 3a_3a_4a_8 + 2a_3a_8b_5 - \\
& -3a_4a_7b_5 + a_7b_5^2 + 2a_4a_8b_4 - a_8b_4b_5)/a_8^2;
\end{aligned} \tag{27}$$

$$b_1 = a_3(b_5 - a_4)/a_8,\ b_4 = a_3 + a_7(b_5 - a_4)/a_8,\ b_8 = a_7. \tag{28}$$

Lemma 3.2. *For cubic differential system* $\{(10), (9)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is at least three if and only if one of the following eight sets of conditions*

2.1) (13), (21); 2.2) (13), (22); 2.3) (13), (23); 2.4) (14), (24);
2.5) (15), (25); 2.6) (15), (26); 2.7) (16), (27); 2.8) (16), (28)

is satisfied.

The invariant straight line $x = 0$ has algebraic multiplicity $m_a \geq 4$ if in each of the cases 2.1)-2.8) the identity $A_3(y) \equiv 0$ holds. Taking into account (9) we have:

2.1) $\Rightarrow A_3(y) = a_6(b_2 + 2b_5y + 3b_9y^2) \cdot A_{11}(y) \equiv 0 \Rightarrow$

$$b_2 = b_5 = b_9 = 0, b_0 \neq 0; \tag{29}$$

2.2) $\Rightarrow A_3(y) = -a_3b_0(2a_3 - b_4 - 2b_8y) \equiv 0 \Rightarrow$

$$b_4 = 2a_3, b_8 = 0, b_0 \neq 0; \tag{30}$$

2.3) $\Rightarrow A_3(y) = -A_{11}(y)(2a_3^2 - a_6b_2 + a_7b_1 - a_3b_4 + 2a_3(2a_7 - b_8)y + a_7(2a_7 - b_8)y^2) \equiv 0 \Rightarrow$

$$b_1 = (a_6b_2 + a_3b_4 - 2a_3^2)/a_7,\ b_8 = 2a_7,\ b_2 \neq 0; \tag{31}$$

2.4) $\Rightarrow A_3(y) = -a_1(3a_6b_0 - 2a_1b_3 - b_0b_7 + 3a_1(a_6 - b_7)y) \equiv 0 \Rightarrow$

$$b_3 = a_6b_0/a_1,\ b_7 = a_6; \tag{32}$$

2.5) $\Rightarrow A_3(y) = -(a_1^2a_3^2a_4 + 2a_1^3a_4a_6 - a_1^3a_3a_7 - 2a_1a_3a_4^2b_1 + a_1^2a_4a_7b_1 + a_4^3b_1^2 - a_1^2a_4^2b_3 - a_1^3a_4b_7)/a_4^2 - 2a_1(3a_1a_4a_6 - a_1a_3a_7 + a_4a_7b_1 - a_4^2b_3 - 2a_1a_4b_7)y/a_4 - (6a_1a_4a_6 - a_1a_3a_7 + a_4a_7b_1 - a_4^2b_3 - 5a_1a_4b_7)y^2 + 2a4^2(b_7 - a6)y^3 \equiv 0 \Rightarrow$

$$b_1 = a_1a_3/a_4,\ b_3 = a_1a_6/a_4,\ b_7 = a_6; \tag{33}$$

2.6) $\Rightarrow A_3(y) = (a_1 + a_4 y)((4a_1^2 a_6 - 5a_1 a_6 b_2 + a_6 b_2^2 + 3a_1 a_4 b_3 - a_4 b_2 b_3 - a_1^2 b_7 + a_1 b_2 b_7) - a_4(2a_1 a_6 + a_6 b_2 - a_4 b_3 - 3a_1 b_7)y + 2a_4^2(a_6 - b_7)y^2)/a_4 \equiv 0 \Rightarrow$

$$b_3 = a_6(b_2 - a_1)/a_4,\ b_7 = a_6; \tag{34}$$

2.7) $\Rightarrow A_3(y) = -(2a_4 - b_5 + a_8 y)(B_0 + B_1 y + B_2 y^2 + B_3 y^3 + B_4 y^4)/a_8^4$, where

$B_0 = 6a_4^4 a_7^2 - 5a_4^4 a_6 a_8 - 11a_3 a_4^3 a_7 a_8 + 5a_3^2 a_4^2 a_8^2 - 3a_4^3 a_8^2 b_3 + 10a_4^3 a_7 a_8 b_4 - 9a_3 a_4^2 a_8^2 b_4 + 4a_4^2 a_8^2 b_4^2 - 19a_4^3 a_7^2 b_5 + 18a_4^3 a_6 a_8 b_5 + 24a_3 a_4^2 a_7 a_8 b_5 - 6a_3^2 a_4 a_8^2 b_5 + 8a_4^2 a_8^2 b_3 b_5 - 21a_4^2 a_7 a_8 b_4 b_5 + 10a_3 a_4 a_8^2 b_4 b_5 - 4a_4 a_8^2 b_4^2 b_5 + 22a_4^2 a_7^2 b_5^2 - 24a_4^2 a_6 a_8 b_5^2 - 17a_3 a_4 a_7 a_8 b_5^2 + 2a_3^2 a_8^2 b_5^2 - 7a_4 a_8^2 b_3 b_5^2 + 14a_4 a_7 a_8 b_4 b_5^2 - 3a_3 a_8^2 b_4 b_5^2 + a_8^2 b_4^2 b_5^2 - 11a_4 a_7^2 b_5^3 + 14a_4 a_6 a_8 b_5^3 + 4a_3 a_7 a_8 b_5^3 + 2a_8^2 b_3 b_5^3 - 3a_7 a_8 b_4 b_5^3 + 2a_7^2 b_5^4 - 3a_6 a_8 b_5^4 + 2a_4^4 a_8 b_7 - 7a_4^3 a_8 b_5 b_7 + 9a_4^2 a_8 b_5^2 b_7 - 5a_4 a_8 b_5^3 b_7 + a_8 b_5^4 b_7$,

$B_1 = 2a_8^2(7a_4^3 a_6 - a_3 a_4^2 a_7 + a_3^2 a_4 a_8 + 3a_4^2 a_8 b_3 + 2a_4^2 a_7 b_4 - 3a_3 a_4 a_8 b_4 + 2a_4 a_8 b_4^2 - 18a_4^2 a_6 b_5 + a_3 a_4 a_7 b_5 - 5a_4 a_8 b_3 b_5 - 3a_4 a_7 b_4 b_5 + a_3 a_8 b_4 b_5 - a_8 b_4^2 b_5 + 15a_4 a_6 b_5^2 + 2a_8 b_3 b_5^2 + a_7 b_4 b_5^2 - 4a_6 b_5^3 - 4a_4^3 b_7 + 10a_4^2 b_5 b_7 - 8a_4 b_5^2 b_7 + 2b_5^3 b_7)$,

$B_2 = a_8^2(3a_4^2 a_7^2 - 12a_4^2 a_6 a_8 - 5a_3 a_4 a_7 a_8 + 2a_3^2 a_8^2 - 3a_4 a_8^2 b_3 + 4a_4 a_7 a_8 b_4 - 3a_3 a_8^2 b_4 + a_8^2 b_4^2 - 5a_4 a_7^2 b_5 + 18a_4 a_6 a_8 b_5 + 4a_3 a_7 a_8 b_5 + 2a_8^2 b_3 b_5 - 3a_7 a_8 b_4 b_5 + 2a_7^2 b_5^2 - 6a_6 a_8 b_5^2 + 9a_4^2 a_8 b_7 - 13a_4 a_8 b_5 b_7 + 4a_8 b_5^2 b_7)$,

$B_3 = 2a_4 a_8^4(a_6 - b_7), \qquad B_4 = a_8^5(a_6 - b_7)$.

In this case the identity $A_3(y) \equiv 0$ holds if at least one of the following three series of conditions is satisfied:

$$a_3 = b_4 - a_7(b_5 - a_4)/a_8,\ b_3 = a_6(b_5 - a_4)/a_8,\ b_7 = a_6; \tag{35}$$

$$a_3 = b_4/2,\ b_5 = 3a_4/2,\ b_7 = a_6; \tag{36}$$

$$a_3 = b_4 - a_4 a_7/(2a_8),\ b_5 = 3a_4/2,\ b_7 = a_6; \tag{37}$$

2.8) $\Rightarrow A_3(y) = (a_1 + a_4 y + a_8 y^2)(a_4^3 a_6 + 3a_1 a_4 a_6 a_8 + a_4^2 a_8 b_3 + 2a_1 a_8^2 b_3 - 2a_4^2 a_6 b_5 - 3a_1 a_6 a_8 b_5 - a_4 a_8 b_3 b_5 + a_4 a_6 b_5^2 - a_1 a_4 a_8 b_7 + a_1 a_8 b_5 b_7 + a_8(3a_4^2 a_6 - 3a_1 a_6 a_8 + 3a_4 a_8 b_3 - 5a_4 a_6 b_5 - 2a_8 b_3 b_5 + 2a_6 b_5^2 + 3a_1 a_8 b_7)y - a_8^2(3a_4 - b_5)(a_6 - b_7)y^2 - a_8^3(a_6 - b_7)y^3)/a_8^2 \equiv 0$
$\Rightarrow$

$$b_7 = a_6,\ b_3 = a_6(b_5 - a_4)/a_8; \tag{38}$$

$$b_7 = a_6,\ b_5 = 3a_4/2,\ a_1 = a_4^2/(4a_8). \tag{39}$$

Lemma 3.3. *For cubic differential system* {(10),(9)} *the algebraic multiplicity of the invariant straight line $x = 0$ is at least four if and only if one of the following eleven series of conditions is satisfied:*

2.9) (13), (21), (29); *2.10)* (13), (22), (30); *2.11)* (13), (23), (31);
2.12) (14), (24), (32); *2.13)* (15), (25), (33); *2.14)* (15), (26), (34);
2.15) (16), (27), (35); *2.16)* (16), (27), (36); *2.17)* (16), (27), (37);
2.18) (16), (28), (38); *2.19)* (16), (28), (39).

In each of the cases 2.10)-2.15), 2.18), the identity $A_4(y) \equiv 0$ and the conditions (9) are not compatible. In the cases 2.9), 2.16), 2.17) and 2.19) we have respectively the implications:

2.9) $\Rightarrow A_4(y) = a_6 b_0(b_4 + 2b_8 y) \equiv 0 \Rightarrow$

$$b_4 = b_8 = 0; \tag{40}$$

2.16) $\Rightarrow A_4(y) = (a_4 + 2a_8 y)(-a_4^3 a_7^3 - 2a_4^3 a_6 a_7 a_8 + 4a_4^2 a_7 a_8^2 b_3 + 4a_4^2 a_7^2 a_8 b_4 + 2a_4^2 a_6 a_8^2 b_4 - 4a_4 a_8^3 b_3 b_4 - 5a_4 a_7 a_8^2 b_4^2 + 2a_8^3 b_4^3 + a_4^3 a_8^2 b_6 2a_8(a_4^2 a_7^3 - 2a_4^2 a_6 a_7 a_8 + 4a_4 a_7 a_8^2 b_3 - 2a_4 a_7^2 a_8 b_4 + 2a_4 a_6 a_8^2 b_4 - 4a_8^3 b_3 b_4 + a_7 a_8^2 b_4^2 + 3a_4^2 a_8^2 b_6)y + 12a_4 a_8^4 b_6 y^2 + 8a_8^5 b_6 y^3)/(16a_8^4) \equiv 0 \Rightarrow$

$$b_6 = 0, b_4 = \frac{a_4 a_7}{a_8}; \tag{41}$$

2.17) $\Rightarrow A_4(y) = (a_4+2a_8y)^2(2a_4^2a_6a_7 - 4a_4a_7a_8b_3 - 2a_4a_6a_8b_4 + 4a_8^2b_3b_4 + a_4^2a_8b_6 + 4a_4a_8^2b_6y + 4a_8^3b_6y^2)/(16a_8^3) \equiv 0 \Rightarrow$ (41);

2.19) $\Rightarrow A_4(y) = (a_4+2a_8y)^2(a_4^2a_6a_7 - 2a_3a_4a_6a_8 - 2a_4a_7a_8b_3 + 4a_3a_8^2b_3 + a_4^2a_8b_6 + 4a_4a_8^2b_6y + 4a_8^3b_6y^2)/(16a_8^3) \equiv 0 \Rightarrow$

$$b_6 = 0, a_3 = \frac{a_4a_7}{2a_8}. \tag{42}$$

It is easy to see that the set of conditions {2.16), (41)}, {2.17), (41)} and {2.19), (42)} are equivalent.

Lemma 3.4. *For cubic differential system* $\{(10),(9)\}$ *the algebraic multiplicity* m_a *of the invariant straight line* $x = 0$ *is at least five if and only if one of the following two series of conditions is satisfied:*

2.20) (13), (21), (29); (40); *2.21)* (16), (27), (36); (41).

In the case 2.20) we have $A_5(y) = -a_6b_0(3a_6 - b_7) \equiv 0 \Rightarrow$

$$b_7 = 3a_6 \tag{43}$$

and in the case 2.21) the polynomial $A_5(y)$ has the form:

$$A_5(y) = -(a_4a_6 - 2a_8b_3)^2(a_4 + 2a_8y)/(4a_8^2) \not\equiv 0.$$

Lemma 3.5. *The algebraic multiplicity of the invariant straight line* $x = 0$ *of system* $\{(10),(9)\}$ *is* $m_a \geq 6$ *if and only if the following five series of conditions* (13), (21), (29), (40), (43) *hold.*

In the conditions of Lemma 3.5 we have: $A_6(y) = a_6^2b_1 \equiv 0 \Rightarrow$

$$b_1 = 0 \tag{44}$$

$\Rightarrow A_7(y) = 2a_6^2(b_3 + 3a_6y) \not\equiv 0$, $m_a = 7$ and the cubic system (10) looks as:

$$\dot{x} = a_6x^3,\ \dot{y} = b_0 + b_3x^2 + b_6x^3 + 3a_6x^2y, \quad a_6b_0 \neq 0. \tag{45}$$

Via the affine transformation of coordinates: $x \to x, y \to -(2b_3 + 3b_6x - 6b_0y)/(6a_6)$ and the rescaling of time $t = \tau/a_6$ the system (45) can be written into the form:

$$\dot{x} = x^3, \qquad \dot{y} = 1 + 3x^2y. \tag{46}$$

In this way we have proved the following theorem.

Theorem 3.6. *In the class of cubic differential systems* $\{(10),(9)\}$ *the maximal algebraic multiplicity of an affine real invariant straight line is equal to 7. Via an affine transformation of coordinates and time rescaling each cubic system which has an invariant straight line of algebraic multiplicity 7 can be written in the form* (46).

In the similar way as the proof of Theorem 3.6 it can be shown, that Conjecture 2.1 is true and in the cases of quadratic ($n = 2$) and quartic ($n = 4$) systems, i.e. $M_a(2) = 4,\ M_a(4) = 10$.

4. Infinitesimal, integrable and geometric maximal multiplicity of an affine invariant straight line for cubic systems

4.1. Infinitesimal multiplicity.

Definition 4.1. [6] Let $f = 0$ be an invariant algebraic curve of degree d of a polynomial vector field $\mathbb{X}$ of degree n. We say that

$$F = f_0 + f_1\epsilon + \cdots + f_{k-1}\epsilon^{k-1} \in \mathbb{C}[x, y, \epsilon]/(\epsilon^k)$$

defines a *generalized invariant algebraic curve of order k based on $f = 0$* if $f_0 = f, \ldots, f_{k-1}$ are polynomials in $\mathbb{C}[x, y]$ of degree at most d, and F satisfies the equation $\mathbb{X}(F) = FL_F$, for some polynomial

$$L_F = L_0 + L_1\epsilon + \cdots + L_{k-1}\epsilon^{k-1} \in \mathbb{C}[x, y, \epsilon]/(\epsilon^k),$$

which must necessarily be of degree at most $n-1$ in x and y. We call L_F the *cofactor* of F.

Definition 4.2. [6] A generalized algebraic curve F based on f is called *nondegenerate* if the polynomial f_1 from Definition 4.1 is not multiple of f.

Definition 4.3. [6] Let $f = 0$ be an invariant algebraic curve of degree d in a polynomial vector field $\mathbb{X}$ of degree n. We say that $f = 0$ is of *infinitesimal multiplicity* m with respect to X if m is the maximal order of all nondegenerate generalized invariant algebraic curves of X based on f. If such a maximum does not exist, then the infinitesimal multiplicity is said to be infinite.

In the work [6] it is shown that the notions of infinitesimal and algebraic multiplicity of an invariant algebraic curve are equivalent. Therefore, for cubic systems the maximal infinitesimal multiplicity of an affine invariant straight line is not greater than seven, and each system which has an invariant affine straight line of infinitesimal multiplicity seven can be written in the form (46). For this system the generalized invariant algebraic curve F of order 7 and the cofactor L_F are described respectively by polynomials f_i and L_i ($i = \overline{0,6}$):

$f_0 = x,$

$f_1 = x\alpha_1 + \gamma_1,$

$f_2 = x\alpha_2 + \gamma_2,$

$f_3 = x\alpha_3 + y\gamma_1^3 + \gamma_3,$

$f_4 = x\alpha_4 + y(-2\alpha_1\gamma_1^3 + 3\gamma_1^2\gamma_2) + \gamma_4,$

$f_5 = x\alpha_5 + y(3\alpha_1^2\gamma_1^3 - 2\alpha_2\gamma_1^3 - 6\alpha_1\gamma_1^2\gamma_2 + 3\gamma_1\gamma_2^2 + 3\gamma_1^2\gamma_3) + \gamma_5,$

$f_6 = x\alpha_6 + y(-4\alpha_1^3\gamma_1^3 + 6\alpha_1\alpha_2\gamma_1^3 - 2\alpha_3\gamma_1^3 + 9\alpha_1^2\gamma_1^2\gamma_2 - 6\alpha_2\gamma_1^2\gamma_2 - 6\alpha_1\gamma_1\gamma_2^2 + \gamma_2^3 - 6\alpha_1\gamma_1^2\gamma_3 + 6\gamma_1\gamma_2\gamma_3 + 3\gamma_1^2\gamma_4) + \gamma_6,$

$L_0 = x^2,$

$L_1 = -x\gamma_1,$

$L_2 = x\alpha_1\gamma_1 + \gamma_1^2 - x\gamma_2,$

$L_3 = -x\alpha_1^2\gamma_1 + x\alpha_2\gamma_1 - 2\alpha_1\gamma_1^2 + 2xy\gamma_1^3 + x\alpha_1\gamma_2 + 2\gamma_1\gamma_2 - x\gamma_3,$

$L_4 = x\alpha_1^3\gamma_1 - 2x\alpha_1\alpha_2\gamma_1 + x\alpha_3\gamma_1 + 3\alpha_1^2\gamma_1^2 - 2\alpha_2\gamma_1^2 - 6xy\alpha_1\gamma_1^3 - y\gamma_1^4 - x\alpha_1^2\gamma_2 + x\alpha_2\gamma_2 - 4\alpha_1\gamma_1\gamma_2 + 6xy\gamma_1^2\gamma_2 + \gamma_2^2 + x\alpha_1\gamma_3 + 2\gamma_1\gamma_3 - x\gamma_4,$

$L_5 = -x\alpha_1^4\gamma_1 + 3x\alpha_1^2\alpha_2\gamma_1 - x\alpha_2^2\gamma_1 - 2x\alpha_1\alpha_3\gamma_1 + x\alpha_4\gamma_1 - 4\alpha_1^3\gamma_1^2 + 6\alpha_1\alpha_2\gamma_1^2 - 2\alpha_3\gamma_1^2 + 12xy\alpha_1^2\gamma_1^3 - 6xy\alpha_2\gamma_1^3 + 4y\alpha_1\gamma_1^4 + x\alpha_1^3\gamma_2 - 2x\alpha_1\alpha_2\gamma_2 + x\alpha_3\gamma_2 + 6\alpha_1^2\gamma_1\gamma_2 - 4\alpha_2\gamma_1\gamma_2 - 18xy\alpha_1\gamma_1^2\gamma_2 - 4y\gamma_1^3\gamma_2 - 2\alpha_1\gamma_2^2 + 6xy\gamma_1\gamma_2^2 - x\alpha_1^2\gamma_3 + x\alpha_2\gamma_3 - 4\alpha_1\gamma_1\gamma_3 + 6xy\gamma_1^2\gamma_3 + 2\gamma_2\gamma_3 + x\alpha_1\gamma_4 + 2\gamma_1\gamma_4 - x\gamma_5,$

$L_6 = x\alpha_1^5\gamma_1 - 4x\alpha_1^3\alpha_2\gamma_1 + 3x\alpha_1\alpha_2^2\gamma_1 + 3x\alpha_1^2\alpha_3\gamma_1 - 2x\alpha_2\alpha_3\gamma_1 - 2x\alpha_1\alpha_4\gamma_1 + x\alpha_5\gamma_1 + 5\alpha_1^4\gamma_1^2 - 12\alpha_1^2\alpha_2\gamma_1^2 + 3\alpha_2^2\gamma_1^2 + 6\alpha_1\alpha_3\gamma_1^2 - 2\alpha_4\gamma_1^2 - 20xy\alpha_1^3\gamma_1^3 + 24xy\alpha_1\alpha_2\gamma_1^3 - 6xy\alpha_3\gamma_1^3 -$

$10y\alpha_1^2\gamma_1^4+4y\alpha_2\gamma_1^4-2y^2\gamma_1^6-x\alpha_1^4\gamma_2+3x\alpha_1^2\alpha_2\gamma_2-x\alpha_2^2\gamma_2-2x\alpha_1\alpha_3\gamma_2+x\alpha_4\gamma_2-8\alpha_1^3\gamma_1\gamma_2+12\alpha_1\alpha_2\gamma_1\gamma_2-4\alpha_3\gamma_1\gamma_2+36xy\alpha_1^2\gamma_1^2\gamma_2-18xy\alpha_2\gamma_1^2\gamma_2+16y\alpha_1\gamma_1^3\gamma_2+3\alpha_1^2\gamma_2^2-2\alpha_2\gamma_2^2-18xy\alpha_1\gamma_1\gamma_2^2-6y\gamma_1^2\gamma_2^2+2xy\gamma_2^3+x\alpha_1^3\gamma_3-2x\alpha_1\alpha_2\gamma_3+x\alpha_3\gamma_3+6\alpha_1^2\gamma_1\gamma_3-4\alpha_2\gamma_1\gamma_3-18xy\alpha_1\gamma_1^2\gamma_3-4y\gamma_1^3\gamma_3-4\alpha_1\gamma_2\gamma_3+12xy\gamma_1\gamma_2\gamma_3+\gamma_3^2-x\alpha_1^2\gamma_4+x\alpha_2\gamma_4-4\alpha_1\gamma_1\gamma_4+6xy\gamma_1^2\gamma_4+2\gamma_2\gamma_4+x\alpha_1\gamma_5+2\gamma_1\gamma_5-x\gamma_6$, where α_i, γ_i, $i=\overline{1,6}$ are parameters.

4.2. Integrable multiplicity.

Definition 4.4. [6] We shall say that the invariant algebraic curve $f=0$ has *integrable multiplicity* m with respect to X if m is the largest integer for which the following is true: there are $m-1$ exponential factors $exp(g_j/f^j)$, $j=1,...,m-1$, with $deg\ g_j \le j\ deg\ f$, such that each g_j is not a multiple of f.

The notions of algebraic and integrable multiplicity are equivalent (see [6]). Therefore, for cubic systems the maximal integrable multiplicity of an invariant affine straight line is not greater than 7. Each cubic system which has an invariant affine straight line can be written in the form (46). For this system we have $f=x$ and invariant exponential factors $exp(g_j/x^j)$, $j=1,...,6$ associated to $x=0$, where

$g_1=x\alpha_1+\gamma_1$,

$g_2=\frac{1}{2}(-x^2\alpha_1^2+2x^2\alpha_2-2x\alpha_1\gamma_1-\gamma_1^2+2x\gamma_2)$,

$g_3=\frac{1}{3}(x^3\alpha_1^3-3x^3\alpha_1\alpha_2+3x^3\alpha_3+3x^2\alpha_1^2\gamma_1-3x^2\alpha_2\gamma_1+3x\alpha_1\gamma_1^2+\gamma_1^3+3x^2y\gamma_1^3-3x^2\alpha_1\gamma_2-3x\gamma_1\gamma_2+3x^2\gamma_3)$,

$g_4=\frac{1}{4}(-x^4\alpha_1^4+4x^4\alpha_1^2\alpha_2-2x^4\alpha_2^2-4x^4\alpha_1\alpha_3+4x^4\alpha_4-4x^3\alpha_1^3\gamma_1+8x^3\alpha_1\alpha_2\gamma_1-4x^3\alpha_3\gamma_1-6x^2\alpha_1^2\gamma_1^2+4x^2\alpha_2\gamma_1^2-4x\alpha_1\gamma_1^3-12x^3y\alpha_1\gamma_1^3-\gamma_1^4-4x^2y\gamma_1^4+4x^3\alpha_1^2\gamma_2-4x^3\alpha_2\gamma_2+8x^2\alpha_1\gamma_1\gamma_2+4x\gamma_1^2\gamma_2+12x^3y\gamma_1^2\gamma_2-2x^2\gamma_2^2-4x^3\alpha_1\gamma_3-4x^2\gamma_1\gamma_3+4x^3\gamma_4)$,

$g_5=\frac{1}{5}(x^5\alpha_1^5-5x^5\alpha_1^3\alpha_2+5x^5\alpha_1\alpha_2^2+5x^5\alpha_1^2\alpha_3-5x^5\alpha_2\alpha_3-5x^5\alpha_1\alpha_4+5x^5\alpha_5+5x^4\alpha_1^4\gamma_1-15x^4\alpha_1^2\alpha_2\gamma_1+5x^4\alpha_2^2\gamma_1+10x^4\alpha_1\alpha_3\gamma_1-5x^4\alpha_4\gamma_1+10x^3\alpha_1^3\gamma_1^2-15x^3\alpha_1\alpha_2\gamma_1^2+5x^3\alpha_3\gamma_1^2+10x^2\alpha_1^2\gamma_1^3+30x^4y\alpha_1^2\gamma_1^3-5x^2\alpha_2\gamma_1^3-15x^4y\alpha_2\gamma_1^3+5x\alpha_1\gamma_1^4+20x^3y\alpha_1\gamma_1^4+\gamma_1^5+5x^2y\gamma_1^5-5x^4\alpha_1^3\gamma_2+10x^4\alpha_1\alpha_2\gamma_2-5x^4\alpha_3\gamma_2-15x^3\alpha_1^2\gamma_1\gamma_2+10x^3\alpha_2\gamma_1\gamma_2-15x^2\alpha_1\gamma_1^2\gamma_2-45x^4y\alpha_1\gamma_1^2\gamma_2-5x\gamma_1^3\gamma_2-20x^3y\gamma_1^3\gamma_2+5x^3\alpha_1\gamma_2^2+5x^2\gamma_1\gamma_2^2+15x^4y\gamma_1\gamma_2^2+5x^4\alpha_1^2\gamma_3-5x^4\alpha_2\gamma_3+10x^3\alpha_1\gamma_1\gamma_3+5x^2\gamma_1^2\gamma_3+15x^4y\gamma_1^2\gamma_3-5x^3\gamma_2\gamma_3-5x^4\alpha_1\gamma_4-5x^3\gamma_1\gamma_4+5x^4\gamma_5)$,

$g_6=\frac{1}{6}(-x^6\alpha_1^6+6x^6\alpha_1^4\alpha_2-9x^6\alpha_1^2\alpha_2^2+2x^6\alpha_2^3-6x^6\alpha_1^3\alpha_3+12x^6\alpha_1\alpha_2\alpha_3-3x^6\alpha_3^2+6x^6\alpha_1^2\alpha_4-6x^6\alpha_2\alpha_4-6x^6\alpha_1\alpha_5+6x^6\alpha_6-6x^5\alpha_1^5\gamma_1+24x^5\alpha_1^3\alpha_2\gamma_1-18x^5\alpha_1\alpha_2^2\gamma_1-18x^5\alpha_1^2\alpha_3\gamma_1+12x^5\alpha_2\alpha_3\gamma_1+12x^5\alpha_1\alpha_4\gamma_1-6x^5\alpha_5\gamma_1-15x^4\alpha_1^4\gamma_1^2+36x^4\alpha_1^2\alpha_2\gamma_1^2-9x^4\alpha 2^2\gamma_1^2-18x^4\alpha_1\alpha_3\gamma_1^2+6x^4\alpha_4\gamma_1^2-20x^3\alpha_1^3\gamma_1^3-60x^5y\alpha_1^3\gamma_1^3+24x^3\alpha_1\alpha_2\gamma_1^3$
$+72x^5y\alpha_1\alpha_2\gamma_1^3-6x^3\alpha_3\gamma_1^3-18x^5y\alpha_3\gamma_1^3-15x^2\alpha_1^2\gamma_1^4-60x^4y\alpha_1^2\gamma_1^4+6x^2\alpha_2\gamma_1^4+24x^4y\alpha_2\gamma_1^4-6x\alpha_1\gamma_1^5-30x^3y\alpha_1\gamma_1^5-\gamma_1^6-6x^2y\gamma_1^6-3x^4y^2\gamma_1^6+6x^5\alpha_1^4\gamma_2-18x^5\alpha_1^2\alpha_2\gamma_2+6x^5\alpha_2^2\gamma_2+12x^5\alpha_1\alpha_3\gamma_2-6x^5\alpha_4\gamma_2+24x^4\alpha_1^3\gamma_1\gamma_2-36x^4\alpha_1\alpha_2\gamma_1\gamma_2+12x^4\alpha_3\gamma_1\gamma_2+36x^3\alpha_1^2\gamma_1^2\gamma_2+108x^5y\alpha_1^2\gamma_1^2\gamma_2-18x^3\alpha_2\gamma_1^2\gamma_2-54x^5y\alpha_2\gamma_1^2\gamma_2+24x^2\alpha_1\gamma_1^3\gamma_2+96x^4y\alpha_1\gamma_1^3\gamma_2+6x\gamma_1^4\gamma_2+30x^3y\gamma_1^4\gamma_2-9x^4\alpha_1^2\gamma_2^2+6x^4\alpha_2\gamma_2^2-18x^3\alpha_1\gamma_1\gamma_2^2-54x^5y\alpha_1\gamma_1\gamma_2^2-9x^2\gamma_1^2\gamma_2^2-36x^4y\gamma_1^2\gamma_2^2+2x^3\gamma_2^3+6x^5y\gamma_2^3-6x^5\alpha_1^3\gamma_3+12x^5\alpha_1\alpha_2\gamma_3-6x^5\alpha_3\gamma_3-18x^4\alpha_1^2\gamma_1\gamma_3+12x^4\alpha_2\gamma_1\gamma_3-18x^3\alpha_1\gamma_1^2\gamma_3-54x^5y\alpha_1\gamma_1^2\gamma_3-6x^2\gamma_1^3\gamma_3-24x^4y\gamma_1^3\gamma_3+12x^4\alpha_1\gamma_2\gamma_3+12x^3\gamma_1\gamma_2\gamma_3+36x^5y\gamma_1\gamma_2\gamma_3-3x^4\gamma_3^2+6x^5\alpha_1^2\gamma_4-6x^5\alpha_2\gamma_4+12x^4\alpha_1\gamma_1\gamma_4+6x^3\gamma_1^2\gamma_4+18x^5y\gamma_1^2\gamma_4-6x^4\gamma_2\gamma_4-6x^5\alpha_1\gamma_5-6x^4\gamma_1\gamma_5+6x^5\gamma_6)$; $\alpha_i, i=\overline{1,4}$, $\gamma_j, j=\overline{1,6}$, are parameters.

4.3. Geometric multiplicity.

Definition 4.5. An invariant algebraic curve $f=0$ of degree d of the vector field X has *weak geometric multiplicity* m if m is the largest integer for which there exists a sequence of vector fields $(\mathbb{X}_i)_{i>0}$ of bounded degree, converging to $h\mathbb{X}$, for some polynomial h, not divisible by f, such that each $\mathbb{X}_r$ has m distinct invariant algebraic curves, $f_{r,1} = 0, ..., f_{r,m} = 0$, of degree at most d, which converge to $f = 0$ as r goes to infinity.

Remark 4.1. In [6] it is proved that the notions of algebraic, infinitesimal, integrable and weak geometric multiplicity are equivalent.

Definition 4.6. If in Definition 4.5 each of the vector field $(\mathbb{X}_i)_{i>0}$ and $\mathbb{X}$ has the degree equal to n and $h = 1$, then we say that the invariant algebraic curve $f = 0$ has the geometric multiplicity m.

It is clear, that the weak geometric multiplicity of an invariant algebraic curve is not less than the geometric algebraic multiplicity of the same curve. Our claim is that in the class of cubic differential systems the notions of weak geometric multiplicity and geometric multiplicity of an invariant straight lines are equivalent.

Example 4.1. We consider the cubic system

$$\begin{aligned} &\dot{x} = x(x-3\epsilon)(x-3\epsilon+6\epsilon^3), \\ &\dot{y} = 1 + 3x^2y - 12xy\epsilon - 3\epsilon^2 + 9y\epsilon^2 + 12xy\epsilon^3 - 12xy^2\epsilon^3 \\ &\quad -6\epsilon^4 - 18y\epsilon^4 + 24y^2\epsilon^4 + 8\epsilon^6 - 24y^2\epsilon^6 + 16y^3\epsilon^6. \end{aligned} \tag{47}$$

This system has the following seven invariant affine straight lines:

$$\begin{aligned} &l_1 = x,\ l_2 = x - 3\epsilon,\ l_3 = x - 3\epsilon + 6\epsilon^3,\ l_4 = x - \epsilon - 2\epsilon^3 - 4y\epsilon^3, \\ &l_5 = x - \epsilon + 4\epsilon^3 - 4y\epsilon^3,\ l_6 = x - 4\epsilon + 4\epsilon^3 - 4y\epsilon^3,\ l_7 = x - 2\epsilon + 2\epsilon^3 - 2y\epsilon^3. \end{aligned}$$

If $\epsilon \to 0$, than (47) tends to the system (46), and the straight lines l_i, $i = 2, ..., 7$ converge to the straight line l_1 which is invariant for both systems (46) and (47).

5. Maximal multiplicity of the line at infinity for cubic differential systems

We consider the cubic system $\{(8), (9)\}$ and its associated homogeneous system

$$\begin{cases} \dot{x} = P_0Z^3 + P_1(x,y)Z^2 + P_2(x,y)Z + P_3(x,y), \\ \dot{y} = Q_0Z^3 + Q_1(x,y)Z^2 + Q_2(x,y)Z + Q_3(x,y). \end{cases} \tag{48}$$

In this section for cubic system $\{(8), (9)\}$ the maximal algebraic multiplicity of the line at infinity $Z = 0$ is calculated.

Without loss of generality we suppose that $b_6 = 1$.

For (48), $E_1(\mathbb{X})$ is a polynomial of degree 8 in x, y, Z. We write $E_1(X)$ in the form:

$$\begin{aligned} E_1(\mathbb{X}) = {} & A_0(x,y) + A_1(x,y)Z + A_2(x,y)Z^2 + A_3(x,y)Z^3 + A_4(x,y)Z^4 \\ & + A_5(x,y)Z^5 + A_6(x,y)Z^6 + A_7(x,y)Z^7 + A_8(x,y)Z^8, \end{aligned} \tag{49}$$

where $A_i(x,y)$, $i = 0, ..., 7$, are polynomials in x and y. Polynomial $A_0(x,y)$ looks as: $A_0(x,y) = -A_{01}(x,y)A_{02}(x,y)$, where $A_{01}(x,y) = -x^4 + (a_6 - b_7)x^3y + (a_7 - b_8)x^2y^2 + (a_8 - b_9)xy^3 + a_9y^4$, $A_{02}(x,y) = (a_6b_7 - a_7)x^4 + 2(a_6b_8 - a_8)x^3y + (3a_6b_9 + a_7b_8 - a_8b_7 - 3a_9)x^2y^2 + 2(a_7b_9 - a_9b_7)xy^3 + (a_8b_9 - a_9b_8)y^4$.

As $A_{01} \not\equiv 0$, we require A_{02} to be identically equal to zero. Assume $A_{02} \equiv 0$, then $a_7 = a_6b_7,\ \ a_8 = a_6b_8,\ \ a_9 = a_6b_9$.

In these conditions we have $A_1(x,y) = -A_{11}(x,y)A_{12}(x,y)$, where

$A_{11}(x,y) = x^3 + b_7x^2y + b_8xy^2 + b_9y^3 \not\equiv 0$,

$A_{12}(x,y) = (a_4 - a_3a_6 - a_6b_4 + a_6^2b_3 - a_3b_7 + a_6b_3b_7)x^4 + 2(a_5 - a_4a_6 - a_6b_5 + a_6^2b_4 - a_3b_8 + a_6b_3b_8)x^3y + (3a_6^2b_5 - 3a_5a_6 + a_5b_7 - a_4a_6b_7 + a_6^2b_4b_7 - a_6b_5b_7 - a_4b_8 +$

$a_3a_6b_8 - a_6^2b_3b_8 + a_6b_4b_8 - 3a_3b_9 + 3a_6b_3b_9)x^2y^2 - 2(a_5a_6b_7 - a_6^2b_5b_7 + a_4b_9 - a_3a_6b_9 + a_6^2b_3b_9 - a_6b_4b_9)xy^3 + (a_6^2b_5b_8 - a_5a_6b_8 - a_5b_9 + a_4a_6b_9 - a_6^2b_4b_9 + a_6b_5b_9)y^4$.

If $A_{12}(x,y) \equiv 0$, then we obtain the following two series of conditions:

1) $a_3 = a_6b_3,\ a_4 = a_6b_4,\ a_5 = a_6b_5$;

2) $a_4 = a_3a_6 + a_6b_4 - a_6^2b_3 + a_3b_7 - a_6b_3b_7,\ a_5 = a_3a_6^2 + a_6b_5 + a_3b_8 - a_6^3b_3 - a_6b_3b_8 + a_3a_6b_7 - a_6^2b_3b_7,\ b_9 = -a_6(a_6^2 + b_8 + a_6b_7),\ a_3 \neq a_6b_3$.

In the conditions 1) we have $A_2(x,y) = -A_{11}(x,y)A_{21}(x,y)$, where

$A_{21} = (a_2 - 2a_1a_6 - a_6b_2 + 2a_6^2b_1 - a_1b_7 + a_6b_1b_7)x^3 + (3a_6^2b_2 - 3a_2a_6 - 2a_1b_8 + 2a_6b_1b_8 - a_1a_6b_7 + a_6^2b_1b_7)x^3y + (3a_6b_9b_1 - 3a_1b_9 - a_2b_8 + a_6b_2b_8 - 2a_2a_6b_7 + 2a_6^2b_2b_7)xy^2 + (a_1a_6b_9 - 2a_2b_9 + 2a_6b_2b_9 - a_6^2b_9b_1 - a_2a_6b_8 + a_6^2b_2b_8)y^3$.

The identity $A_{21}(x,y) \equiv 0$ leads us to the following two series of conditions:

$$a_1 = a_6b_1,\ a_2 = a_6b_2; \tag{50}$$

$$\begin{aligned} &a_2 = 2a_1a_6 + a_6b_2 - 2a_6^2b_1 + a_1b_7 - a_6b_1b_7,\\ &b_8 = -a_6(3a_6 + 2b_7),\ b_9 = a_6^2(2a_6 + b_7), a_1 \neq a_6b_1. \end{aligned} \tag{51}$$

In conditions (50) we have $A_3(x,y) = \alpha A_{11}(x,y)A_{31}(x,y)$, where $\alpha = a_0 - a_6b_0$, $A_{31}(x,y) = (3a_6 + b_7)x^2 + 2(b_8 + a_6b_7)xy + (3b_9 + a_6b_8)y^2)$.

If $\alpha = 0$, then $deg(GCD(P,Q)) > 0$ (see (9)). Let $\alpha \neq 0$ and $A_{31}(x,y) \equiv 0$, i.e. $b_7 = -3a_6,\ b_8 = 3a_6^2,\ b_9 = -a_6^3$, then $A_4(x,y) = \alpha A_{11}(x,y)((b_4 + 2a_6b_3)x + (2b_5 + a_6b_4)y)$.

The identity $A_4(x,y) \equiv 0$ holds if $b_4 = -2a_6b_3$ and $b_5 = a_6^2b_3$. In these conditions $A_5(x,y) = \alpha A_{11}(x,y)(b_2 + a_6b_1) \equiv 0 \Rightarrow b_2 = -a_6b_1 \Rightarrow A_6(x,y) = 3\alpha^2(a_6y - x)^2 \not\equiv 0$.

We have obtained that $E_1(x,y) = \alpha^2Z^6(3x^2 - 6a_6xy + 3a_6^2y^2 + 2b_3xZ - 2a_6b_3yZ + b_1Z^2)$, and therefore the algebraic multiplicity of the line $Z = 0$ is equal to seven. In this case the cubic system (8) looks as

$$\begin{aligned} &\dot{x} = a_6b_0 + \alpha + a_6b_1x - a_6^2b_1y + a_6b_3x^2 - 2a_6^2b_3xy + a_6^3b_3y^2 + a_6x^3 -\\ &3a_6^2x^2y + 3a_6^3xy^2 - a_6^4y^3, \qquad \dot{y} = b_0 + b_1x - a_6b_1y + b_3x^2 - 2a_6b_3xy +\\ &a_6^2b_3y^2 + x^3 - 3a_6x^2y + 3a_6^2xy^2 - a_6^3y^3. \end{aligned} \tag{52}$$

Via the transformation of coordinates $X = (b_3 + 3x - 3a_6y)/(3\alpha)$, $Y = -((27b_0 - 9b_1b_3 + 2b_3^3)x - (27a_6b_0 - 9a_6b_1b_3 + 2a_6b_3^3 + 27\alpha)y)/(27\alpha^4)$ the system (52) can be written in the form

$$\dot{X} = 1, \qquad \dot{Y} = aX + X^3, \tag{53}$$

where $a = (3b_1 - b_3^2)/(3\alpha^2)$.

In the case of conditions (51) we have $A_3(x,y) = A_{11}(x,y)A_{31}(x,y)$, where

$A_{31}(x,y) = (3a_0a_6 - 3a_6^2b_0 + a_0b_7 - a_6b_0b_7 + b_4\beta - a_6b_3\beta - b_3b_7\beta)x^2 - 2(3a_0a_6^2 - 3a_6^3b_0 + a_0a_6b_7 - a_6^2b_0b_7 - b_5\beta - 2a_6^2b_3\beta - a_6b_3b_7\beta)xy + (3a_0a_6^3 - 3a_6^4b_0 + a_0a_6^2b_7 - a_6^3b_0b_7 + a_6b_5\beta + 2a_6^2b_4\beta + b_5b_7\beta + a_6b_4b_7\beta)y^2$, $\beta = a_1 - a_6b_1 \not\equiv 0$.

The identity $A_{31}(x,y) \equiv 0$ yields $b_4 = (-3a_0a_6 + 3a_6^2b_0 - a_0b_7 + a_6b_0b_7 + a_6b_3\beta + b_3b_7\beta)/\beta$ and $b_5 = -a_6(-3a_0a_6 + 3a_6^2b_0 - a_0b_7 + a_6b_0b_7 + 2a_6b_3\beta + b_3b_7\beta)/\beta$.

Therefore $A_4(x,y) = A_{41}(x,y)A_{42}(x,y)/\beta$, where

$A_{41} = (-x + a_6y)(x + 2a_6y + b_7y) \not\equiv 0$,

$A_{42}(x,y) = (3a_0^2a_6 - 6a_0a_6^2b_0 + 3a_6^3b_0^2 + a_0^2b_7 - 2a_0a_6b_0b_7 + a_6^2b_0^2b_7 - 3a_0a_6b_3\beta + 3a_6^2b_0b_3\beta - a_0b_3b_7\beta + a_6b_0b_3b_7\beta - b_2\beta^2 + 2a_6b_1\beta^2 + b_1b_7\beta^2 - 2\beta^3)x^2 - 2(3a_0^2a_6^2 - 6a_0a_6^3b_0 + 3a_6^4b_0^2 + a_0^2a_6b_7 - 2a_0a_6^2b_0b_7 + a_6^3b_0^2b_7 - 3a_0a_6^2b_3\beta + 3a_6^3b_0b_3\beta - a_0a_6b_3b_7\beta + a_6^2b_0b_3b_7\beta - a_6b_2\beta^2 + 2a_6^2b_1\beta^2 + a_6b_1b_7\beta^2 + 4a_6\beta^3 + 2b_7\beta^3)xy + (3a_0^2a_6^3 - 6a_0a_6^4b_0 + 3a_6^5b_0^2 + a_0^2a_6^2b_7 - 2a_0a_6^3b_0b_7 + a_6^4b_0^2b_7 - 3a_0a_6^3b_3\beta + 3a_6^4b_0b_3\beta - a_0a_6^2b_3b_7\beta + a_6^3b_0b_3b_7\beta - a_6^2b_2\beta^2 + 2a_6^3b_1\beta^2 + a_6^2b_1b_7\beta^2 - 8a_6^2\beta^3 - 8a_6b_7\beta^3 - 2b_7^2\beta^3)y^2$.

If $A_{42}(x,y) \equiv 0$, then $b_2 = -a_6b_1 - 2\beta$, $b_7 = -3a_6$ and $A_5(x,y) = \beta(3a_0 - 3a_6b_0 - b_3\beta)A_{11}(x,y)$.

The identity $A_5(x,y)\equiv 0$ yields $a_0=(3a_6b_0+b_3\beta)/3$. In the above conditions $A_6(x,y)=2\beta^2(x-a_6y)((b_3^2-3b_1)x+(9\beta-a_6(b_3^2-3b_1))y)/3\not\equiv 0$ and $E_1(X)=Z^6\beta^2(3x-3a_6y+b_3Z)(2b_3^2x-6b_1x+6a_6b_1y-2a_6b_3^2y-9b_0Z+b_1b_3Z+18y\beta)/9$. Therefore $m_a(Z=0)=7$ and the cubic system (8) has the form

$$\begin{aligned}\dot{x}&=(3a_6b_0+b_3\beta+3(a_6b_1+\beta)x-3a_6(a_6b_1+3\beta)y+3a_6b_3x^2-6a_6^2b_3xy\\&\quad+3a_6^3b_3y^2+3a_6x^3-9a_6^2x^2y+9a6^3xy^2-3a_6^4y^3)/3,\\ \dot{y}&=b_0+b_1x-(a_6b_1+2\beta)y+b_3x^2-2a_6b_3xy+a_6^2b_3y^2+x^3-3a_6x^2y\\&\quad+3a_6^2xy^2-a_6^3y^3.\end{aligned}\tag{54}$$

The transformation of coordinates $X=(b_3+3x-3a_6y)/3\quad Y=(9b_0-b_1b_3+2(3b_1-b_3^2)x+2(a_6b_3^2-3a_6b_1-9\beta)y)/18$ and time rescaling $t=-\tau/\beta$ reduce (54) to the system

$$\dot{X}=-X,\qquad \dot{Y}=2Y+X^3.\tag{55}$$

In the case 2) we have $A_2(x,y)=-A_{21}(x,y)A_{22}(x,y)$, where

$A_{21}(x,y)=(x^2+a_6xy+b_7xy+a_6^2y^2+b_8y^2+a_6b_7y^2)\not\equiv 0$,

$A_{22}(x,y)=(a_2-2a_1a_6-a_6b_2+2a_6^2b_1-a_1b_7+a_6b_1b_7-b_4\gamma+a_6b_3\gamma+b_3b_7\gamma-\gamma^2)x^4-2(2a_2a_6-a_1a_6^2-2a_6^2b_2+a_6^3b_1+a_1b_8-a_6b_1b_8+b_5\gamma-a_6b_4\gamma-b_8b_3\gamma+a_6\gamma^2+b_7\gamma^2)x^3y+(3a_2a_6^2+3a_1a_6^3-3a_6^3b_2-3a_6^4b_1-a_2b_8+5a_1a_6b_8+a_6b_2b_8-5a_6^2b_1b_8-2a_2a_6b_7+4a_1a_6^2b_7+2a_6^2b_2b_7-4a_6^3b_1b_7+3a_6b_5\gamma+b_4b_8\gamma-3a_6^3b_3\gamma-4a_6b_8b_3\gamma-b_5b_7\gamma+a_6b_4b_7\gamma-3a_6^2b_3b_7\gamma-3a_6^2\gamma^2-2b_8\gamma^2-4a_6b_7\gamma^2-b_7^2\gamma^2)x^2y^2+2(a_2a_6^3-2a_1a_6^4-a_6^4b_2+2a_6^5b_1+a_2a_6b_8-2a_1a_6^2b_8-a_6^2b_2b_8+2a_6^3b_1b_8+2a_2a_6^2b_7-2a_1a_6^3b_7-2a_6^3b_2b_7+2a_6^4b_1b_7-a_6^3b_4\gamma-a_6b_4b_8\gamma+a_6^4b_3\gamma+a_6^2b_8b_3\gamma+a_6b_5b_7\gamma-a_6^2b_4b_7\gamma+a_6^3b_3b_7\gamma-a_6^3\gamma^2-a_6b_8\gamma^2-2a_6^2b_7\gamma^2-b_8b_7\gamma^2-a_6b_7^2\gamma^2)xy^3+(a_1a_6^5-2a_2a_6^4+2a_6^5b_2-a_6^6b_1-a_2a_6^2b_8+a_1a_6^3b_8+a_6^3b_2b_8-a_6^4b_1b_8-2a_2a_6^3b_7+a_1a_6^4b_7+2a_6^4b_2b_7-a_6^5b_1b_7-a_6^3b_5\gamma+a_6^4b_4\gamma+a_6^2b_4b_8\gamma-a_6^2b_5b_7\gamma+a_6^3b_4b_7\gamma-a_6^4\gamma^2-2a_6^2b_8\gamma^2-b_8^2\gamma^2-2a_6^3b_7\gamma^2-2a_6b_8b_7\gamma^2-a_6^2b_7^2\gamma^2)y^4$, $\gamma=a_3-a_6b_3\not\equiv 0$.

The identity $A_{22}(x,y)\equiv 0$ yields $a_2=2a_1a_6+a_6b_2-2a_6^2b_1+a_1b_7-a_6b_1b_7+b_4\gamma-a_6b_3\gamma-b_3b_7\gamma+\gamma^2$, $b_5=-a_6b_4-a_6^2b_3-3a_6\gamma-b_7\gamma$, $b_8=-a_6(3a_6+2b_7)$. Taking into account these conditions the polynomial $A_3(x,y)$ looks as $A_3(x,y)=A_{21}(x,y)A_{31}(x,y)$, where

$A_{31}(x,y)=(-3a_0a_6+3a_6^2b_0-a_1b_4+a_6b_1b_4+a_1a_6b_3-a_6^2b_1b_3-a_0b_7+a_6b_0b_7+a_1b_3b_7-a_6b_1b_3b_7-3a_1\gamma-b_2\gamma+5a_6b_1\gamma+b_4b_3\gamma-a_6b_3^2\gamma+b_1b_7\gamma-b_3^2b_7\gamma+2b_3\gamma^2)x^3+(9a_0a_6^2-9a_6^3b_0+3a_1a_6b_4-3a_6^2b_1b_4-3a_1a_6^2b_3+3a_6^3b_1b_3+3a_0a_6b_7-3a_6^2b_0b_7-3a_1a_6b_3b_7+3a_6^2b_1b_3b_7-3a_1a_6\gamma+3a_6b_2\gamma-3a_6^2b_1\gamma-3a_6b_4b_3\gamma+3a_6^2b_3^2\gamma-4a_1b_7\gamma+a_6b_1b_7\gamma+3a_6b_3^2b_7\gamma-2b_4\gamma^2+2a_6b_3\gamma^2+4b_3b_7\gamma^2-4\gamma^3)x^2y+(9a_6^4b_0-9a_0a_6^3-3a_1a_6^2b_4+3a_6^3b_1b_4+3a_1a_6^3b_3-3a_6^4b_1b_3-3a_0a_6^2b_7+3a_6^3b_0b_7+3a_1a_6^2b_3b_7-3a_6^3b_1b_3b_7-3a_1a_6^2\gamma-3a_6^2b_2\gamma+9a_6^3b_1\gamma+3a_6^2b_4b_3\gamma-3a_6^3b_3^2\gamma-4a_1a_6b_7\gamma+7a_6^2b_1b_7\gamma-3a_6^2b_3^2b_7\gamma-2a_1b_7^2\gamma+2a_6b_1b_7^2\gamma+a_6b_4\gamma^2+2a_6^2b_3\gamma^2-b_4b_7\gamma^2+2a_6b_3b_7\gamma^2+2b_3b_7^2\gamma^2-7a_6\gamma^3-5b_7\gamma^3)xy^2+(3a_0a_6^4-3a_6^5b_0+a_1a_6^3b_4-a_6^4b_1b_4-a_1a_6^4b_3+a_6^5b_1b_3+a_0a_6^3b_7-a_6^4b_0b_7-a_1a_6^3b_3b_7+a_6^4b_1b_3b_7+9a_1A_6^3\gamma+a_6^3b_2\gamma-11a_6^4b_1\gamma-a_6^3b_4b_3\gamma+a_6^4b_3^2\gamma+8a_1a_6^2b_7\gamma-9a_6^3b_1b_7\gamma+a_6^3b_3^2b_7\gamma+2a_1a_6b_7^2\gamma-2a_6^2b_1b_7^2\gamma+a_6^2b_4\gamma^2-6a_6^3b_3\gamma^2+a_6b_4b_7\gamma^2-6a_6^2b_3b_7\gamma^2-2a_6b_3b_7^2\gamma^2-7a_6^2\gamma^3-7a_6b_7\gamma^3-2b_7^2\gamma^3)y^3$.

If $A_{31}(x,y)\equiv 0$, then $b_2=-a_1,\ b_4=-2(a_6b_3+\gamma),\ b_7=-3a_6$ and $A_4(x,y)=A_{21}(x,y)A_{41}(x,y)$, where

$A_{41}(x,y)=(a_1^2-2a_1a_6b_1+a_6^2b_1^2+2a_0\gamma-2a_6b_0\gamma-2a_1b_3\gamma+2a_6b_1b_3\gamma-b_1\gamma^2+b_3^2\gamma^2)x^2-2(a_1^2a_6-2a_1a_6^2b_1+a_6^3b_1^2+2a_0a_6\gamma-2a_6^2b_0\gamma-2a_1a_6b_3\gamma+2a_6^2b_1b_3\gamma-3a_1\gamma^2+2a_6b_1\gamma^2+a_6b_3^2\gamma^2+2b_3\gamma^3)xy+(a_1^2a_6^2-2a_1a_6^3b_1+a_6^4b_1^2+2a_0a_6^2\gamma-2a_6^3b_0\gamma-2a_1a_6^2b_3\gamma+2a_6^3b_1b_3\gamma-6a_1a_6\gamma^2+5a_6^2b_1\gamma^2+a_6^2b_3^2\gamma^2+4a_6b_3\gamma^3+3\gamma^4)y^2$. It is easy to show that $A_{41}(x,y)\not\equiv 0$, and therefore $m_a\{Z=0\}=5$.

In this way we have proved the following theorem.

Theorem 5.1. *For cubic differential systems the algebraic multiplicity of the line at infinity is not greater than seven. Via an affine transformation of coordinates and time rescaling each cubic system for which the line at infinity has the algebraic multiplicity seven can be written in the form of system* (53) *or* (55)*.*

The following two examples show that in the class of cubic systems the maximal geometric multiplicity of the line at infinity is also equal to seven.

Example 5.1. The cubic system

$$\begin{aligned} \dot{X} &= 1 - 4\epsilon^2 + \epsilon(a - 3 + 14\epsilon^2 - 4a\epsilon^2)X + 2\epsilon^2(1 - 7\epsilon^2)X^2 \\ &\quad + \epsilon(1 - 4\epsilon^2 + 4\epsilon^4)X^3, \\ \dot{Y} &= aX + X^3 - 4a\epsilon^2 X - \epsilon(3 - 14\epsilon^2)Y + 4\epsilon^2(1 - 7\epsilon^2)XY \\ &\quad - 2\epsilon^3(1 - 7\epsilon^2)Y^2 - 4\epsilon^2 X^3 + 12\epsilon^5 X^2 Y - 12\epsilon^6 XY^2 + 4\epsilon^7 Y^3 \end{aligned} \tag{56}$$

has six invariant affine straight lines $l_j,\ j = 1, 2, \dots, 6$:
$l_1 \cdot l_2 \cdot l_3 = 1 - 4\epsilon^2 + \epsilon(a - 3 + 14\epsilon^2 - 4a\epsilon^2)X + 2\epsilon^2(1 - 7\epsilon^2)X^2 + \epsilon(1 - 4\epsilon^2 + 4\epsilon^4)X^3$,
$l_4 = 1 - \epsilon X + \epsilon^2 Y,\ l_5 = 1 - 2\epsilon X + 2\epsilon^2 Y,\ l_6 = -1 + 4\epsilon^2 - 2\epsilon^3 X + 2\epsilon^4 Y.$

If $\epsilon \to 0$, then (56) tends to the system (53) and the invariant straight lines $l_j,\ j = 1, ..., 6$ tend to the infinity.

Example 5.2. The cubic system

$$\begin{aligned} \dot{X} &= X(-1 + 3\epsilon X)(1 - 3\epsilon X + 6\epsilon^3 X), \\ \dot{Y} &= 2Y + X^3 - 6\epsilon(1 - \epsilon^2)XY - 12\epsilon^3 Y^2 - \epsilon^2(3 + 6\epsilon^2 - 8\epsilon^4)X^3 \\ &\quad + 24\epsilon^4(1 - \epsilon^2)XY^2 + 16\epsilon^6 Y^3 \end{aligned} \tag{57}$$

has seven invariant affine straight lines:
$l_1 = X,\ \ l_2 = -1 + 3\epsilon X,\ \ l_3 = 1 - 3\epsilon X + 6\epsilon^3 X,\ \ l_4 = 1 - 4\epsilon X + 4\epsilon^3 X - 4\epsilon^3 Y,$
$l_5 = 1 - \epsilon X + 4\epsilon^3 X - 4\epsilon^3 Y,\ \ l_6 = 1 - 2\epsilon X + 2\epsilon^3 X - 2\epsilon^3 Y,\ \ l_7 = -1 + \epsilon X + 2\epsilon^3 X + 4\epsilon^3 Y.$

When $\epsilon \to 0$ the system (57) converges to the system (55) and the straight lines $l_2, l_3, ..., l_7$ tend to the infinity, i.e. for system (55) the line at infinity has the geometric multiplicity seven.

6. Classification of cubic differential systems with two invariant straight lines of maximal total multiplicity

Definition 6.1. We shall say that $(\mu_1; \mu_2; ...; \mu_k; \mu_\infty)$, where $\mu_j \in \mathbb{N}^*,\ j = 1, 2, \dots, k, \infty$ is in the class of cubic systems a sequence of multiplicities of invariant straight lines if there exists a cubic system with k invariant affine straight lines $l_1, ..., l_k$ which have respectively the multiplicities $\mu_1; \mu_2; ...; \mu_k$ and the line at infinity has the multiplicity μ_∞.

Definition 6.2. The sequence of multiplicities $(\mu_1; \mu_2; ...; \mu_k; \mu_\infty)$ is called *maximal with respect the component* $j, j \in \{1, 2, ..., k, \infty\}$ if $(\mu_1; \mu_2; ...; \mu_j + 1; ...; \mu_k; \mu_\infty)$ is not in the class of cubic systems a sequence of multiplicities of invariant straight lines. We will denote this sequence by $m_j(\mu_1; \mu_2; ...; \mu_k; \mu_\infty)$. The sequence of the type $m_j(\mu_1; \mu_2; ...; \mu_k; \mu_\infty)$ is called partial maximal. If the sequence $(\mu_1; \mu_2; ...; \mu_k; \mu_\infty)$ is maximal with respect to all component, then it is called maximal (or total maximal) and is denoted by $m(\mu_1; \mu_2; ...; \mu_k; \mu_\infty)$.

According to Sections 3 and 5, in the class of cubic differential systems $\{(10), (9)\}$ with an invariant affine straight line we have $m(7, 1)$ and $m(1, 7)$ (see Theorem 3.6 and 5.1). Note that $m(1, 7)$ is realizable by system (55).

In this section for cubic differential systems $\{(10),(9)\}$ with an invariant affine straight line we establish all the partial sequence of the type $m_\infty(\mu_1;\mu_\infty)$.

Without loss of generality, we can consider that the invariant affine straight line is described by equation $x=0$. This allows us to use the Lemmas 3.1-3.5. For $\mu_1\in\{2,...,6\}$ we establish μ_∞ such as the sequence (μ_1,μ_∞) to be maximal with respect to the component ∞.

1. $\mu_1=6$ (Lemma 3.5). *Algebraic multiplicity.* Let for system (10) the conditions (9), (13), (21), (29), (40) and (43) are satisfied. Then it looks as

$$\dot{x}=a_6x^3,\quad \dot{y}=b_0+b_1x+b_3x^2+b_6x^3+3a_6x^2y. \tag{58}$$

For (58) the invariant straight line $x=0$ has the algebraic multiplicity at least six. The condition (9) gives $a_6b_0\neq 0$ and the equality $E_1(X)=a_6^2x^6(b_1+2b_3x+3b_6x^2+6a_6xy)$ shows us that the multiplicity of the invariant straight line $x=0$ is exactly equal to six, if $a_6b_1\neq 0$.

Let $a_6b_0b_1\neq 0$. Then for homogeneous system

$$\dot{x}=a_6x^3,\quad \dot{y}=b_0Z^3+b_1xZ^2+b_3x^2Z+b_6x^3+3a_6x^2y,$$

the polynomial $E_1(\mathbb{X})=a_6^2x^6(3b_6x^2+6a_6xy+2b_3xZ+b_1Z^2)$ is not divided by Z. Therefore, the line at infinity has multiplicity exactly equal to one.

Denote $a=b_1/b_0$. The affine transformation $X=x, Y=(2b_3+3b_6x+6a_6y)/(6b_0)$ and the time rescaling $\tau=a_6t$ reduce (58) to the system

$$\dot{x}=x^3,\quad \dot{y}=1+ax+3x^2y, a\neq 0. \tag{59}$$

The equality $E_1(X)=x^6(a+6xy)$ shows that $x=0$ is a single invariant affine straight line for (59). The system (59) is Darboux integrable and has the first integral $F(x,y)=(20x^2y+5ax+4)/(20x^5)$.

Geometric multiplicity. The following example shows that for system (59) the algebraic and geometric multiplicities of the invariant straight line $x=0$ are equal.

Example 6.1. The perturbed system

$$\begin{aligned}&\dot{x}=x(-\epsilon^2-2a\epsilon^3-2x\epsilon+x^2),\\&\dot{y}=1+ax+3x^2y-\epsilon(\epsilon y+4xy+4\epsilon^3y^2-6\epsilon^2xy^2+-4\epsilon^5y^3)\end{aligned} \tag{60}$$

has the invariant straight lines: $l_1=x$, $l_{2,3}=\pm\epsilon+x+2\epsilon^3y$, $l_4=-2\epsilon+x+2\epsilon^3y$, $l_{5,6}=-\epsilon^2-2a\epsilon^3-2x\epsilon+x^2$. If $\epsilon\to 0$ then (60) tend to the system (59) and the straight lines $l_i, i=2,..,6$ tend to the straight line $x=0$.

In this way we have proved the following theorem.

Theorem 6.1. *Via an affine transformation of coordinates and time rescaling any cubic differential system* $\{(10),(9)\}$ *with an invariant straight line of multiplicity six can be written in the form* (59). *This system has a single invariant affine straight line, is Darboux integrable and for it we have* $m_\infty(6;1)$.

2. $\mu_1=5$ (Lemma 3.4). *Algebraic multiplicity.* In the conditions 2.20): (13), (21), (29), (40), the system $\{(10),(9)\}$ looks as

$$\dot{x}=a_6x^3,\quad \dot{y}=b_0+b_1x+b_3x^2+b_6x^3+b_7x^2y,\ \ a_6b_0\neq 0. \tag{61}$$

For (61) $A_j(y)\equiv 0, j=\overline{1,4}$ and $A_5(y)=a_6b_0(b_7-3a_6)$ (see (11)). If $b_7-3a_6\neq 0$, then the invariant straight line $x=0$ of system (61) has algebraic multiplicity exactly equal to five.

We consider the homogeneous system

$$\dot{x}=a_6x^3,\quad \dot{y}=b_0Z^3+b_1xZ^2+b_3x^2Z+b_6x^3+b_7x^2y,\ \ a_6b_0\neq 0, \tag{62}$$

associated to the system (61). We calculate $E_1(X)$ and write it in the form (49). Thus we have: $A_0(x,y) = a_6b_7x^7(b_6x - a_6y + b_7y)$. The polynomial $A_0(x,y)$ is identic zero if $b_7 = a_6$ and $b_6 = 0$ or if $b_7 = 0$. If $b_7 = a_6$ and $b_6 = 0$ then the cubic system (61) has the degenerate infinity. If $b_7 = 0$, then: $A_1(x,y) = -a_6^2b_3x^7 \equiv 0 \Rightarrow b_3 = 0 \Rightarrow A_2(x,y) = -2a_6^2b_1x^6 \equiv 0 \Rightarrow b_1 = 0 \Rightarrow A_3(x,y) = -3a_6^2b_0x^5 \not\equiv 0$, $\mu_\infty = 4$. The cubic system (61) obtains the form

$$\dot{x} = a_6x^3, \quad \dot{y} = b_0 + b_6x^3. \tag{63}$$

The transformation $X = x, Y = (-b_6x + a_6y)/b_0$, $\tau = a_6t$, reduces (63) to the following system

$$\dot{x} = x^3, \quad \dot{y} = 1. \tag{64}$$

The obtained system is Darboux integrable and has the first integral:

$$F(x,y) = (2x^2y + 1)/(2x^2).$$

In conditions 2.21) ((16), (27), (36); (41)) we have the cubic system

$$\begin{aligned} \dot{x} &= x(a_4^2 + 2a_4a_7x + 4a_4a_8y + 4a_6a_8x^2 + 4a_7a_8xy + 4a_8^2y^2)/(4a_8), \\ \dot{y} &= (a_4^3 + 2a_4^2a_7x + 6a_4^2a_8y + 8a_8^2b_3x^2 + 8a_4a_7a_8xy + 12a_4a_8^2y^2 \\ &\quad + 8a_6a_8^2x^2y + 8a_7a_8^2xy^2 + 8a_8^3y^3)/(8a_8^2). \end{aligned} \tag{65}$$

For this system the polynomial $E_1(\mathbb{X})$ looks as $E_1(X) = -x^5(a_4a_6 - 2a_8b_3)^2(a_4 + a_7x + 2a_8y)/(4a_8^2)$. Therefore, if $a_4a_6 - 2a_8b_3 \neq 0$, then the invariant straight line $x = 0$ of system (65) has algebraic multiplicity exactly equal to five.

For the line at infinity of (65) we have: $A_0(x,y) = A_1(x,y) = 0$, $A_2(x,y) = -x^5((a_4a_6 - 2a_8b_3)^2(a_7x + 2a_8y))/(4a_8^2) \not\equiv 0$, and consequently this line has the multiplicity 3, i.e. $\mu_\infty = 3$.

Geometric multiplicity. The geometric multiplicity of the line at the infinity of the system (64) is equal to four. This is confirmed by the following example.

Example 6.2. [10] The system

$$\begin{aligned} \dot{x} &= x(x + 3\epsilon)(x + 6\epsilon), \\ \dot{y} &= (1 - 2\epsilon^2y)(1 + 4\epsilon^2y)(1 - 8\epsilon^2y) \end{aligned} \tag{66}$$

has the invariant straight lines: $l_1 = x$, $l_2 = x + 3\epsilon$, $l_3 = x + 6\epsilon$, $l_4 = x + 8\epsilon^3y + 2\epsilon$, $l_5 = x - 8\epsilon^3y + 4\epsilon$, $l_6 = 1 - 2\epsilon^2y$, $l_7 = 1 + 4\epsilon^2y$, $l_8 = 1 - 8\epsilon^2y$. If $\epsilon \to 0$, then the system (66) tend to (64) and $l_{1,\dots,5} \to x$, $l_{6,7,8} \to \infty$.

Theorem 6.2. *In the class of cubic differential systems* $\{(10),(9)\}$ *with an invariant affine straight line of multiplicity five the maximal multiplicity of the line at infinity is equal to four, i.e.* $m_2(5;4)$. *Any system of this class has a single invariant affine straight line, is Darboux integrable and via an affine transformation of coordinates and time rescaling it can be written in the form* (64).

3. $\mu_1 = 4$ (Lemma 3.3). *Algebraic multiplicity.* In the case 2.9) ((13), (21), (29)) the system $\{(10),(9)\}$ looks as

$$\dot{x} = a_6x^3, \;\; \dot{y} = b_0 + b_1x + b_3x^2 + b_4xy + b_6x^3 + b_7x^2y + b_8xy^2, \;\; a_6b_0 \neq 0. \tag{67}$$

For (67): $A_j(y) = 0$, $j = \overline{1,3}$, $A_4(y) = a_6b_0(b_4 + 2b_8y)$. If $(b_4, b_8) \neq 0$, then the algebraic multiplicity of the straight line $x = 0$ is exactly equal to four.

Assume $(b_4, b_8) \neq 0$ and consider the homogeneous system associated to the system (67). For this system we have $A_0(x,y) = a_6x^5(b_7x + 2b_8y)(b_6x^2 + (b_7 - a_6)xy + b_8y^2)$. If $b_7 = b_8 = 0$, then $A_1(x,y) = a_6x^6((b_4b_6 - a_6b_3)x - 2a_6b_4y) \not\equiv 0$ and $\mu_\infty = 2$. If $b_6 = b_7 - a_6 = b_8 = 0$, then $A_0(x,y) \equiv 0$, $A_1(x,y) \equiv 0$, $A_2(x,y) = a_6x^5((b_3b_4 - a_6b_1)x + b_4^2y) \not\equiv 0$ and $\mu_\infty = 3$.

In condition 2.10) ((13), (22), (30)) the cubic system $\{(10),(9)\}$ has the form

$$\dot{x}=x^2(a_3+a_6x),\quad \dot{y}=b_0+b_1x+b_3x^2+b_6x^3+2a_3xy+b_7x^2y,\quad a_3b_0\neq 0. \tag{68}$$

For (68): $A_j(y)=0,\ j=\overline{1,3},\ A_4(y)=a_3(a_3b_1+b_0b_7-3a_6b_0+2a_3^2y)\neq 0$ and $m_a(x=0)=4$.

For the homogeneous system associated to the system (68) we have $A_0(x,y)=a_6b_7x^7(b_6x-a_6y+b_7y)$. If $b_6x-a_6y+b_7y\equiv 0$, then the infinity of system (68) is degenerate, contrary to (9). If $a_6=0$, then $A_1(x,y)=a_3b_7x^6(b_6x+b_7y)\equiv 0$ $\Rightarrow b_7=0\Rightarrow A_2(x,y)=3a_3^2b_6x^6\not\equiv 0\Rightarrow \mu_\infty=3$. Let $b_7=0$ and $a_6\neq 0$. Then $A_1(x,y)=-a_6x^6(a_6b_3x-3a_3b_6x+4a_3a_6y)\not\equiv 0\Rightarrow \mu_\infty=2$.

It is easy to check that in the cases 2.11), 2.15), 2.16), 2.17), 2.18) and 2.19) the multiplicity of the line at infinity of system $\{(10),(9)\}$ is equal to one because $A_0(x,y)\not\equiv 0$.

In the case 2.12) ((14), (24), (32)) we have the following system

$$\begin{aligned}\dot{x}=x(a_1+a_3x+a_6x^2+a_7xy),\quad \dot{y}=(&a_1b_0+a_3b_0x+a_1^2y+a_6b_0x^2\\ &+(a_1a_3+a_7b_0)xy+a_1b_6x^3+a_1a_6x^2y+a_1a_7xy^2)/a_1\end{aligned} \tag{69}$$

and the polynomials $A_j(y)=0,\ j=1,2,3;\ A_4(y)=3a_1^2b_6$. If $b_6=0$, then the infinity is degenerate. If $b_6\neq 0$, then $A_4(y)\neq 0$ and $m_a(x=0)=4$.

For homogeneous system associated to the system (69) we have $A_0(x,y)=-b_6x^6((a_7b_6-a_6^2)x^2-2a_6a_7xy-a_7^2y^2)$. Because $b_6\neq 0$ the identity $A_0(x,y)\equiv 0$ gives $a_6=a_7=0$. Then, $A_0(x,y)\equiv 0$, $A_1(x,y)\equiv 0$ and $A_2(x,y)=2a_3^2b_6x^6\equiv 0\Rightarrow$ $a_3=0\Rightarrow A_2(x,y)\equiv 0$, $A_3(x,y)\equiv 0$, $A_4(x,y)=3a_1^2b_6x^4\not\equiv 0\Rightarrow \mu_\infty=5$. The cubic system (69) looks as

$$\dot{x}=a_1x,\quad \dot{y}=b_0+a_1y+b_6x^3, a_1\neq 0. \tag{70}$$

The transformations $X=x$, $Y=(b_0+a_1y)/b_6$, $\tau=a_1t$ reduce (70) to the system

$$\dot{x}=x,\quad \dot{y}=y+x^3. \tag{71}$$

This system has the single invariant affine straight line $x=0$, is Darboux integrable and has the first integral $F(x,y)=(2y-x^3)/x$. Note that the system (71) was examined in [10].

In conditions of the case 2.13) ((15), (25), (33)) we have the system

$$\begin{aligned}\dot{x}=x(a_1+a_3x+a_6x^2+a_4y+a_7xy),\ \ \dot{y}=(&a_1^2+a_1a_3x+2a_1a_4y+a_1a_6x^2\\ &+(a_3a_4+a_1a_7)xy+a_4^2y^2+a_4b_6x^3+a_4a_6x^2y+a_4a_7xy^2)/a_4,\end{aligned} \tag{72}$$

$A_j(y)=0,\ j=1,2,3$ and $A_4(y)=2b_6(a_1+a_4y)^2$. If $b_6\neq 0$ then the invariant straight line $x=0$ has the algebraic multiplicity exactly equal to four.

For homogeneous system associated to the system (72) the polynomial $A_0(x,y)=-b_6x^6(-a_6^2x^2+a_7b_6x^2-2a_6a_7xy-a_7^2y^2)$ is identic zero if $a_6=a_7=0$. Then $A_1(x,y)=-a_4b_6^2x^7\not\equiv 0$ and therefore $\mu_\infty=2$.

In the last case 2.14) ((15), (26), (34)) the cubic system has the form

$$\begin{aligned}&\dot{x}=x(a_1+a_3x+a_4y+a_6x^2+a_7xy),\\ &\dot{y}=(-a_1^2+a_1b_2+a_3(b_2-a_1)x+a_4b_2y+a_6(b_2-a_1)x^2\\ &\qquad+(a_3a_4-a_1a_7+a_7b_2)xy+a_4^2y^2+a_4b_6x^3+a_4a_6x^2y+a_4a_7xy^2)/a_4\end{aligned} \tag{73}$$

and for it $A_j(y)=0,\ j=\overline{1,3}$, $A_4(y)=b_6(a_1+a_4y)(4a_1-b_2+2a_4y)$. Let $b_6\neq 0$. Then $A_4(y)\not\equiv 0$ and consequently the algebraic multiplicity of the invariant straight line $x=0$ is exactly equal to four. For the homogeneous system associated to the system (73) we have: $A_0(x,y)=-b_6x^6(-a_6^2x^2+a_7b_6x^2-2a_6a_7xy-a_7^2y^2)\Rightarrow a_6=a_7=0$ $\Rightarrow A_1(x,y)=-a_4b_6^2x^7\not\equiv 0\Rightarrow \mu_\infty=2$.

Geometric multiplicity. The following example shows that the geometric multiplicity of the invariant affine straight line $x = 0$ of the system (71) is equal to four and the line at infinity has the multiplicity equal to five.

Example 6.3. [10] The system

$$\dot{x} = x - 4\epsilon^2 x^3, \quad \dot{y} = y + x^3 - 3\epsilon^2 x^2 y - 9\epsilon^4 xy^2 - 9\epsilon^6 y^3, \tag{74}$$

possesses eight invariant affine straight lanes: $l_1 = x$, $l_{2,3} = x \pm 3\epsilon^2 y$, $l_4 = x + \epsilon^2 y$, $l_{5,6} = 2\epsilon x \pm 1$, $l_{7,8} = \epsilon x + 3\epsilon^3 y \pm 1$, and (74)$\to$ (71), $l_{1,2,3,4} \to x$, $l_{5,6,7,8} \to \infty$ if $\epsilon \to 0$.

Theorem 6.3. *In the class of cubic differential systems* $\{(10), (9)\}$ *with a real invariant affine straight line of multiplicity four the maximal multiplicity of the line at infinity is equal to five, i.e.* $m_2(4; 5)$. *Any system of this class has a single invariant affine straight line, is Darboux integrable and via an affine transformation of coordinates and time rescaling it can be written in the form* (71).

4. $m_1 = 3$ (Lemma 3.2). *Algebraic multiplicity.*

In Case 2.1)((13), (21)) the cubic system $\{(10), (9)\}$ has the form:

$$\begin{aligned} \dot{x} = a_6 x^3, \quad \dot{y} = {} & b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 \\ & + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3, a_6 \neq 0. \end{aligned} \tag{75}$$

For it $A_1(y) \equiv 0$, $A_2(y) \equiv 0$ and $A_3(y) = a_6(b_2 + 2b_5 y + 3b_9 y^2)(b_0 + b_2 y + b_5 y^2 + b_9 y^3)$. If $(b_2, b_5, b_9) \neq 0$ then for system (75) the invariant straight line $x = 0$ has the algebraic multiplicity equal to three. Under this condition we will establish the maximal multiplicity of the line at infinity. For homogeneous system associated to (75) we have $A_0(x, y) = -a_6 x^3(b_7 x^2 + 2b_8 xy + 3b_9 y^2)(-b_6 x^3 + (a_6 - b_7)x^2 y - b_8 xy^2 - b_9 y^3)$ (see (49)). The identity $A_0(x, y) \equiv 0$ holds if $b_6 = b_8 = b_9 = 0$, $b_7 = a_6$ or $b_7 = b_8 = b_9 = 0$. If $b_6 = b_8 = b_9 = 0$, $b_7 = a_6$, then the infinity for cubic system (10) is degenerate. Let $b_7 = b_8 = b_9 = 0$. Relations (9) and equalities $b_7 = b_8 = b_9 = 0$ give $A_1(x, y) = -a_6 x^5((a_6 b_3 - b_4 b_6)x^2 + 2(a_6 b_4 - b_5 b_6)xy + 3a_6 b_5 y^2) \equiv 0 \Rightarrow b_3 = b_4 = b_5 = 0$ and $A_2(x, y) = -a_6 x^5((2a_6 b_1 - b_2 b_6)x + 3a_6 b_2 y) \not\equiv 0$. Therefore, $\mu_\infty = 3$.

In Case 2.2) ((13), (22)) we have the system

$$\begin{aligned} & \dot{x} = x^2(a_3 + a_6 x), \\ & \dot{y} = b_0 + b_1 x + b_3 x^2 + b_4 xy + b_6 x^3 + b_7 x^2 y + b_8 xy^2, \;\; a_3 \neq 0. \end{aligned} \tag{76}$$

For (76): $A_1(y) = A_2(y) = 0$ and $A_3(y) = -a_3 b_0(2a_3 - b_4 - 2b_8 y)$. If $a_3 b_0(|b_4 - 2a_3| + |b_8|) \neq 0$, i.e. $A_3(y) \not\equiv 0$, then the invariant straight line $x = 0$ of the system (76) has the algebraic multiplicity exactly equal to three. For homogeneous system associated to (76) the polynomial $A_0(x, y) = a_6 x^5(b_7 x + 2b_8 y)(b_6 x^2 + (b_7 - a_6)xy + b_8 y^2)$ is identically equal to zero if at

$$a_6 = 0; \tag{77}$$

$$b_7 = b_8 = 0, a_6 \neq 0; \tag{78}$$

$$b_6 = b_8 = 0, b_7 = a_6, a_6 \neq 0 \tag{79}$$

holds. Let us consider (77). In this case: $\{(9), A_1(x, y) = a_3 x^4(b_7 x + 2b_8 y)(b_6 x^2 + b_7 xy + b_8 y^2) \equiv 0\} \Rightarrow b_8 = b_7 = 0$, $a_3 b_6 \neq 0 \Rightarrow A_2(x, y) = a_3(a_3 + b_4)b_6 x^6 \equiv 0 \Rightarrow b_4 = -a_3 \Rightarrow A_3(x, y) = a_3^2 x^4(-b_3 x + 2a_3 y) \not\equiv 0$, $\mu_\infty = 4$.

Assume (78) is realized, then $A_1(x, y) = -a_6 x^6((a_6 b_3 - a_3 b_6 - b_4 b_6)x + 2a_6 b_4 y) \equiv 0 \Rightarrow b_4 = 0$, $b_3 = a_3 b_6 / a_6 \Rightarrow A_2(x, y) = -2a_6^2 b_1 x^6 \equiv 0 \Rightarrow b_1 = 0 \Rightarrow A_3(x, y) = -3a_6^2 b_0 x^5 \not\equiv 0$, $\mu_\infty = 4$.

Under conditions (79) the system (76) has the degenerate infinity.

In Case 2.3) ((13), (23)) we have the system

$$\dot{x} = x^2(a_3 + a_6x + a_7y), \quad \dot{y} = (a_3b_2 + a_7b_1x + a_7b_3x^2 + a_7b_6x^3 + \\ + a_7b_2y + a_7b_4xy + a_7b_7x^2y + a_7b_8xy^2)/a_7 \tag{80}$$

and the polynomials: $A_1(y) = A_2(y) = 0$ and $A_3(y) = -b_2(a_3 + a_7y)(2a_3^2 - a_6b_2 + a_7b_1 - a_3b_4 + 2a_3(2a_7 - b_8)y + a_7(2a_7 - b_8)y^2)/a_7$. If

$$b_2 \neq 0 \tag{81}$$

and $(|2a_7 - b_8| + |2a_3^2 - a_6b_2 + a_7b_1 - a_3b_4|) \neq 0$, then the invariant straight line $x = 0$ of the system (80) has the algebraic multiplicity exactly equal to three. For homogeneous system associated to (80) the polynomial $A_0(x, y)$ look as: $A_0(x, y) = A_{01}(x, y)A_{02}(x, y)$, where $A_{01}(x, y) = x^4(-b_6x^2 + (a_6 - b_7)xy + (a_7 - b_8)y^2)$ and $A_{02}(x, y) = (a_7b_6 - a_6b_7)x^2 - 2a_6b_8xy - a_7b_8y^2$. If $A_{01}(x, y) \equiv 0$, then the infinity for (80) is degenerate. Let $A_{01}(x, y) \not\equiv 0$ and $A_{02}(x, y) \equiv 0$. Then $b_6 = a_6b_7/a_7$ and $b_8 = 0$. In these conditions $A_1(x, y) = x^4(a_6x + a_7y)((a_6a_7b_3 - a_3a_6b_7 + a_7b_3b_7 - a_6b_4b_7 - a_3b_7^2)x^2 + 2a_6a_7b_4xy + a_7^2b_4y^2)$. If $A_1(x, y) \equiv 0$, then $b_4 = a6 + b_7 = 0$ or $b_4 = a_3b_7 - a_7b_3 = 0$. In both cases the identity $A_2(x, y) \equiv 0$ contradicts (81).

In Case 2.4) ((14), (24)) the system $\{(10), (9)\}$ takes the form

$$\dot{x} = x(a_1 + a_3x + a_6x^2 + a_7xy), \quad \dot{y} = (a_1b_0 + a_3b_0x + a_1^2y + a_1b_3x^2 \\ + (a_1a_3 + a_7b_0)xy + a_1b_6x^3 + a_1b_7x^2y + a_1a_7xy^2)/a_1. \tag{82}$$

For (82): $A_1(y) \equiv 0$, $A_2(y) \equiv 0$ and $A_3(y) = -a_1(3a_6b_0 - 2a_1b_3 - b_0b_7 + 3a_1(a_6 - b_7)y)$. The algebraic multiplicity of the straight line $x = 0$ is equal to three if the following inequality $|a_6 - b_7| + |3a_6b_0 - 2a_1b_3 - b_0b_7| \neq 0$ holds. Taking into account this inequality we calculate the maximal multiplicity of the line at infinity $Z = 0$. The identity $A_0(x, y) \equiv 0$, where $A_0(x, y) = x^5(b_6x + (b_7 - a_6)y)((a_6b_7 - a_7b_6)x^2 + 2a_7a_6xy + a_7^2y^2)$, yields the following three sets of conditions:

$$b_6 = 0, b_7 = a_6; \tag{83}$$

$$a_6 = a_7 = 0; \tag{84}$$

$$a_7 = 0, b_7 = 0, a_6 \neq 0. \tag{85}$$

In conditions (83) the cubic system has the degenerate infinity. If the equalities (84) hold, then $A_1(x, y) = a_3b_7x^6(b_6x + b_7y)$. The identity $A_1(x, y) \equiv 0$ yields $a_3 = 0$ or $b_7 = 0, a_3 \neq 0$. If $a_3 = 0$ then $A_2(x, y) = a_1b_7x^5(b_6x + b_7y) \equiv 0 \Rightarrow b_7 = 0 \Rightarrow A_3(x, y) \equiv 0, A_4(x, y) = 3a_1^2b_6x^4 \not\equiv 0$ (see (9)) $\Rightarrow \mu_\infty = 5$.

In this case the system $\{(10), (9)\}$ looks as

$$\dot{x} = a_1x, \quad \dot{y} = b_0 + a_1y + b_3x^2 + b_6x^3, a_1b_3b_6 \neq 0. \tag{86}$$

The affine transformation $X = b_6x/b_3$, $Y = b_6^2(b_0 + a_1y))/b_3^3$ and time rescaling $\tau = a_1t$ reduce the system (86) to the following system

$$\dot{X} = X, \quad \dot{Y} = Y + X^2 + X^3. \tag{87}$$

This system is Darboux integrable and has the following first integral

$$F(X, Y) = (2Y - 2X^2 - X^3)/(2X).$$

If $b_7 = 0, a_3 \neq 0$ then $A_2(x, y) = 2a_3^2b_6x^6 \not\equiv 0 \Rightarrow \mu_\infty = 3$.

If conditions (85) hold, then $A_1(x, y) = -a_6x^6((a_6b_3 - 2a_3b_6)x + 2a_3a_6y) \equiv 0 \Rightarrow a_3 = 0, b_3 = 0 \Rightarrow A_2(x, y) = 3a_1a_6x^5(b_6x - a_6y) \not\equiv 0$, $\mu_\infty = 3$.

In Cases 2.5) and 2.6) we have the systems respectively:

$$\dot{x} = x(a_1 + a_3x + a_4y + a_6x^2 + a_7xy), \quad \dot{y} = (a_1^2 + a_4b_1x + 2a_1a_4y + a_4b_3x^2 \\ + (a_3a_4 + a_1a_7)xy + a_4^2y^2 + a_4b_6x^3 + a_4b_7x^2y + a_4a_7xy^2)/a_4; \tag{88}$$

$$\begin{aligned}&\dot{x}=x(a_1+a_3x+a_4y+a_6x^2+a_7xy),\quad \dot{y}=((b_2-a_1)(a_1+a_3x+a_7xy)\\&\quad+a_4b_2y+a_4b_3x^2+a_4a_3xy+a_4^2y^2+a_4b_6x^3+a_4b_7x^2y+a_4a_7xy^2)/a_4.\end{aligned}\tag{89}$$

For these systems $A_1(y)\equiv 0$, $A_2(y)\equiv 0$ and $A_3(y)$ look as respectively:

$$A_3(y)=-(B_0+B_1y+B_2y^2+B_3y^3)/a_4^2,$$
$$A_3(y)=-(a_1+a_4y)(B_0'+B_1'y+B_2'y^2)/a_4,$$

where $B_0=a_1^2a_3^2a_4-a_1^3a_3a_7+2a_1^3a_4a_6-2a_1a_3a_4^2b_1+a_1^2a_4a_7b_1+a_4^3b_1^2-a_1^2a_4^2b_3-a_1^3a_4b_7$, $B_1=2a_1a_4(3a_1a_4a_6-a_1a_3a_7+a_4a_7b_1-a_4^2b_3-2a_1a_4b_7)$, $B_2=a_4^2(6a_1a_4a_6-a_1a_3a_7+a_4a_7b_1-a_4^2b_3-5a_1a_4b_7)$, $B_3=2a_4^4(a_6-b_7)$, $B_0'=-4a_1^2a_6+5a_1a_6b_2-a_6b_2^2-3a_1a_4b_3+a_4b_2b_3+a_1^2b_7-a_1b_2b_7$, $B_1'=a_4(2a_1a_6+a_6b_2-a_4b_3-3a_1b_7)$, $B_2'=2a_4^2(a_6-b_7)$.

For (88) ((89)) the invariant straight line $x=0$ has the algebraic multiplicity exactly equal to three if $|B_0|+|B_1|+|B_2|+|B_3|\neq 0\quad(|B_0'|+|B_1'|+|B_2'|\neq 0)$.

For both homogeneous systems associated to the systems (88) and (89) we have $A_0(x,y)=x^5(b_6x+(b_7-a_6)y)((a_6b_7-a_7b_6)x^2+2a_6a_7xy+a_7^2y^2)$. In conditions (9) the identity $A_0(x,y)\equiv 0$ holds if $a_7=b_7=0, a_6\neq 0$ or $a_7=a_6=0$. These relations give respectively: $A_1(x,y)=-x^5(a_6^2b_3x^2-2a_3a_6b_6x^2+a_4b_6^2x^2+2a_3a_6^2xy-4a_4a_6b_6xy+3a_4a_6^2y^2)\not\equiv 0$, $\mu_\infty=2$ and $A_1(x,y)=(a_3b_7-a_4b_6)x^6(b_6x+b_7y)\equiv 0\Rightarrow b_6=a_3b_7/a_4$, $A_2(x,y)=b_7x^3(a_3x+a_4y)(2a_3^2x^2-a_4b_3x^2+a_1b_7x^2+4a_3a_4xy+2a_4^2y^2)/a_4\not\equiv 0$, $\mu_\infty=3$.

In each of Cases 2.7) and 2.8) the algebraic multiplicity of the line at infinity of the cubic system $\{(10),(9)\}$ is equal to one because $A_0(x,y)\neq 0$.

Geometric multiplicity. We consider the cubic differential system

$$\begin{aligned}&\dot{X}=X(1+\epsilon)(1+\epsilon+2x\epsilon^2)(1+4X\epsilon^2),\\&\dot{Y}=Y+X^2+X^3+\epsilon((2+\epsilon)(Y+X^2+X^3)-2\epsilon(1+\epsilon)XY\\&\quad+16\epsilon^3(1+\epsilon)Y^2+4\epsilon(1+\epsilon)(-3-3\epsilon+2\epsilon^2)X^2Y\\&\quad+16\epsilon^3(3+6\epsilon+2\epsilon^2)XY^2-64\epsilon^5(1+2\epsilon)Y^3).\end{aligned}\tag{90}$$

This system has seven invariant straight lines:

$l_1=X$, $l_2=X-4\epsilon^2Y$, $l_3=(1+\epsilon)X-4\epsilon^2Y$, $l_4=1+4\epsilon^2X$, $l_5=1+\epsilon+2\epsilon^2X$, $l_6=(1+\epsilon)(1+2\epsilon X)-8\epsilon^3Y$, $l_7=(1+\epsilon)(1-2\epsilon X)+8\epsilon^3(1+2\epsilon)Y$.

If $\epsilon\to 0$ then the system (90) converges to the system (87), the straight lines l_2, l_3 tend to the straight line l_1 and the straight lines l_4,l_5,l_6,l_7 tend to the line at infinity.

In this way, we have proved the following theorem.

Theorem 6.4. *In the class of cubic differential systems* $\{(10),(9)\}$ *with a real invariant affine straight line of multiplicity three the maximal multiplicity of the line at infinity is equal to five, i.e.* $m_2(3;5)$. *Any system of this class has a single invariant affine straight line, is Darboux integrable and via an affine transformation of coordinates and time rescaling it can be written in the form* (87).

5. $\mu_1=2$ (Lemma 3.1).

In the conditions (13) the cubic system (10) looks as

$$\begin{aligned}&\dot{x}=x^2(a_3+a_6x+a_7y),\quad \dot{y}=b_0+b_1x+b_2y+b_3x^2+b_4xy\\&+b_5y^2+b_6x^3++b_7x^2y+b_8xy^2+b_9y^3,\end{aligned}\tag{91}$$

and for which $A_1(y)=0$, $A_2(y)=(b_0+b_2y+b_5y^2+b_9y^3)(a_3b_2-a_7b_0+2a_3b_5y+(a_7b_5+3a_3b_9)y^2+2a_7b_9y^3)$. Therefore, the invariant straight line $x=0$ of (91) has the algebraic multiplicity exactly equal to two if

$$(|b_0|+|b_2|+|b_5|+|b_9|)(|a_3b_2-a_7b_0|+|a_3b_5|+|a_7b_5+3a_3b_9|+|a_7b_9|)\neq 0.\tag{92}$$

For the homogeneous system associated to the system (91) the polynomial $A_0(x,y)$ has the form $A_0(x,y)=x^2A_{01}(x,y)A_{02}(x,y)$, where $A_{01}(x,y)=b_6x^3-a_6x^2y+b_7x^2y-$

$a_7xy^2 + b_8xy^2 + b_9y^3$, $A_{02} = a_6b_7x^3 - a_7b_6x^3 + 2a_6b_8x^2y + 3a_6b_9xy^2 + a_7b_8xy^2 + 2a_7b_9y^3$. If $A_{01}(x,y) \equiv 0$, then the cubic system (91) has the degenerate infinity. Let $A_{01}(x,y) \not\equiv 0$ and $A_{02}(x,y) \equiv 0$. The identity $A_{02}(x,y) \equiv 0$ takes place if at least one of the following three series of conditions

$$a_6 = a_7 = 0, \tag{93}$$

$$a_7 = b_7 = b_8 = b_9 = 0,\ a_6 \neq 0, \tag{94}$$

$$b_6 = a_6b_7/a_7, b_8 = b_9 = 0 \tag{95}$$

holds.

In conditions (93) we have $\{A_1(x,y) = a_3x^2A_{01}(x,y)(b_7x^2 + 2b_8xy + 3b_9y^2) \equiv 0, (9)\} \Rightarrow b_7 = b_8 = b_9 = 0, a_3 \neq 0 \Rightarrow A_2(x,y) = a_3x^2A_{01}(x,y)((a_3 + b_4)x + 2b_5y) \Rightarrow b_4 = -a_3, b_5 = 0, a_3b_6 \neq 0 \Rightarrow A_3(x,y) = a_3x^4((b_2b_6 - a_3b_3)x + 2a_3^2y) \not\equiv 0,\ \mu_\infty = 4$.

In the case (94) we have $A_1(x,y) = -a_6x^5(a_6b_3x^2 - a_3b_6x^2 - b_4b_6x^2 + 2a_6b_4xy - 2b_5b_6xy + 3a_6b_5y^2) \equiv 0 \Rightarrow b_5 = b_4 = 0, b_3 = a_3b_6/a_6 \Rightarrow A_2(x,y) = -a_6x^5(2a_6b_1x - b_2b_6x + 3a_6b_2y)$. Taking into account the inequality (92) the polynomial $A_2(x,y)$ is not identic zero and therefore $\mu_\infty = 3$.

Assume equalities (95) are satisfied, then the identity $A_1(x,y) = -x^3(a_6x + a_7y)((a_6a_7b_3 - a_3a_6b_7 + a_7b_3b_7 - a_6b_4b_7 - a_3b_7^2)x^3 + 2a_6(a_7b_4 - b_5b_7)x^2y + a_7(a_7b_4 + 3a_6b_5 - b_5b_7)xy^2 + 2a_7^2b_5y^3)/a_7 \equiv 0$ give us the following two series of conditions:

$$b_4 = b_5 = 0, b_7 = -a_6; \tag{96}$$

$$b_3 = a_3b_7/a_7, b_4 = b_5 = 0. \tag{97}$$

In conditions (96) we have $\{A_2(x,y) = -x^3((a_3^2a_6^2 + a_6^2a_7b_1 + a_6^3b_2 + 2a_3a_6a_7b_3 + a_7^2b_3^2)x^3 + 2a_6a_7(a_7b_1 + 2a_6b_2)x^2y + a_7^2(a_7b_1 + 5a_6b_2)xy^2 + 2a_7^3b_2y^3)/a_7 \equiv 0,\ (9)\} \Rightarrow b_1 = b_2 = 0, b_3 = -a_3a_6/a_7, b_0 \neq 0 \Rightarrow A_3(x,y) = -2b_0x^3(a_6x + a_7y)^2 \not\equiv 0,\ \mu_\infty = 4$, and in conditions (97): $\{A_2(x,y) = -x^3(a_6x + a_7y)((2a_6a_7b_1 + a_7b_1b_7 - a_6b_2b_7)x^2 + a_7(a_7b_1 + 3a_6b_2)xy + 2a_7^2b_2y^2)/a_7 \equiv 0,\ (9)\} \Rightarrow b_1 = b_2 = 0, b_0 \neq 0 \Rightarrow A_3(x,y) = -b_0x^3(a_6x + a_7y)(3a_6x + b_7x + 2a_7y) \not\equiv 0, \mu_\infty = 4$.

In conditions (14) from Lemma 3.1 the cubic system (10) has the form

$$\begin{aligned}&\dot{x} = x(a_1 + a_3x + a_6x^2 + a_7xy), \quad \dot{y} = b_0 + b_1x + a_1y + b_3x^2 + b_4xy\\ &+b_6x^3 + b_7x^2y + b_8xy^2,\ a_1 \neq 0.\end{aligned} \tag{98}$$

For this system: $A_1(y) \equiv 0$ and $A_2(y) = B_0 + B_1y + B_2y^2$, where $B_0 = -2a_1a_3b_0 - a_7b_0^2 + a_1^2b_1 + a_1b_0b_4$, $B_1 = -2a_1(a_1a_3 + 2a_7b_0 - a_1b_4 - b_0b_8)$, $B_2 = -3a_1^2(a_7 - b_8)$. The invariant straight line $x = 0$ of the system (98) has the multiplicity exactly equal to two if the following inequality $|B_0| + |B_1| + |B_2| \neq 0$ holds.

Next, we consider the homogeneous system associated to the system (98). We have: $A_0(x,y) = x^4A_{01}(x,y)A_{02}(x,y)$, where $A_{01}(x,y) = b_6x^2 + (b_7 - a_6)xy + (b_8 - a_7)y^2$, $A_{02}(x,y) = (a_6b_7 - a_7b_6)x^2 + 2a_6b_8xy + a_7b_8y^2$. If $A_{01}(x,y) \equiv 0$, then for (98) the infinity is degenerate. The identity $A_{02}(x,y) \equiv 0$ holds if at least one of the following three series of conditions (93), (99), (100):

$$a_7 = b_7 = b_8 = 0, a_6 \neq 0; \tag{99}$$

$$b_6 = a_6b_7/a_7, b_8 = 0 \tag{100}$$

is satisfied.

Let the equalities (93) be verified. Then $A_1(x,y) = a_3x^4(b_7x + 2b_8y)(b_6x^2 + b_7xy + b_8y^2) \equiv 0 \Rightarrow b_7 = b_8 = 0, a_3 \neq 0$ or $a_3 = 0$. If $b_7 = b_8 = 0, a_3 \neq 0$, then $\{A_2(x,y) = a_3b_6(a_3 + b_4)x^6 \equiv 0,\ (9)\} \Rightarrow b_4 = -a_3 \Rightarrow A_3(x,y) = -a_3x^4(a_3b_3x - 3a_1b_6x - 2a_3^2y) \not\equiv 0$, $\mu_\infty = 4$.

If $a_3 = 0$ we have: $A_2(x,y) = a_1x^3(b_7x + 2b_8y)(b_6x^2 + b_7xy + b_8y^2) \equiv 0 \Rightarrow b_8 = b_7 = 0 \Rightarrow \{A_3(x,y) = a_1b_4b_6x^5 \equiv 0,\ (9)\} \Rightarrow b_4 = 0, b_6 \neq 0 \Rightarrow A_4(x,y) = 3a_1^2b_6x^4 \not\equiv 0$, $\mu_\infty = 5$. The cubic system $\{(98),(9)\}$ takes the form

$$\dot{x} = a_1x, \quad \dot{y} = b_0 + b_1x + a_1y + b_3x^2 + b_6x^3,\ a_1b_6 \neq 0. \tag{101}$$

The affine transformation $X = \sqrt[3]{b_6/a_1}x$, $Y = y + b_0/a_1$ and time rescaling $\tau = a_1t$ reduce (101) to the system

$$\dot{x} = x, \quad \dot{y} = y + ax + bx^2 + x^3, a \neq 0, \tag{102}$$

where $a = b_1/\sqrt[3]{a_1^2b_6}$, $b = b_3/\sqrt[3]{a_1b_6^2}$. This system has: a) the single affine invariant straight line: $x = 0$; b) the integrating factor of the Darboux form: $\mu(x,y) = 1/x^2$; and c) the first integral: $F(x,y) = (2y - x^3 - 2ax^2 - 2bx\ln x)/x$.

În conditions (99) we have: $A_1(x,y) = -a_6x^6((a_6b_3 - a_3b_6 - b_4b_6)x + 2a_6b_4y) \equiv 0 \Rightarrow b_4 = 0, b_3 = a_3b_6/a_6 \Rightarrow A_2(x,y) = -a_6x^5((2a_6b_1 - 3a_1b_6)x + 3a_1a_6y) \not\equiv 0,\ \mu_\infty = 3$, and in conditions (100):

$$\begin{aligned}&A_1(x,y) = x^4(a_6x + a_7y)((a_6b_4b_7 - (a_6 + b_7)(a_7b_3 - a_3b_7))x^2 + 2a_6a_7b_4xy\\ &+a_7^2b_4y^2)/a_7 \equiv 0 \Rightarrow b_4 = 0, b_3 = a_3b_7/a_7 \Rightarrow A_2(x,y) = -x^3(a_6x + a_7y)((2a_6a_7b_1 -\\ &3a_1a_6b_7 + a_7b_1b_7 - a_1b_7^2)x^2 + a_7(3a_1a_6 + a_7b_1 - a_1b_7)xy + 2a_1a_7^2y^2)/a_7 \not\equiv 0,\ \mu = 3;\end{aligned}$$

or

$$\begin{aligned}&b_4 = 0, b_7 = -a_6 \Rightarrow A_2(x,y) = -x^3((a_3^2a_6^2 + 2a_1a_6^3 + a_6^2a_7b_1 + 2a_3a_6a_7b_3 + a_7^2b_3^2)x^3\\ &+2a_6a_7(3a_1a_6 + a_7b_1)x^2y + a_7^2(6a_1a_6 + a_7b_1)xy^2 + 2a_1a_7^3y^3)/a_7 \not\equiv 0,\ \mu = 3.\end{aligned}$$

In conditions (15) from Lemma 3.1 the cubic system (10) takes the form

$$\begin{aligned}&\dot{x} = x(a_1 + a_3x + a_4y + a_6x^2 + a_7xy), \quad \dot{y} = (a_1b_2 - a_1^2 + a_4b_1x\\ &+a_4b_2y + a_4b_3x^2 + a_4b_4xy + a_4^2y^2 + a_4b_6x^3 + a_4b_7x^2y + a_4b_8xy^2)/a_4.\end{aligned} \tag{103}$$

For this system: $A_1(y) = 0$, $A_2(y) = -(a_1 + a_4y)(B_0 + B_1y + B_2y^2 + B_3y^3)/a_4^2$, where $B_0 = a_1^3a_7 - 3a_1^2a_3a_4 - 2a_1a_4^2b_1 + 4a_1a_3a_4b_2 - 2a_1^2a_7b_2 + a_4^2b_1b_2 - a_3a_4b_2^2 + a_1a_7b_2^2 + a_1^2a_4b_4 - a_1a_4b_2b_4$, $B_1 = 2a_1a_4(a_3a_4 - 2a_1a_7 + 2a_7b_2 - a_4b_4 + a_1b_8 + b_2b_8)$, $B_2 = a_4^2(a_3a_4 + a_1a_7 + 2a_7b_2 - a_4b_4 - 2a_1b_8 - b_2b_8)$, $B_3 = 2a_4^3(a_7 - b_8)$. The multiplicity of the invariant straight line $x = 0$ is exactly equal to two if the following inequality $|B_0| + |B_1| + |B_2| + |B_3| \neq 0$ holds.

The homogeneous system associated to the system (103) gives us: $A_0(x,y) = x^4A_{01}(x,y)A_{02}(x,y)$, where $A_{01}(x,y) = b_6x^2 + (b_7 - a_6)xy + (b_8 - a_7)y^2$, $A_{02}(x,y) = (a_6b_7 - a_7b_6)x^2 + 2a_6b_8xy + a_7b_8y^2$, and $\{A_0(x,y) \equiv 0,\ (9)\} \Rightarrow A_{02}(x,y) \equiv 0 \Rightarrow (93)$, (99) or (100).

In conditions (93) we have $A_1(x,y) = x^3A_{01}(x,y)((a_3b_7 - a_4b_6)x^2 + 2a_3b_8xy + a_4b_8y^2) \equiv 0 \Rightarrow b_8 = 0, b_6 = a_3b_7/a_4 \Rightarrow A_2(x,y) = b_7x^3(a_3x + a_4y)((a_3^2 + a_3b_4 - a_4b_3 + a_1b_7)x^2 + 4a_4a_3xy + 2a_4^2y^2)/a_4 \not\equiv 0$, $\mu_\infty = 3$. Note that (9) imposes $b_7 \neq 0$.

In each of the cases (99) and (100) we have respectively

$A_1(x,y) = -x^5((a_6^2b_3 - a_3a_6b_6 - a_6b_4b_6 + a_4b_6^2)x^2 + 2a_6(a_6b_4 - 2a_4b_6)xy + 3a_4a_6^2y^2) \not\equiv 0$;

$A_1(x,y) = -x^3(a_6x + a_7y)((a_6a_7^2b_3 - a_3a_6a_7b_7 + a_7^2b_3b_7 - a_6a_7b_4b_7 + a_4a_6b_7^2 - a_3a_7b_7^2)x^3 + 2a_6a_7(a_7b_4 - 2a_4b_7)x^2y + a_7^2(3a_4a_6 + a_7b_4 - 2a_4b_7)xy^2 + 2a_4a_7^3y^3)/a_7^2 \not\equiv 0$, and, consequently, in both of the cases the multiplicity of the line at infinity is equal to two.

In the last case (16) from Lemma 3.1 we have the cubic differential system

$$\begin{aligned}&\dot{x} = x(a_1 + a_3x + a_4y + a_6x^2 + a_7xy + a_8y^2),\\ &\dot{y} = (a_1(b_5 - a_4) + a_8b_1x + (a_1a_8 + a_4b_5 - a_4^2)y + a_8b_3x^2 + a_8b_4xy\\ &\qquad +a_8b_5y^2 + a_8b_6x^3 + a_8b_7x^2y + a_8b_8xy^2 + a_8^2y^3)/a_8\end{aligned} \tag{104}$$

for which $A_1(y) \equiv 0$ and $A_2(y) = -(a_1 + a_4 y + a_8 y^2)(B_0 + B_1 y + B_2 y^2 + B_3 y^3 + B_4 y^4)/a_8^2$, where $B_0 = a_1 a_4^2 a_7 - a_3 a_4^3 - 2a_1 a_3 a_4 a_8 - a_4^2 a_8 b_1 - a_1 a_8^2 b_1 + a_1 a_4 a_8 b_4 + 2a_3 a_4^2 b_5 - 2a_1 a_4 a_7 b_5 + 2a_1 a_3 a_8 b_5 + a_4 a_8 b_1 b_5 - a_1 a_8 b_4 b_5 - a_3 a_4 b_5^2 + a_1 a_7 b_5^2$, $B_1 = -2a_8(a_3 a_4^2 + 2a_1 a_4 a_7 - a_1 a_3 a_8 + a_4 a_8 b_1 + a_1 a_8 b_4 - 2a_3 a_4 b_5 - 2a_1 a_7 b_5 - a_8 b_1 b_5 + a_3 b_5^2 - a_1 a_4 b_8 + a_1 b_5 b_8)$, $B_2 = -a_8(3a_4^2 a_7 - 3a_3 a_4 a_8 - 3a_1 a_7 a_8 - a_8^2 b_1 + 2a_4 a_8 b_4 - 4a_4 a_7 b_5 + 2a_3 a_8 b_5 - a_8 b_4 b_5 + a_7 b_5^2 - a_4^2 b_8 + 3a_1 a_8 b_8 + a_4 b_5 b_8)$, $B_3 = 2a_4 a_8^2(a_7 - b_8)$, $B_4 = a_8^3(a_7 - b_8)$. Let $A_2(y) \not\equiv 0$. Then the algebraic multiplicity of the invariant line $x = 0$ of the system (104) is exactly equal to two.

For homogeneous system associated to the system (104) the polynomial $A_0(x, y)$ has the form: $A_0(x, y) = x^2 A_{01}(x, y) A_{02}(x, y)$, where $A_{01} = (a_6 b_7 - a_7 b_6)x^4 + 2(a_6 b_8 - a_8 b_6)x^3 y + (3a_6 a_8 - a_8 b_7 + a_7 b_8)x^2 y^2 + 2a_7 a_8 x y^3 + a_8^2 y^4 \not\equiv 0$ and $(9) \Rightarrow A_{02} = b_6 x^2 + (b_7 - a_6)xy + (b_8 - a_7)y^2 \not\equiv 0$. Therefore the line at infinity has the multiplicity equal to one.

Geometric multiplicity.

We show that the invariant straight line (including the line at infinity) of the system (102) has the same algebraic and geometric multiplicities.

The system

$$\begin{aligned} &\dot{x} = x(3 + 2a\epsilon + b\epsilon x)(3 + 2a\epsilon - b\epsilon x), \\ &\dot{y} = \epsilon(3 + 2a\epsilon)(9 + 6a\epsilon - b^2\epsilon)x^2 y + \epsilon^2(3x + \epsilon y)(9 + 12a\epsilon - b^2\epsilon + 4a^2\epsilon^2 \\ &\qquad -ab^2\epsilon^2)y^2 + (3 + 2a\epsilon)^2(ax + (1 + a\epsilon)y + b\epsilon xy + bx^2 + x^3) \end{aligned} \tag{105}$$

has the invariant straight lines: $l_1 = x$, $l_2 = x + \epsilon y$, $l_{3,4} = 3 + 2a\epsilon \pm b\epsilon x$, $l_5 \cdot l_6 = \epsilon(3 + 2a\epsilon)^2 x^2 + (1 + a\epsilon)(3 + 2a\epsilon)(3 + 2a\epsilon + 2bx\epsilon) + y\epsilon^2(2x + y\epsilon)(9 + 12a\epsilon - b^2\epsilon + 4a^2\epsilon^2 - ab^2\epsilon^2)$, and $(105) \to (102)$, $l_2 \to l_1$, $l_{3,4,5,6} \to \infty$ if $\epsilon \to 0$.

Thus it has been proved the following theorem.

Theorem 6.5. *In the class of cubic differential systems* $\{(10), (9)\}$ *with a real invariant affine straight line of multiplicity two the maximal multiplicity of the line at infinity is equal to five, i.e.* $m_2(2; 5)$*. Any system of this class has a single invariant affine straight line, is Darboux integrable and via an affine transformation of coordinates and time rescaling it can be written in the form* (102).

References

[1] J. Artes, B. Grünbaum and J. Llibre, On the number of invariant straight lines for polynomial differential systems, *Pacific J. of Math.* **184** (1998), no. 2, 207–230.

[2] J. Artes and J. Llibre, On the number of slopes of invariant straight lines for polynomial differential systems, *J. of Nanjing University* **13** (1996), 143–149.

[3] C. Bujac, One subfamily of cubic systems with invariant lines of total multiplicity eight and with two distinct real infinite singularities. *Bul. Acad. Ştiinţe Repub. Mold.,* Mat., 2015, no. 1(77), 1–39.

[4] C. Bujac and N. Vulpe, Cubic systems with invariant straight lines of total multiplicity eight and with three distinct infinite singularities, *Qual. Theory Dyn. Syst.* **14** (2015), no. 1, 109-137.

[5] C. Bujac and N. Vulpe, Cubic systems with invariant lines of total multiplicity eight and with four distinct infinite singularities, *Journal of Mathematical Analysis and Applications,* **423** (2015), no. 2, 1025–1080.

[6] C. Christopher, J. Llibre and J.V. Pereira, Multiplicity of invariant algebraic curves in polynomial vector fields, *Pacific J. of Math.* **229** (2007), no. 1, 63–117.

[7] D. Cozma and A. Şubă, The solution of the problem of center for cubic differential systems with four invariant straight lines. *An. Ştiinţ. Univ. "Al. I. Cuza"(Iaşi)*, **44** (1998), suppl., 517–530.

[8] G. Suo and J. Sun, The n-degree differential system with $(n-1)(n+1)/2$ straight line solutions has no limit cycles, *Proc. of Ordinary Differential Equations and Control Theory,* Wuhan. 1987, 216–220 (in Chinese).

[9] R. Kooij, Cubic systems with four line invariants, including complex conjugated lines, *Math. Proc. Camb. Phil. Soc.* **118** (1995), no. 1, 7–19.

[10] J. Llibre and N. Vulpe, Planar cubic polynomial differential systems with the maximum number of invariant straight lines, *Rocky Mountain J. Math.* **36** (2006), no. 4, 1301–1373.

[11] R.A. Lyubimova, About one differential equation with invariant straight lines, *Differential and integral equations*, Gorky Universitet., **8** (1984), 66–69; **1** (1997), 19–22 (in Russian).

[12] V.I. Mironenko, *Linear dependence of functions along solutions of differential equations,* Beloruss. Gos. Univ., Minsk, 1981, –104 pp. (in Russian).

[13] V. Puţuntică and A. Şubă, The cubic dif ferential system with six real invariant straight lines along two directions, *Studia Universitatis.* 2008, no. 8(13), 5–16.

[14] V. Puţuntică and A. Şubă, The cubic differential system with six real invariant straight lines along three directions, *Bul. Acad. Ştiinţe Repub. Mold.,* Mat., 2009, no. 2(60), 111–130.

[15] G.S. Rychkov, The limit cycles of the equation $u(x+1)du = (-x + ax^2 + bxu + cu + du^2)dx$, *Differentsial'nye Uravneniya,* **8** (1972), no. 12, 2257–2259 (in Russian).

[16] A. Şubă and D. Cozma, Solution of the problem of the center for cubic differential system with three invariant straight lines in generic position, *Qual. Th. of Dyn. Systems,* **6** (2005), 45–58.

[17] A. Şubă, V. Repeşco and V. Puţuntică, Cubic systems with invariant affine straight lines of total parallel multiplicity seven, *Electron. J. Diff. Equ.*, Vol. 2013 (2013), no. 274, 1–22. http://ejde.math.txstate.edu/

[18] X. Zhang, Number of integral lines of polynomial systems of degree three and four, *J. of Nanjing University,* Math. Biquartely. **10** (1993), 209–212.

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