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# Multiple solutions for a Robin problem involving the $p(x)$-biharmonic operator

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Abstract. This article is devoted to the solvability of Robin boundary problem involving the $p(x)$-biharmonic operator with two parameters. Using as main tool a result due to Ricceri, we obtain the existence of at least three nontrivial solutions.

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## 1. Introduction

In recent years, various mathematical problems with variable exponent growth condition have been received considerable attention (see [5, 10, 16]). The interest in studying such problems arise from nonlinear elasticity theory, electrorheological fluids (cf. [19, 22]) and image processing (cf. [4]). We point out that, this kind of problems have been the subject of a large literature and many results have been obtained. We can cite, among others, the articles [1, 2, 3, 9, 13, 17, 21] and references therein for details.

Here, we are concerned with the following fourth-order quasilinear elliptic equation with Robin boundary conditions

$$\begin{aligned} \Delta^2_{p(x)} u = \lambda f(x,u) + \mu g(x,u), \quad \text{in } \Omega, \\ |\Delta u|^{p(x)-2}\frac{\partial u}{\partial \nu} + m(x)|u|^{p(x)-2}u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, $p \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$, $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator of fourth order, $f,\ g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, $\lambda, \mu > 0$ are real numbers and $m \in L^\infty(\Omega)$ with $ess\, inf_{x\in\Omega} m(x) = m_0 > 0$.

Precise that elliptic equations involving the $p(x)$-biharmonic equations are not trivial generalizations of similar problems studied in the constant case since the $p(x)$-biharmonic operator is not homogeneous and, thus, some techniques which can be applied in the case of the $p$-biharmonic operators will fail in that new situation, such as the Lagrange Multiplier Theorem.

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To our best of knowledge, there seems few results about multiple solutions to $p(x)$-biharmonic equation. Although a natural extension of the theory, the problem addressed here is a natural continuation of recent papers. In [15], for the $p(x)$-laplacian Neumann problem, authors have obtained at least three weak solutions, which generalizes the corresponding result of [12]. In [6], the authors show the existence of at least three solutions to a Navier boundary problem involving the $p(x)$-biharmonic operator.

Motivated by the above papers and the ideas introduced in [15], the purpose of this work is to extend the results of [15] to the case of $p(x)$-biharmonic equation with Robin boundary condition. Our technical approach is an adaptation of variational method. More precisely, we assume $f(x,u)$ and $g(x,u)$ satisfies the following conditions:

**($\mathbf{f_1}$)**

$$|f(x,s)| \leq a_1 + a_2|s|^{\alpha(x)-1}, \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

**($\mathbf{g_1}$)**

$$|g(x,s)| \leq b_1 + b_2|s|^{\beta(x)-1}, \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

for some $\alpha, \beta \in C_+(\overline{\Omega})$ with $\alpha^+ < p^-$ and $a_i, b_i$ $(i = 1, 2)$ are positive constants, where

$$C_+(\overline{\Omega}) := \Big\{ p \in C(\overline{\Omega}) : \quad p(x) > 1, \forall x \in \overline{\Omega} \Big\},$$

and

$$h^- = \min_{x \in \overline{\Omega}} h(x), \quad h^+ = \max_{x \in \overline{\Omega}} p(x) \text{ for any } \in C_+(\overline{\Omega}).$$

**($\mathbf{f_2}$)**

$$\begin{aligned} &|f(x,s)| < 0, \quad \text{for } s \in (0, s_0), \\ &|f(x,s)| > M > 0, \quad \text{for } s \in (s_0, +\infty), \end{aligned}$$

where $M$ and $s_0$ are positive constants.

Using the three critical points theorem of Ricceri [18] which is a powerful tool to study boundary problem of differential equation (see, for example, [3, 14]), we prove that problem 1 has at least three weak solutions for $\lambda$ sufficiently large and requiring $\mu$ small enough.

The paper consists of three sections. In the the second section, we list some well known definitions, basic properties, recall some background facts concerning generalized Lebesgue-Sobolev spaces and introduce some notations used below. In third section, we recall Ricceri's three critical points theorem at first, then prove our main result.

## 2. Preliminaries and main result

For completeness, we introduce some theories of Lebesgue-Sobolev space with variable exponent. The detailed description can be found in, for example, [7, 8, 11, 20, 21].

For any $p \in C_+(\overline{\Omega})$, as in the constant exponent case, define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) := \Big\{ u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \Big\}.$$

Equipped with the so called Luxemburg norm

$$|u|_{p(x)} = \inf\big\{\nu > 0 : \int_\Omega |\frac{u(x)}{\nu}|^{p(x)}dx \le 1\big\}$$

the space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a separable and reflexive Banach space.

For any positive integer $k$, the generalized Sobolev space $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \le k\big\}.$$

Endowed with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\le k} |D^\alpha u|_{p(x)},$$

$W^{k,p(x)}(\Omega)$ is also a separable and reflexive Banach space.

For any $x \in \overline{\Omega}$ and $k \ge 1$,

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N,\\ \infty & \text{if } kp(x) \ge N,\end{cases}$$

denote the critical exponent. Obviously, $p(x) < p_k^*(x)$ for all $x \in \overline{\Omega}$

**Proposition 2.1.** [7] *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \le p_k^*(x)$ for all $x \in \overline{\Omega}$), there is a continuous and compact embedding $W^{k,p(x)}(\overline{\Omega}))$ into $L^{r(x)}(\overline{\Omega}))$.*

Define

$$\|u\|_m = \inf\big\{\nu > 0 : \rho\Big(\frac{u}{\nu}\Big) \le 1\big\} \qquad \text{for } u \in X,$$

with

$$\rho_m(u) = \int_\Omega |\Delta u|^{p(x)}dx + \int_{\partial\Omega} m(x)|u|^{p(x)}d\sigma, \qquad \text{for } u \in X,$$

where $d\sigma$ is the measure on the boundary $\partial\Omega$. In view of $m_0 > 0$, it is easy to see that $\|.\|_m$ which will be used, is a norm equivalent to the norm $\|.\|_{2,p(x)}$. Moreover, similar to [7, Theorem 3.1], we have

**Proposition 2.2.** *The following statements hold true:*

(1) $\rho_m\Big(u\Big/|u|_{p(x)}\Big) = 1$.

(2) $\|u\|_m < 1 (= 1, > 1) \iff \rho_m(u) < 1 (= 1 > 1)$.

(3) $\|u\|_m < 1 \Longrightarrow \|u\|_m^{p^+} \le \rho_m(u) \le \|u\|_m^{p^-}$.

(4) $\|u\|_m > 1 \Longrightarrow \|u\|_m^{p^-} \le \rho_m(u) \le \|u\|_m^{p^+}$.

Here, problem (1) is stated in the framework of the generalized Sobolev space $X := W^{2,p(x)}(\Omega)$. A function $u \in X$ is said to be a weak solution of problem (1) if

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx + \int_{\partial\Omega} m(x)|u|^{p(x)-2}uv d\sigma = \lambda\int_\Omega f(x,u)v dx + \mu\int_\Omega g(x,u)v dx,$$

for all $v \in X$.

Now, we can state our main result as follows.

**Theorem 2.3.** *If* $(\mathbf{f_1})$, $(\mathbf{f_2})$ *hold and* $\frac{N}{2} < p^-$. *Then, there exist an open interval $\Lambda \subseteq (0, +\infty)$ and a positive real number $\rho > 0$ such that each $\lambda \in \Lambda$ and every function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ which satisfying* $(\mathbf{g_1})$, *there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$ problem* (1) *has at least three solutions whose norms are less than $\rho$.*

## 3. Proof of main result

Throughout the sequel, the letters $a_i, i = 1, 2, ...,$ denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process.

To prove the existence of at least three weak solutions for each of the given problem (1), we will use the revised form of Ricceri's three critical points theorem stated as follows.

**Theorem 3.1.** [18] *Let $X$ be a reflexive real Banach space. $\Phi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $X'$ and $\Phi$ is bounded on each bounded subset of $X$; $\Psi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbb{R}$ an interval. Assume that*

(i) $\lim_{\|x\|\to+\infty}(\Phi(x) + \lambda\Psi(x)) = +\infty,$ *for all* $\lambda > 0$,

(ii) *there exist* $r \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that* $\Phi(u_0) < r < \Phi(u_1)$,

(iii) $\inf\limits_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \dfrac{\big(\Phi(u_1) - r\big)\Psi(u_0) + \big(r - \Phi(u_0)\big)\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}.$

*Then, there exists an open interval $\Lambda \subseteq (0, \infty)$ and a positive real number $\rho$ with the following property: for every $\lambda \in \Lambda$ and every $C^1$ functional $J\colon X \mapsto \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the equation*

$$\Phi'(x) + \lambda\Psi'(x) + \mu J'(x) = 0$$

*has at least three solutions in $X$ whose norms are less than $\rho$.*

Let $H : X \to \mathbb{R}$ be the energy functional corresponding to problem (1) defined by

$$H(u) = \Phi(u) + \lambda\Psi(u) + \mu J(u), \tag{2}$$

where

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx + \int_{\partial\Omega} \frac{m(x)}{p(x)}|u|^{p(x)}d\sigma, \tag{3}$$

$$\Psi(u) = -\int_\Omega F(x, u)dx, \tag{4}$$

$$J(u) = -\int_\Omega G(x, u)dx, \tag{5}$$

where $F(x, u) = \int_0^u f(x, s)ds$ and $G(x, u) = \int_0^u g(x, s)ds$.

It is well known that $\Phi, \Psi, J \in C^1(X, R)$ with the derivatives given by

$$\langle\Phi'(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx + \int_{\partial\Omega} m(x)|u|^{p(x)-2}uvdx,$$

$$\langle\Psi'(u), v\rangle = -\int_\Omega f(x, u)vdx,$$

$$\langle J'(u), v\rangle = -\int_\Omega g(x, u)vdx,$$

for any $u, v \in X$.

Arguments similar to those used in the proof of [1, Proposition 4.2], we have the following

**Proposition 3.2.** $\Phi^{'}: X \to X^{'}$ *is a*

1. *continuous, bounded, of type* $(S)^+$ *and strictly monotone operator,*
2. *homeomorphism.*

Now, it is enough to verify that $\Phi, \Psi$ and J satisfy the hypotheses of Theorem 3.1. Obviously, by proposition 3.2, $(\Phi^{'})^{-1}: X^{'} \to X$ exists and continuous. Moreover, in view of $(f1)$ and [11], $\Psi^{'}, J^{'}: X \to X^{'}$ are completely continuous, which imply $\Psi^{'}$ and $J^{'}$ are compact. Thus, the precondition of Theorem 3.1 is satisfied. It remains to verify that the conditions (i), (ii) and (iii) are fulfilled.

First, we claim that condition (i) is satisfied. In fact, by Proposition 2.2, we have

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx + \int_{\partial\Omega} \frac{m(x)}{p(x)}|u|^{p(x)}d\sigma \geq \frac{1}{p^+}\|u\|_m^{p^-}, \tag{6}$$

for every $u \in X$ with $\|u\|_m > 1$.

On the other hand, due to the assumption $(f1)$, we have

$$|F(x,s)| \leq a_1|s| + \frac{a_2}{\alpha(x)}|s|^{\alpha(x)}, \quad \text{a.e.} x \in \Omega, \forall s \in \mathbb{R}.$$

Therefore

$$\begin{aligned}\Psi(u) &= -\int_\Omega F(x,u)dx \geq -a_1\int_\Omega |u|dx - a_2\int_\Omega \frac{1}{\alpha(x)}|u|dx^{\alpha(x)}\\ &\geq -a_3\|u\|_m - \frac{a_2}{\alpha^+}\int_\Omega \big(|u|^{\alpha^+} + |u|^{\alpha^-}\big)dx = -a_3\|u\|_m - a_4\big(|u|_{\alpha^+}^{\alpha^+} + |u|_{\alpha^-}^{\alpha^-}\big).\end{aligned}$$

Since $X$ is continuously embedded in $L^{\alpha^+}(\Omega)$ and $L^{\alpha^-}(\Omega)$, it follows

$$\Phi(u) \geq -a_3\|u\|_m - a_5\big(\|u\|_m^{\alpha^+} + \|u\|_m^{\alpha^-}\big). \tag{7}$$

So, combining the two inequalities (6) and (7), for any $\lambda > 0$ we obtain

$$\Phi(u) + \lambda\Psi(u) \geq \frac{1}{p^+}\|u\|_m^{p^-} - \lambda a_3\frac{1}{p^+}\|u\|_m - \lambda a_5\big(\|u\|_m^{\alpha^+} + \|u\|_m^{\alpha^-}\big),$$

for $u \in X$ with $\|u\|_m > 1$. As $1 < \alpha^+ < p^-$, one has $\lim\limits_{\|u\|_m \to \infty} \Phi(u) + \lambda\Psi(u) = \infty$ and the condition (i) is verified.

Secondly, we will verify the conditions (ii). Precise that, from assumption $(\mathbf{f_2})$, $F(x,t)$ is increasing for $t \in (s_0, 1)$ and decreasing for $t \in (0, s_0)$, uniformly with respect to $x$. Moreover, $F(x,t) \to \infty$ as $t \to \infty$, so, there exists a real number $\delta > s_0$ such that

$$F(u,t) \geq 0 = F(u,0) \geq F(u,s), \quad \forall u \in X, t > \delta, s \in (0, s_0).$$

Furthermore, since $\frac{N}{2} < p^-$, there is a continuous embedding of X into $W^{2,p^-}(\Omega)$ which is continuously embedded in $C(\overline{\Omega})$. Then, there exists a constant $k > 0$ such that

$$\|u\|_\infty := \sup_{x \in \overline{\Omega}} |u(x)| \leq k\|u\|_m, \quad \forall u \in X. \tag{8}$$

Let choose $A$ and $B$ two real numbers such that $0 < A < \min\{t_0, k\}$ and $B > \delta$ satisfying

$$B^{p^\pm}\|m\|_{L^1(\partial\Omega)} > 1, \quad \text{where} \quad p^\mp = \begin{cases} p^-, & \text{if } B > 1,\\ p^+, & \text{if } B < 1.\end{cases}$$

Thus, for $t \in [0, A]$, we have $F(x,t) \leq F(x,0)$ which implies

$$\int_\Omega \sup_{t\in[0,A]} F(x,t)dx \leq \int_\Omega F(x,0)dx = 0. \tag{9}$$

Since $B > \delta$, we can get $\int_\Omega F(x,B)dx > 0$ and so,

$$\frac{A^{p^+}}{k^{p^+}B^{p^\mp}} \int_\Omega F(x,B)dx > 0. \tag{10}$$

Next, consider $u_0, u_1 \in X$ with $u_0(x) = 0$ and $u_1(x) = B$ for any $x \in \Omega$. Obviously, $\Phi(u_0) = \Psi(u_0) = 0$ and

$$\Phi(u_1) = \int_{\partial\Omega} \frac{1}{p(x)} m(x) B^{p(x)} dx \geq \frac{B^{p^\mp}}{p^+} \|m\|_{L^1(\partial\Omega)} > \frac{1}{p^+} > \frac{1}{p^+}\left(\frac{A}{k}\right)^{p^+}$$

Consequently, if we put $r = \frac{1}{p^+}\left(\frac{A}{k}\right)^{p^+}$, it follows

$$\Phi(u_0) < r < \Phi(u_1),$$

which ensures the condition $(ii)$.

Finally, we will show the condition $(iii)$. A simple calculation yields

$$\begin{aligned}
-\frac{\big(\Phi(u_1) - r\big)\Psi(u_0) + \big(r - \Phi(u_0)\big)\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} &= -r\frac{\Psi(u_1)}{\Phi(u_1)} \\
&= r\frac{\int_\Omega F(x,B)dx}{\int_{\partial\Omega} \frac{1}{p(x)} m(x) B^{p(x)} dx} > 0
\end{aligned} \tag{11}$$

Now, let $u \in \Phi^{-1}((-\infty, r])$. Then, $I_m(u) \leq rp^+ = \left(\frac{A}{k}\right)^{p^+} < 1$ which, by Proposition 3.2, implies $\|u\|_m < 1$. Consequently,

$$\frac{1}{p^+}\|u\|_m^{p^+} \leq \Phi(u) < r.$$

Therefore, by 8, we infer that

$$|u(x)| \leq \|u\|_\infty \leq k\|u\|_m \leq k\Big(rp^+\Big)^{1/p^+} = A, \quad \forall x \in \Omega,$$

for all $u \in X$ with $\Phi(u) \leq r$. The above inequality shows that

$$-\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) = \sup_{u\in\Phi^{-1}((-\infty,r])} -\Psi(u) \leq \int_\Omega \sup_{t\in[0,A])} F(x,t)dx \leq 0.$$

From (11), we deduce that

$$-\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) < r\frac{\int_\Omega F(x,B)dx}{\int_{\partial\Omega} \frac{1}{p(x)} m(x) B^{p(x)} dx},$$

that is,

$$\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{\big(\Phi(u_1) - r\big)\Psi(u_0) + \big(r - \Phi(u_0)\big)\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}$$

which means that condition (iii) holds. At this point, conclusion follows from Theorem 3.1.

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