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On the data completion problem for Laplace's equation

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Abstract. The purpose of this paper is the study and the resolution of the inverse problem for the Laplace equation, including the case of data completion problem where it is to cover the missing data on the inaccessible part of the boundary of a domain from measurements on the accessible part. Furthermore, we present a survey of the inverse problem of reconstructing the missing data for the Laplace equation. We describe the notion of ill-posed problems; namely, the results concerning the existence, uniqueness and stability of their solutions. In addition, we present several areas and fields of applications of this kind of problem. We also include the different developed methods for solving this problem, discussing their advantages and inconveniences. Numerical results with the iterative KMF algorithm and the developed variant are presented.

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1. Introduction

Many problems of engineering and industry can be considered as inverse problems; which explains the importance of this type of problems [1]. However, there are generally ill-posed in the Hadamard sense, since the existence or uniqueness or the continuous dependence on the data of their solutions may not be ensured.

Several definitions can be given to an inverse problem. Among them, we propose that given in [2]:

In science, an inverse problem is a situation in which we try to determine the causes of a phenomenon from experimental observations of its effects. For example, in seismology, the location of origin of an earthquake from measurements made by several seismic stations distributed over the surface of the earth is an inverse problem.

In general, when a number of conditions, partially or completely unknown then an inverse problem must be formulated to determine the unknown from measurements or data system. Taking into account the quantities searched, there are two types of inverse problems:

- Problems of reconstruction to find the parameters or unknown data to the system from overabundant data.
- Problems of identification which consist in finding an unknown property of an object or environment.

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The modeling of the phenomenon studied is the first step to solve an inverse problem. This is called direct problem which describes how the model parameters result in experimentally observable effects. Then, on the basis of the measurements obtained on the real phenomenon, the aim is to approach as closely as possible the parameters which make it possible to realize these measurements. This resolution can be done by numerical simulation or analytically.

A data completion problem is a class of inverse problems which consists to reconstruct the missing data on the inaccessible part of the boundary of the domain, that cannot be evaluated because of physical difficulties or geometric inaccessibility, from measured data on the known parts of the boundary. In other words, unlike the direct problem, in the data completion problem the geometry of the domain is determined, but the conditions on the boundary are not all known. The goal is to find the unknown boundary conditions based on the additional information provided on the boundary of domain.

This kind of problem has a great importance and these applications are not reduced only to the Laplace equation; but, it also occurs in several types of equations as the biharmonic equation [3, 4, 5], the Helmholtz equation [6, 7, 8, 9, 10, 11, 12, 13], the stocks equation [14, 15], heat conduction equation [16, 17, 18, 19] and elasticity [20, 21], where the applications are very important.

The problem of our interest concern the data completion problem for the Laplace equation which can be formulated as follows:
Let Ω be an open subset of $\mathbb{R}^2$ with regular boundary Γ. We consider the partition of the boundary as $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $mes(\Gamma_1) \neq 0$. We consider the problem where experimental measurements are available on the boundary Γ_0 and the conditions on Γ_1 are unknown, such that:

$$\begin{cases} \Delta u = 0 & in \quad \Omega \\ u = f & on \quad \Gamma_0 \\ \partial_n u = g & on \quad \Gamma_0 \end{cases} \tag{1}$$

where, u is the potential (or the temperature) at each point of the domain Ω,

$\partial_n u$ is the normal derivative of u,

and f and g are respectively the known values of the function u and its flow on Γ_0.

This Cauchy problem arises in many areas of engineering and can be considered as challenge in many fields of industry such as geophysics, medical imaging, structural mechanics, non-destructive testing of structure $\cdots$ We refer for example to [22] and [23]. However, it is known to be severely ill-posed in Hadamard sense; indeed, experimental measurements are not sufficient to correctly determine the model parameters and a small perturbation of these measures influence the solution, which makes its resolution by direct methods very difficult and leads to very unstable solutions. Hence, the investigation of many researchers to develop regularizing methods and efficient numerical approaches.

This type of problem has been the subject of several studies for many years and they still are. This is due to several factors; indeed, several physical problems of engineering, industry, medicine, ... are modeled in the form of this type of problem which must be studied theoretically and resolved numerically. The purpose of this work is to present an overview of the data completion problem with the Laplace equation gathering most of the results and research that has been devoted to the study and resolution of this problem.

This paper is organized as follows: The second section is devoted to some application fields in which occurs this type of problem. In the third section, we present and study its ill-posed nature, existence, uniqueness and stability of its solution. The forth section is centered on the different methods of resolution, where we explain the principle of each method, we present the scheme of the existing algorithms and results concerning their convergence and the developed variants to accelerate convergence; in addition to the advantages and disadvantages of each method. At the end, we present some numerical results using the KMF method and its developped variant.

2. Fields of application

The data completion problems are extensively studied and still interest many researchers. Their importance derives from the fact that they arise in several areas of industry, engineering and many other fields of science. Physical phenomenon involved and the measurements can be thermal or electrostatic. In the following, we present the mathematical modeling of the inverse problem for the Laplace equation with some interesting problems, most frequently encountered. Particularly, Electroencephalography, Electrocardiography, Crack detection, Corrosion detection, Identification of the boundary, Medical imaging...

It should be noted that several configuration examples are used in the performed studies, especially in the numerical simulations; in particular the disk, the square and the annular domain.

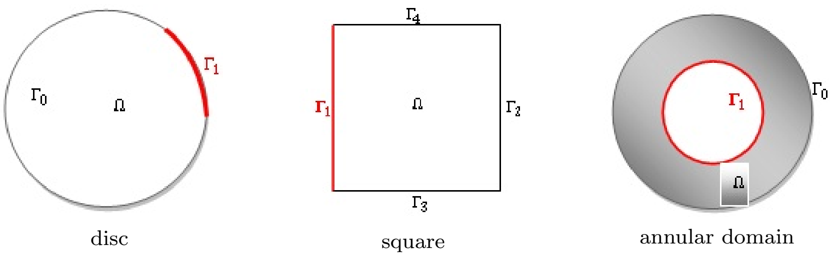


FIGURE 1. Examples of configuration.

2.1. Electroencephalography (EEG). The Electroencephalography (EEG) is a technique for functional cerebral imaging to directly measure cerebral electrical activity.

In the direct problem in EEG, from the knowledge of the geometry of the head, and the properties of conduction in the brain tissue, it is possible to calculate the potentials obtained on the skin, engendered a configuration of a known sources.

Conversely, the EEG inverse problem is to estimate the current source produced by neuronal activity, given the electrical conductivity of tissue and measures the potential u in a few points on the surface of the head. In other words, the EEG inverse problem is to detect epileptic foci or tumors in the brain from measurements on the scalp (see [24], [25], [26] and [27]).

A first step is to construct the Cauchy data on the surface of the brain from the data of the measured potential by the EEG to the surface of the head. A widely accepted model of the head three concentric layers $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ representing the outside inwards: Ω_2 for the Scalp, Ω_1 for the Skull and Ω_0 for the Brain. The three domains are considered homogeneous and isotropic (scalar constant conductivity).

We denote by Γ_0 the interface between Ω_0 and Ω_1, Γ_1 the interface between Ω_1 and Ω_2, and by Γ_2 the outer surface of the Scalp and by S the portion of Γ_2, where we know potential measurements obtained by the EEG.

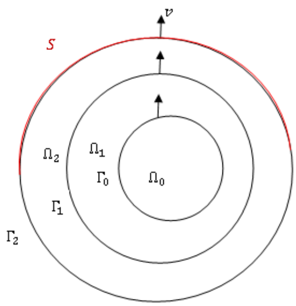


FIGURE 2. Representative scheme of the head model.

In this conductive medium (Air insulated), the electric field E and the magnetic field B satisfy the following equations:

$$\begin{cases} \Delta \times E &= \partial B/\partial t & (Maxwell - Faraday) \\ \Delta . E &= \rho/\epsilon & (Maxwell - Gauss) \\ \Delta \times B &= \mu(J + \epsilon(\partial E)\partial t) & (Maxwell - Amper) \\ \Delta . B &= 0 & (Flow\ Conservation) \end{cases} \tag{2}$$

where J, ρ, μ and ϵ respectively denote the volume density of current, volume density of charge, magnetic permeability and electrical permittivity.

Taking into account the low frequency signal and low capacitance of the tissues of the head, which appear as passive conductors, the behavior of electric currents and magnetic fields can be considered stationary at all times [28]; this explains the use of quasi-static in studies in the field of EEG. Thus, Maxwell's equations that govern the electrical behavior can conclude that the scalar potential satisfies the Laplace equation in the case of a homogeneous area of constant conductivity and absence of current sources.

So the problem of determining the potential and the flux on the surface of the brain Γ_0 from the Cauchy data on the scalp surface Γ_2 is a data completion problem for the Laplace equation which can be written as the following form:

$$\begin{cases} -\Delta u = 0 & in & \Omega_i, i = 1,2 \\ u = f & on & S \subset \Gamma_2 \\ \partial_n u = g & on & \Gamma_2 \\ [u] = 0 & on & \Gamma_1 \\ [\partial_n u] = 0 & on & \Gamma_1 \end{cases} \tag{3}$$

where $[u] = u^- - u^+$ with u^- denote the inner limit and u^+ the outer limit.

2.2. Electrocardiography. A problem of great interest in electrocardiography is the computation of the electric potential on a closed surface surrounding the heart, given the potential part of the body surface and the geometry of the heart and thorax [29], [30] and [31]. This inverse problem leads to a Cauchy problem for the Laplace equation in the case of constant conductivity.

A mathematical model of the electric field associated with the bioelectric activity of the heart is given by Maxwell's equations. It can be shown from the relative values of the coefficients that time derivatives in the equation may be neglected in first approximation. Consequently the electric field E and the current density J satisfy the equations:

$$\Delta \times E = 0, \qquad \Delta . J = 0 \qquad \text{outside the region of the heart} \tag{4}$$

The physiological tissue can be considered as a linear medium resistance and therefore we can take:

$$J = \sigma E \tag{5}$$

Since E is irrotational, it admits a scalar potential u. So, if the conductivity σ is assumed constant, u satisfies:

$$-\Delta u = 0 \qquad \text{outside the heart} \tag{6}$$

Then, we obtain a Cauchy problem for u in the bounded domain Ω with L the exterior surface and l the interior surface:

$$\begin{cases} -\Delta u = 0 & in \quad \Omega \\ \partial_n u = 0 & on \quad L \\ u = f & on \quad L_0 \in L \end{cases} \tag{7}$$

where : L_0 an open part of L.

Note that we have made $\partial_n u = 0$ on L the outer boundary of the body as the surrounding medium is air that is electrically insulating.

2.3. Detection of fissure. The problem of detection of fissure is an inverse problem which involves geometric detect and locate the presence of cracks in a material from measurements on the boundary. This type of problem occurs in several industrial applications under non-destructive testing (imaging and tomography; basements surveys, monitoring, ...).

Physical phenomena involved and the measurements can be thermal, electrostatic, acoustic or elastic. We focus here on the physical phenomena governed by the Laplace equation in dimension 2.

This problem has been the subject of several studies over the last twenty years where it was particularly interested in the problem of identifiability (Can we identify an unknown geometry (crack) for temperature measurements? If yes, how many measures are needed to identify?), stability (Since measurements are taken on board raised experimentally, it is important to show that perturbation data lead to two close geometries) in addition to problem identification (different methods to locate and define the shape of the crack).

The principle of these methods is to impose a condition on the outer edge of the area concerned (for example by imposing a heat flux Φ in the case of thermal and u measure the response of the material (temperature in the same case) [32, 97, 34, 35, 36, 37, 38, 39].

Consider a fissure interior modeled by a simple curve $\mathtt{Y}$ oriented of class $C^{(1,\alpha)}, 0 < \alpha < 1$ (for a crack of class C^2, you can see [32]) included in a simply connected domain Ω of $\mathbb{R}^2$, with a boundary Γ having the same regularity of $\mathtt{Y}$. Consider a heat flux density $\Phi \in L^2(\Gamma)$ not identically zero along Γ and satisfying the condition:

$$\int_\Gamma \Phi ds = 0 \tag{8}$$

The problem consists to:

Given Ω, $\mathtt{Y} \subset \Omega$ and Φ as above, find u solution of the following problem:

$$\begin{cases} -\Delta u = 0 & in \quad \Omega/\mathtt{Y} \\ \partial u/\partial_{n_\Gamma} = \Phi & on \quad \Gamma \\ \partial u_-^+/\partial_{n_\mathtt{Y}} = 0 & on \quad \mathtt{Y}^0 = \mathtt{Y}/\{\mathtt{Y}_0, \mathtt{Y}_1\} \end{cases} \tag{9}$$

where $\mathtt{Y}_0$ and $\mathtt{Y}_1$ are the endpoints of the fissure $\mathtt{Y}$, n_Γ is the unit vector normal outside Γ, $n_\mathtt{Y}$ is one of the two normal vectors of $\mathtt{Y}$.

It is known that this problem has a unique solution when we add the following normalization condition:

$$\int_\Gamma u ds = 0 \tag{10}$$

The inverse problem associated with the presented direct problem is to determine the unknown fissure $\mathtt{Y} \subset \Omega$ from overdetermined measurements on the boundary (the data flow Φ and a measure u on a part $\mathtt{K}$ of the boundary, assumed to be available and of positive measurement.

That is to say: Given Ω, $\mathtt{K} \subset \partial\Omega$ and Φ et $u_\mathtt{K}$, find $\mathtt{Y} \subset \Omega$ such that the solution of the problem verify: $u_{/\mathtt{K}} = u_\mathtt{K}$.

2.4. Detection of corrosion. The nondestructive evaluation of a boundary corrosion continues to be a very interesting topic in engineering and mathematics. Unfortunately, in many practical situations, the data on the known boundary is not complete, and corrosion has occurred on an inaccessible part of the boundary. The problem is to determine the shape of the corrosive border from the measured data.

Corrosion occurs in many different forms and multiple models may be found in the literature [40, 41]. We are intersted here to the potential model introduced by Inglese [42] where the interior domain is governed by the Laplace equation.

We consider a connected bounded domain Ω of $\mathbb{R}^2$, representing the specimen to be tested. We assume that the boundary $\partial\Omega$ is at least smooth a piece. Γ_0 and $\Gamma_1$are two disjointed closed parts of $\partial\Omega$, where Γ_0 is the accessible part, however Γ_1 is the part where corrosion has occurred.

Then :

$$-\Delta u(x) = 0 \quad in \quad x \in \Omega \tag{11}$$

To generate static data, we prescribe a flow given by the Neumann condition:

$$\partial_n u(x) = g(x) \quad on \quad x \in \Gamma_0 \tag{12}$$

where $\partial_n u$ is the normal derivative of u .

On the corroded part, the condition in the part Γ_0 is given by:

$$\partial_n u(x) + \gamma(x) u(x) = 0 \quad on \quad x \in \Gamma_1 \tag{13}$$

here,$\gamma(x) \geq 0$ represents the corrosion damage. Conventionally, it is interpreted as energy transfer coefficient known as Robin coefficient [43]. The remaining part of the boundary is assumed, for simplicity, to be isolated:

$$\partial_n u(x) = 0 \quad on \quad x \in \partial\Omega/\Gamma_0/\Gamma_1 \tag{14}$$

In the direct problem, $\gamma(x)$ and $g(x)$ will be given and the aim is to find $u(x)$.

The inverse problem consists in finding $\gamma(x)$ from the knowledge of $u(x)$ (and of course g) on the accessible part of the boundary.

We note that the principle of conservation of charge requires:

$$\int_{\Gamma_0} g - \int_{\Gamma_1} \gamma u \quad = \quad 0 \tag{15}$$

This means that the flow of the current density through Γ_0 is not necessarily zero. In fact, we can choose $g \geq 0$ to produce a positive solution in Ω, with conditions as the supports γ and g have a positive Lebesgue measure.

In many applications, Ω is an annular domain where Γ_0 is outside the accessible part, however, Γ_1 is the interior part whose corrosion has occurred [44, 45, 46].

So, the problem of corrosion detection resolves into two steps:

The first step is to complete the data on the inaccessible part, which leads to solving a data completion problem given in the form:

$$\begin{cases} \Delta u = 0 & in \quad \Omega \\ u = f & on \quad \Gamma_0 \\ \partial_n u = g & on \quad \Gamma_0 \end{cases} \tag{16}$$

The second step is to calculate the coefficient of Robin representing corrosion damage from the condition on Γ_0 as the ratio of the two found conditions:

$$\gamma(x) = -\frac{\partial_n u(x)}{u(x)}_{|\Gamma_1} \tag{17}$$

3. Study of the data completion problem

3.1. Ill-posed problem. In 1902, J. Hadamard [47] introduced the notion of a well-posed problem by giving the following definition:

A problem is well posed if it satisfies the following three properties:

- Existence, i.e that the problem has a solution,
- Uniqueness, i.e that the problem has at most one solution;
- Stability, i.e that the solution depends continuously on the data of the problem.

J. Hadamard considered a particular example to illustrate the fact that the Cauchy problem for elliptic partial differential equations, which is to cover the data on a part of the boundary, are ill-posed (see section 3.4). He showed by considering a Laplace's operator on a square domain that the solution does not depend continuously on the boundary data. This result was recently generalized to any domain [48].

Mathematically, we can formulate the notion of well-posed in this form:

Definition 3.1. Let X and Y be two normed spaces, $A : X \to Y$ be a linear application. The equation $Ax = y$ is said to be well posed if the following properties are satisfied:

- For all $y \in Y$ there exists $x \in X$ such that $Ax = y$,

- For all $y \in Y$ there exists at most $x \in X$ such that $Ax = y$,
- The solution x depends continuously of y, ie that for every sequence (x_n) in X, $limKx_n = Kx$, we have $limx_n = x$.

These three properties are not always assured for a Cauchy problem for the Laplace equation, and then the latter is ill-posed. Of course, these concepts must be specified by the choice of spaces (and topologies) in which data and the solution exist.

3.2. Existence. The existence of the solution of the data completion problem is not always guaranteed. However, it exists under certain compatibility conditions of data problem. The challenge is to identify the suitable space with the compatibility conditions to ensure the existence of the solution.

Let Ω be an open set of $\mathbb{R}^2$ with a regular boundary Γ. We consider the partition of the boundary as $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \cap \Gamma_1 = \phi$ with Γ_0 and Γ_1 are two parts of Γ that have positive measures.

We are interested to the problem that is to reconstruct a harmonic function u, which corresponds to certain prescribed data on Γ_0 and belongs to the set $H(\Omega)$ defined as follows:

$$H(\Omega) = \{u \in (H^1(\Omega)), \Delta u = 0 \text{ in } \Omega\}.$$

$H(\Omega)$ is a closed subset of $H^1(\Omega)$, which makes it a Hilbert space when it is equipped with the scalar product of $H^1(\Omega)$.
Moreover, $H(\Omega)$ is a subspace of $H_\Delta(\Omega)$ defined as:

$$H_\Delta(\Omega) = \{u \in H^1(\Omega), \Delta u \in L^2(\Omega)\}$$

This allows to define the trace of the normal derivative belonging to this space as part of $H^{-\frac{1}{2}}(\Gamma)$ [49].
Thus, the traces $(v_{/\Gamma}, \partial_n v_{/\Gamma})$ of $H(\Omega)$ cover the space $H(\Gamma)$ by pairs of compatible data defined as:

$$H(\Gamma) = \{\Phi = (\phi, \psi) \in H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma), \exists v \in H(\Omega), v_{/\Gamma} = \phi, \partial_n v_{/\Gamma} = \psi\}$$

Lemma 3.1. *$H(\Gamma)$ is a closed subspace of $X = H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ and therefore it is a Hilbert space when equipped with the scalar product of $X = H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$:*

$$(\Phi, \Phi') = (\phi, \phi')_{\frac{1}{2},\Gamma} + (\psi, \psi')_{-\frac{1}{2},\Gamma}$$

where, $\Phi = (\phi, \psi)$ and $\Phi' = (\phi', \psi')$ are two elements of $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$.

For Γ_0 the inaccessible part of the boundary Γ, we introduce the following spaces [50]:

- $H^{\frac{1}{2}}(\Gamma_0)$ space of restrictions on Γ_0 the elements of $H^{\frac{1}{2}}(\Gamma)$.
- $H^{\frac{1}{2}}_{00}(\Gamma_0)$ subspace of $H^{\frac{1}{2}}(\Gamma_0)$ whose elements extended by 0 on Γ belongs to $H^{\frac{1}{2}}(\Gamma)$.
- $H^{-\frac{1}{2}}(\Gamma_0)$ space of restrictions on Γ_0 the elements of $H^{-\frac{1}{2}}(\Gamma)$., which is the dual space of $H^{\frac{1}{2}}_{00}(\Gamma_0)$.
- $H(\Gamma_0)$ is the space of restrictions of the pairs of compatible data on Γ_0 defined by:

$$H(\Gamma_0) = \{\Phi = (f, g) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0), \exists v \in H(\Omega), v_{/\Gamma_0} = f, \partial_n v_{/\Gamma_0} = g\}$$

We note $X_0 \subset H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$ the set of the compatible Cauchy data (f, g) for the problem (1) defined as follows:

Definition 3.2. Let $(\alpha,\beta) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$. The Cauchy data (α,β) are compatible for the Cauchy problem (1) if it exist a function $u \in H^1(\Omega)$ such that:

$$\Delta u = 0 \text{ in } \Omega \ , \ u_{/\Gamma_0} = \alpha \ , \ \partial_n u_{/\Gamma_0} = \beta.$$

With this notation, an equivalent formulation of the data completion problem (1) studied can be written as:

Given the compatible data $\phi = (f,g) \in H(\Gamma_0)$,

$$\text{Find } U = (u, \partial_n u) \in H(\Gamma) \text{ such that } U = \phi \text{ on } \Gamma_0$$

3.3. Unicity (identifiability). We consider the Cauchy problem for Laplace's equation which consists for $(f,g) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$ to find $u \in H^1(\Omega)$ such that:

$$\begin{cases} \Delta u = 0 & in \quad \Omega \\ u = f & on \quad \Gamma_0 \\ \partial_n u = g & on \quad \Gamma_0 \end{cases} \tag{18}$$

Based on the foregoing, this problem does not always have a solution. But for compatible data, it has a solution. However, if the operator of the problem verifies the property of unique extension, then, this solution is unique.

We will need the three spheres inequality, the proof can be found in [51].

Proposition 3.2 (Three spheres inequality). *Let $x \in \Omega$ and $0 < r_0 < r_1 < r_2$ satisfying : $B(x,r_2) \subset \Omega$,it exists $C > 0, s > 0$ such that $\forall u \in H_\Delta(\Omega)$.*

$$\|u\|_{H^1(B(x,r_1))} \leq C \|\Delta u\|_{L^2(B(x,r_2))} + \|u\|^{\frac{s}{s+1}}_{H^1(B(x,r_0))} + \|u\|^{\frac{1}{s+1}}_{H^1(B(x,r_1))}$$

We also need the following result, which is proved in [52]:

Proposition 3.3 (Theorem through customs). *If E is a topological vector space and A is a part of E. Any path joining an point of the interior of A to an point outside of A, necessarily encounter the boundary of A.*

Thus, we can cite a Theorem of unique extension as follows:

Theorem 3.4 (unique extension). *Let $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω connected and such as it exists an open set $\omega \subset \Omega, |\omega| \neq 0$ such that $u = 0$ in ω. Then $u = 0$ in Ω.*

To prove the theorem, the connectedness of Ω is used [53].

As we noted earlier, the unique extension property, brings the uniqueness of the solution of the Cauchy problem provided by the following theorem whose demonstration can be found in [54].

Theorem 3.5 (*Uniqueness of Cauchy problem*). *Let Ω a bounded open set of $\mathbb{R}^d$, Γ_0 an open part, nonzero measure and Lipschitz of the boundary of Ω. Let $k \in \mathbb{R}$. and $u \in H^1(\Omega)$ satisfying:*

$$\begin{cases} \Delta u = 0 & in \quad \Omega \\ u = 0 & on \quad \Gamma_0 \\ \partial_n u = 0 & on \quad \Gamma_0 \end{cases} \tag{19}$$

Then $u = 0$ in Ω.

Thus, we can mention the following lemma, ensuring the uniqueness of the solution of the Cauchy problem for the Laplace equation.

Lemma 3.6. *There is at most one solution $u \in H^1(\Omega)$ of the Cauchy problem for Laplace's equation.*

Proof. Let $f \in H^{\frac{1}{2}}(\Gamma_0)$ and $g \in H^{-\frac{1}{2}}(\Gamma_0)$ two given functions and let u and v two solutions of the Cauchy problem (1) corresponding to f and g.
We pose $w = u - v$. then, we have $\Delta w = 0$ in Ω and $\partial_n w_{/\Gamma_0} = 0$ and $w_{/\Gamma_0} = 0$, which involves by the preceding theorem that $w = 0$ in Ω. ☐

Remark 3.1.
- This result is, for example, a consequence of a Theorem shown in ([55] appendix B, p.75).
- The uniqueness of the inverse problem in bioelectric field especially in electroencephalography is shown in [56].
- The uniqueness of the inverse problems has been the subject of several studies [57, 58, 59, 60], where the theorem of Holmgreen, Carleman estimates are used, as well as the theorem Kowalewsky.

3.4. Stability. This is the continuous dependence of the solution in relation with the data problem. For an inverse problem for the Laplace equation a small perturbation on the data (very low additional noise to the measures) can create a large gap between the solution obtained by noisy data and the obtained one by undisturbed data [61, 62]. So it is clear that the stability of the inverse problem is the most difficult problem due to his important implication in the algorithms used to calculate the solution. This problem has encouraged several authors to develop several regularization schemes to ensure that the calculation procedures does not diverge and to obtain more relevant results.

The most well-known example, to illustrate the fact that the inverse problem is ill-posed is the Cauchy problem for the Laplace equation given by Hadamard [63, 64, 65]; it is to find a solution u of the Laplace equation:

$$\Delta u(x,y) = \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \text{ in } \mathbb{R} \times [0,\infty)$$

which satisfies the following initial conditions:

$$u(x,0) = f(x), \quad \frac{\partial}{\partial y} u(x,0) = g(x),\ x \in \mathbb{R}$$

where f and g are two given fonctions.

The unique solution for $f(x) = 0$ and $g(x) = \frac{1}{n} sin(nx)$ is given by:

$$u(x,y) = \frac{1}{n^2} sin(nx) sinh(ny),\ x \in \mathbb{R},\ y > 0$$

Therefore, we have:

$$sup_{x\in\mathbb{R}}(|f(x)| + |g(x)|) = \frac{1}{n} \to 0\ ,\ n \to \infty$$

But,

$$sup_{x\in\mathbb{R}}(|u(x,y)|) = \frac{1}{n^2} sinh(ny) \to \infty\ ,\ n \to \infty \text{ for all } y > 0.$$

The data error converges to zero; however, the error u converges to infinity. Then, the solution does not depends continuously of the data and the problem is ill-posed.

This example shows that the norm of the solution may blow up even if the standard data converge to zero.

It is very interesting to mention here the work of J. Alessandrini [62] where he highlights most of the work dedicated to the stability of elliptic Cauchy problems, since Hadamard.

3.5. Identification. Among the important questions to ask about the Cauchy problems for the Laplace equation sight their ill-posed nature that relates to identification methods to determine their solution.

It is an identification method to determine the solution?

We know that this problem has at most one solution; then, we can try to find the solution. But, the ill-posed aspect of the cauchy problem for the Laplace equation and especially the instability makes resolution by the direct methods very delicate and produce very unstable solutions. Hence, the need to regulate the problem in order to develop robust numerical methods of resolution.

In the literature, there are two types of methods: Probabilistic methods [56] and deterministic methods that will make the object of the section (4).

4. Methods for solving the data completion problem

In this part, we will cite most developed methods to solve this kind of problem and we will present the advantages and disadvantages of each of them.

Most of the methods developed are based on a control approach, ie. the minimization of a functional by taking functions of the unknown part of the boundary as minimization parameters.

There are also other methods which are distinguished as follows:

4.1. Thikhonov method. Among the most known methods, we find methods that regulate the Cauchy problem using Tikhonov parametric functions. There is an extensive literature on this regularization technique [66, 67, 68, 69, 70].

The Tikhonov regularization method is a minimization problem (in the sense of least squares) which is added a penalty term which depends on a parameter called regularization parameter.

The data completion problem consists to find an harmonic function u such that:

$$u_{|\Gamma_0} = f, \qquad \partial_n u_{|\Gamma_0} = g \tag{20}$$

i.e.: given compatible data $\Phi_0 = (f, g) \in X_0$, the problem is to find:

$$U = (u, \partial_n u) \in H(\Gamma) \quad \text{such that} \quad U = \Phi_0 \quad \text{on} \quad \Gamma_0 \tag{21}$$

where the spaces X_0 and $H(\Gamma)$ are defined in (3.2).
A cost function is defined as follows:

$$\text{For all} \quad v \in X, J_c(v) = \|v - \Phi_0\|^2_{\Gamma_0} + c\,\|v - \Phi\|^2_{\Gamma} \tag{22}$$

where c is a positive coefficient and Φ is a pair in $H(\Gamma)$.

And we are addressed in the problem :

$$\text{Find} \quad u \in H(\Gamma) \quad \text{such} \quad J_c(u) \leq J_c(v), \forall v \in H(\Gamma) \tag{23}$$

which is a well-posed problem, its solution depends continuously on the data Φ_0 and the choice of c and Φ, and has a unique solution characterized by:

$$\langle u - \Phi_0, v\rangle_{\Gamma_0} + c\langle u - \Phi, v\rangle_{\Gamma} = 0, \quad \forall v \in H(\Gamma) \tag{24}$$

It is shown that the required solution is the only fixed point of a suitable operator, which naturally gives rise to an iterative process that is shown to be convergent [66].

There are several methods for the optimal choice of the parameter c, as the L-curve method and the principle of Mozorov [71]. Among the works Applying this method to the Cauchy problem, we cite the work of Falk & Monk [68], the work of Cimitiere and al. [66] where the penalty term covers the distance between two successive iterative solutions.

Tikhonov methods have the disadvantage of disturbing the operator problem. In addition, they require a priori information on the solution of the inverse problem.

4.2. Quasireversibility method. Among the most interesting methods, there is the method of quasi-reversibility, which is a non-iterative method, introduced for the first time by Lattes and Lions in 1967 [72], which is to replace the inverse problem by a well-posed problem in the sense of Hadamard introducing a certain parameter, the convergence of the original problem is assured when this parameter tends to 0. This method has been adopted by several authors to solve a Cauchy problem, including Klibanov and Santosa [73] and more recently Bourgeois [74] and others [75, 76, 77].

The Quasi-reversibility method is a method of regularization, its main characteristic is that, unlike most of regularization methods, it is not based on solving an optimization problem and is written in a variational form, that can exploited numerically by a Galerkin method.

Several formulations of this method for the Cauchy problem for the Laplace equation have been developed including the so-called classical formulation and mixed formulation.

This method allows to directly solve the problem and to obtain accurate and robust results. However, this method has some disadvantages such as the difficulty of taking into account physical constraints which may be related to the problem and the particular choice of the introduced parameter that can be difficult to achieve in real circumstances.

In the following; we will present the various formulations for the Laplace equation. We first define some spaces that will be useful later.

$$H_\Delta(\Omega) = \{u \in (H^1(\Omega))/\Delta u \in L^2(\Omega)\}$$
$$H^1(\Delta, \Omega, \Gamma_0) = \{u \in H_\Delta(\Omega)/u_{/\Gamma_0} = 0, \partial_n u_{/\Gamma_0} = 0\}$$
$$\tilde{H}^1(\Delta, \Omega, \Gamma_0) = \{u \in H_\Delta(\Omega)/u_{/\Gamma_0} = f, \partial_n u_{/\Gamma_0} = g\}$$

4.2.1. *Classical formulation H^1 (in $H_\Delta(\Omega)$).* This formulation is slightly different from those in [72]; this difference is related to the variational formulation.
Thus, the method of Quasi-reversibility is to find an approximation u_ϵ of the solution u of the weak formulation for $\epsilon > 0$.
Find $u \in \tilde{H}^1(\Delta, \Omega, \Gamma_0)$ such that:

$$\int_\Omega \Delta u.\Delta v dx + \epsilon \int_\Omega \nabla u.\nabla v dx + \epsilon \int_\Omega u.v dx = 0, \forall v \in H^1(\Delta, \Omega, \Gamma_0). \tag{25}$$

And we have the following results of existence and convergence (the demonstrations can be found in [74]):

Proposition 4.1. *Quasi-reversibility problem H^1 has a unique solution u_ϵ if and only if $\tilde{H}^1(\Delta, \Omega, \Gamma_0)$ is not empty.*

Theorem 4.2. *For $(f, g) \in H \subset H^{\frac{1}{2}}(\Gamma_0) \times H^{(-\frac{1}{2})}(\Gamma_0)$, The solution u_ϵ of the Quasi-reversibility problem H^1 converge to u in $H_\Delta(\Omega)$ when ϵ tends to 0.*

The major drawback of this formulation is that any finite element formulation of u_ϵ must be calculated in a finite dimensional space of $H_\Delta(\Omega)$ and then, one has to use the C^1 finite element.
However there is another formulation (mixed formulation), for which the use of usual finite elements class C^0 is possible, that we present in the following section.

4.2.2. *Mixte Formulation.* The mixed formulation is proposed in the sense that the first Equation of order 4 of the classic formulation containing a function u in a type space $H_\Delta(\Omega)$ is replaced by a system of two equations of second order with u and λ both in type spaces $H^1(\Omega)$.
However, this transformation requires the introduction of a second regularization parameter δ, in addition to classical parameter ϵ.
The spaces are defined:

$$\begin{aligned} H_0 &= \{v \in H^1(\Omega)/v_{/\Gamma_0} = 0\} \\ \tilde{H}_0 &= \{v \in H^1(\Omega)/v_{/\Gamma_0} = f\} \\ H_1 &= \{v \in H^1(\Omega)/v_{/\Gamma_1} = 0\} \end{aligned}$$

We are interested in problem:
For $\epsilon > 0$ and $\delta > 0$, find $(u, \lambda) \in \tilde{H}_0 \times H_1$ such that:

$$\epsilon \int_\Omega \nabla u.\nabla v dx + \epsilon \int_\Omega u.v dx + \int_\Omega \nabla v.\nabla \lambda dx = 0, \forall v \in H_0. \tag{26}$$

$$\int_\Omega \nabla u.\nabla \mu dx - \delta \int_\Omega \nabla \lambda.\nabla \mu dx - (1+\delta) \int_\Omega \lambda.\mu dx = \int_{\Gamma_0} g.\mu dx, \forall \mu \in H_1 \tag{27}$$

In [64], the integral on Γ_0 is defined in the sense of duality between $H^{-\frac{1}{2}}(\Gamma_0)$ and $H_{00}^{\frac{1}{2}}(\Gamma_0)$.
We can find a more general mixed formulation in [37].
However, we include in this section some results of existence and convergence of this formulation where one will find these demonstrations in [74].

Theorem 4.3. *For $(f, g) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$, the problem (mixed formulation) admit an unique $(u_\alpha, \lambda_\alpha)$ in $\tilde{H}_0 \times H_1$ with certain estimations.*

Theorem 4.4. *For $(f, g) \in H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$, for which there existe $u \in H^1(\Omega)$ satisfying the Cauchy problem, and if δ is a function of ϵ as: $lim_{\epsilon \to 0} \frac{\epsilon}{\delta(\epsilon)} = 0$ then, the solution $(u_\alpha(\epsilon), \lambda_\alpha(\epsilon))$ of the problem (mixed formulation) converges to $(u, 0)$ in $H^1(\Omega) \times H^1(\Omega)$ where $\epsilon \to 0$.*

It is interesting to also mention the result showing the relationship between mixed formulation and classical formulation.

Proposition 4.5. *For $(f, g) \in H \subset H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$, and for a fixed , the solution $(u_\alpha, \lambda_\alpha)$ of the problem (mixed formulation) converge to $(u_\epsilon, -\Delta u_\epsilon)$ in $H^1(\Omega) \times L^2(\Omega)$ when $\delta \to 0$, where u_ϵ is the solution of the classical formulation.*

The advantage of this formulation is that it ensures the existence of the solution and the convergence of the solution to the exact solution of the Cauchy problem and even if we have the noisy data (slightly disturbed), which is not always the case for classical formulation.

It is interesting to note that there are other formulations of the Quasi-reversibility method, including the formulation in $H^2(\Omega)$, which allows a more regular solution from data of the Cauchy problem and the more regular part of the boundary Γ_0 with a single regularization parameter to fix. However, it requires adapted finite element to the space $H^2(\Omega)$.

Remark 4.1. Conditional Stability results for the Cauchy problem for giving an estimate of the speed of convergence of the Quasi-reversible method in the case of $C^{1,1}$ bounbary area and lipchitzienn can be found in the work [78, 79].

4.3. Iterative Methods. Another class of methods includes the iterative methods that have a regulating character. These methods have been widely applied to inverse problems. Among them, we cite the one proposed by Kozlov et al. [80] whose regularization parameters depend on the number of iterations and the initial choice of the iterative scheme. This method consists in alternatively solve a sequence of well-posed problems with mixed boundary conditions, until some stopping criterion defined in advance is satisfied. We also include in this group of methods, the iterative method of Mann developed by Engl and Leitao [81] and the method of Backus-Gilbert applied to a formulation in a moment of problem such as the work of Hon and Wei [82].

4.3.1. *Advantages.* The iterative methods have several advantages, which encourages many researchers to invest to use their performance. Among the advantages of such methods, we mention:
- Easy to implement computations schemes to get a sequence as direct numerical solutions of well posed problems.
- The similarity of the schemes for the problems with linear and nonlinear operators.
- The high accuracy and stability of the solution.
- It allows physical restraint to be easily taken into account directly in the scheme of the iterative algorithm.

4.3.2. *Disadvantages.* The possible disadvantage of iterative algorithms is the large number of iterations required to achieve convergence. So; we easily used relaxation algorithms to improve the speed of convergence.

4.4. The KMF method and its variants. This method called here the KMF standard is called also the method (or solver) Dirichlet-to-Neumann [83], where several questions, remarks and discussions can be addressed on the convergence of this algorithm. In particular; the choice of the initial data and the stopping criterion, the regularity of the data of the problem needed to have convergence, the most suitable numerical methods to use; in addition to the choice of the relaxation methods to accelerate convergence and the choice of the relaxation parameters for faster convergence.

4.4.1. *Principle of the method.* This is to determine the traces $u_{/\Gamma_1}$ and $\partial_n u_{/\Gamma_1}$ on part Γ_1. We note respectively f^* and g^*, thus amounts to determining u solution of the following problem:

$$\begin{cases} -\Delta u = 0 & in \quad \Omega \\ u = f, \partial_n u = g & on \quad \Gamma_0 \\ u = f^*, \partial_n u = g^* & on \quad \Gamma_1 \end{cases} \tag{28}$$

This problem can be divided into two well-posed sub-problems, one with a Dirichlet condition on Γ_0 and Neumann condition on Γ_1 and the other with a Neumann condition on Γ_1 and Dirichlet condition on Γ_0 defined as follows:

$$\text{(a)} \quad \begin{cases} -\Delta\tilde{u} = 0 & in \quad \Omega \\ \tilde{u} = f & on \quad \Gamma_0 \\ \partial_n\tilde{u} = g^* & on \quad \Gamma_1 \end{cases} \quad \text{and} \quad \text{(b)} \quad \begin{cases} -\Delta\bar{u} = 0 & in \quad \Omega \\ \partial_n\bar{u} = g & on \quad \Gamma_0 \\ \bar{u} = f^* & on \quad \Gamma_1 \end{cases} \tag{29}$$

The main idea of the KMF method consists to:
To solve the Cauchy problem (1), it is necessary to determine u which verifies the problem (28), which is ensured when $\tilde{u}$ and $\bar{u}$ coincide. From an initial estimate of solution $u = f^*$ on Γ_1, then the method is to alternatively solve two well- posed problems type (a) and (b) where each of these problems allows a condition on the part Γ_1 that will be introduced in the other problem to find the other condition. Thus, a sequence of well posed problems with mixed boundary conditions is constructed using an alternation of the Dirichlet and Neumann data, on the part of the boundary containing the data, and the iterative process is stopped when a certain criterion stop predefined in advance is satisfied.

4.4.2. *Scheme of the KMF Algorithm.* The KMF iterative algorithm originally developed by Kozlov and al. is to replace the Cauchy problem by a series of well-posed mixed problems. It allows to approach the Dirichlet and Neumann conditions on the inaccessible part Γ_1 of the boundary, alternately using the given Dirichlet and Neumann data on the boundary portion containing the data.

The KMF algorithm consists of the following steps:
`Step 1:` Specify an initial guess u_0 on Γ_1 and solve:

$$\begin{cases} -\Delta u^{(0)} = 0 & in \quad \Omega \\ u^{(0)} = u_0 & on \quad \Gamma_1 \\ \partial_n u^{(0)} = g & on \quad \Gamma_0 \end{cases} \tag{30}$$

to obtain $v_0 = \partial_n u^0_{|\Gamma_1}$
`Step 2:` For $n \geq 0$, solving alternatively the following two mixed well-posed boundary value problem until a prescribed stopping is satisfied:

$$\begin{cases} -\Delta u^{(2n+1)} = 0 & in \quad \Omega \\ \partial_n u^{(2n+1)} = v_n & on \quad \Gamma_1 \\ u^{(2n+1)} = f & on \quad \Gamma_0 \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u^{(2n+2)} = 0 & in \quad \Omega \\ \partial_n u^{(2n+2)} = g & on \quad \Gamma_0 \\ u^{(2n+2)} = u_{n+1} & on \quad \Gamma_1 \end{cases} \tag{31}$$

$$\text{to obtain} \quad u_{n+1} = u^{(2n+1)}_{/\Gamma_1} \qquad \qquad \text{to obtain} \quad v_{n+1} = \partial_n u^{(2n+2)}_{/\Gamma_1}$$

In what follows, we present the theorem of the convergence of the algorithm proposed in the theoretical study by [80], in addition to other results given by [84].

4.4.3. *Convergence result.* In this section, we recall the convergence results proposed in the work of Kozlov and al. in the case of a connected open domain.

Theorem 4.6 (Convergence)**.** *For a compatible data, the sequence $(u^k)_k$ converge in $H^1(\Omega)$ to the solution of the Cauchy problem (1) for any initial choice $u_0 \in H^{\frac{1}{2}}(\Gamma_1)$.*

Let us also recall the convergence result given by Baumeister and al. where the demonstration can be found in [84].

Theorem 4.7.
- *If the Cauchy problem has a unique solution $u \in H^1(\Omega)$ then the sequence $(u_n)_{n\geq 0}$ defined in the algorithm converges to $u_{/\Gamma_1}$ for the norm of $H^{\frac{1}{2}}(\Gamma_1)$.*
- *If the sequence $(u_n)_{n\geq 0}$ defined in the algorithm converges in $H^{\frac{1}{2}}(\Gamma_1)$ then it converges to $u_{/\Gamma_1}$ where $u \in H^1(\Omega)$ is the unique solution of the Cauchy problem.*

4.4.4. *Regularizing character of the method.* We consider the problem defined in (1), f and g represent the exact data set on Γ_0, $\tilde{f}$ and $\tilde{g}$ perturbations of f and g. We denote by U the exact solution of the Cauchy problem corresponding to f and g and we consider for a given u_0, the family of operators $\mathtt{R}_k(.,.,u_0) : H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0) \to H^1(\Omega)$ which associates with (f, g) the solution $u^{(k)}$ at iteration k of the iterative algorithm $(k = 0, 1, \cdots\cdots)$.

Our goal in this section is to examine the nature of the Regularizing algorithm as in [80].

Definition 4.1 (regularizing family)**.** The family of operators $\mathtt{R}_k(.,.,u_0) : H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0) \to H^1(\Omega)$ for $(k = 0, 1, \cdots\cdots)$ is called regularizing for the Cauchy problem (1) of exact solution U if there is a positive real δ_0 and functions $k(\delta)$ and $\epsilon(\delta)$ defined in $(0, \delta_0)$ as $\epsilon(\delta) \to 0$ and inequality

$$\left\|f - \tilde{f}\right\|_{H^{\frac{1}{2}}(\Gamma_0)} + \|g - \tilde{g}\|_{H^{-\frac{1}{2}}(\Gamma_0)} \leq \delta$$

involves the following estimate:

$$\left\|\mathtt{R}_{k(\delta)}(\tilde{f}, \tilde{g}, u_0) - U\right\|_{H^1(\Omega)} \leq \epsilon(\delta)$$

Here u_0 acts as parameter for the family of operators $(\mathtt{R}_k)_k$.
The sequence of operators $(\mathtt{R}_k)_k$ of the algorithm is regularizing for the Cauchy problem and we have the following estimate:

$$\left\|\mathtt{R}_k(\tilde{f}, \tilde{g}, u_0) - U\right\|_{H^1(\Omega)} \leq \rho_k\delta + \left\|r_k(u_0 - U_{/\Gamma_1})\right\|_{H^1(\Omega)}$$

where $r_k(u_0 - U_{/\Gamma_1}) = \mathtt{R}_k(0, 0, u_0 - U_{/\Gamma_1})$ and ρ_k is the norm of the operator $\mathtt{R}_k(.,.,0)$ whose estimate below is given in [80].

$$\rho_k \leq ck$$

with c is a constant that depends on the geometry and the elliptic operator of the Cauchy problem. So finally we have:

$$\left\|\mathtt{R}_k(\tilde{f}, \tilde{g}, u_0) - U\right\|_{H^1(\Omega)} \leq ck\delta + \left\|r_k(u_0 - U_{/\Gamma_1})\right\|_{H^1(\Omega)}$$

and that $k(\delta)$ is defined as the smallest iteration such that:

$$ck(\delta) + \left\|r_k(u_0 - U_{/\Gamma_1})\right\|_{H^1(\Omega)} \leq \epsilon(\delta)$$

This estimation shows the delicacy of the choice of stopping criterion of the algorithm. It's about finding the optimal iteration k_0 that depends on δ and u_0 and for which this estimation is satisfied.

Remark 4.2. We can note that:
- The KMF described algorithm is initialized by a type of condition $u_0 \in H^{\frac{1}{2}}(\Gamma_0)$, however, we can take the form of a condition $v_0 \in H^{-\frac{1}{2}}(\Gamma_0)$ by modifying in the second step of the algorithm the two mixed problems in such a manner they were resolved.

- The algorithm does not converge if the two mixed problems are replaced by two Dirichlet or Neumann problems.

4.4.5. *Bibliographic synthesis.* Given the many advantages of the KMF method, this method has been the basis of several studies in recent years and which covered several theoretical and numerical aspects. In 1991, Kozlov and al. [80] proposed this method and they studied only theoretically without presenting any numerical experiment. This method was taken and tested by other researchers and in several applications, especially in the case where the problem is generated by the Laplace equation. Principally, we cite the work of Lesnic and al. [85, 86] where the numerical implementation of the iterative algorithm is made by the boundary element method, Jourhmane and al. [87, 88, 89], Leitao and al. [90, 81], Weikel and al. [91], Nachaoui [92] and more recently Azaiez and al. [83]. These work has treated the iterative method KMF for the Laplace equation, numerically and theoretically. The applications presented have been all implemented in 2D with analytical tests on homogeneous areas. However, we can find applications in the nonlinear case with numerical simulations in the case of a square [93], applications in the case of linear elasticity [94] and even for the Helmholtz equation in the case where k is imaginary number that ensures the convergence of the method requires that the operator is defined positive [95].

This algorithm is used in practical applications especially as a first step to complete the missing conditions in a pipe to detect eventual corrosion [96]. In another context, this algorithm is implemented in 3D on a homogeneous piecewise domain (spherical geometry) from a realistic problem of Electroencephalography (EEG) [63].

Given the importance of this method and the interesting results obtained, works are devoted to the comparison betwen results obtained with this method and those obtained by the best known methods, studying the link between these different methods [6] and also to study the possible relationship between the KMF algorithm and existing algorithms to solve this problem [97].

For the numerical implementation of the iterative algorithm KMF, some authors have chosen the finite element method [83, 87, 90, 98, 99], and other the boundary element method [85, 88, 89], which uses only the conditions on the border which reduces the dimension by one, in addition to the mesh that will be simpler since we discretize only the boundary, unlike the first method where one must discretize the entire domain.

All these studies show the effectiveness of the method and his regularizing character for slightly perturbed data, but point to a slow numerical convergence especially for an initial choice relatively far from the exact solution.

On the other hand, note that the results of Lesnic and al. in [85] show that the flow $\frac{\partial u}{\partial n}$ is more difficult to calculate with good precision as the potential u; in additon, where the surfaces of the crown are not regular, the results are less good.

To accelerate convergence, some authors have sought to relaxed the algorithm to accelerate the convergence. Thus, Jourhmane et al. [87, 89] have relaxed the calculation of flux at each iteration, and Jourhmane et al. [88] have relaxed calculating the Dirichlet condition and presented a relaxation scheme of the two conditions Dirichlet and Neumann, and a comparison between these different schemes. Furthermore, a strategy for choosing the optimal relaxation parameter is also presented in [87, 88] which will be presented in the following section.

4.4.6. *Relaxation methods for KMF algorithm.* The different work already cited and numerical results show that this algorithm is effective, it provides satisfactory and stable results against weak perturbation. However; its only disadvantage is the high number of iterations required convergence. To accelerate the convergence of the algorithm, variants have been developed called relaxed methods which consists in introducing a relaxation parameter in the KMF algorithm.

The first relaxation method was proposed by Jourhmane et al. [89] since 1999 and has been the basis of several studies and especially the convergence of the proposed method in [87], in addition to a study on the choice of the optimal relaxation parameter; Then, in 2004 Jourhmane et al. [88] proposed two other relaxation schemes they have studied and compared theoretically and numerically.

- The first relaxation algorithm has the same scheme as the KMF standard algorithm, but the calculation of the Neumann condition is relaxed by:

$$v_n = \theta^{(n)}\partial_n u^{(2n)}_{/\Gamma_1} + (1-\theta^{(n)})v_{n-1} \quad , \quad n \geq 1 \tag{32}$$

  where $\theta^{(n)}$, $n \geq 1$ is a positive relaxation factor. Note that if $\forall n \geq 1$ we have $\theta^{(n)} = 1$, then in this case the considered algorithm is reduced to the standard KMF algorithm.
- The second relaxation algorithm has the same scheme as the standard KMF algorithm, but in this case the calculation of the Dirichlet boundary condition is relaxed by:

$$u_n = \delta^{(n)} u^{(2n-1)}_{/\Gamma_1} + (1-\delta^{(n)})u_{n-1} \quad , \quad n \geq 1 \tag{33}$$

  where $\delta^{(n)}$, $n \geq 1$ is a positive relaxation factor.
- In this third relaxation algorithm, we relaxes both conditions v_n and u_n in standard KMF algorithm by replacing them:

$$\begin{aligned} u_n &= \beta^{(n)} u^{(2n-1)}_{/\Gamma_1} + (1-\beta^{(n)})u_{n-1} \quad , \quad n \geq 1 \\ v_n &= \alpha^{(n)}\partial_n u^{(2n)}_{/\Gamma_1} + (1-\alpha^{(n)})v_{n-1} \quad , \quad n \geq 1 \end{aligned} \tag{34}$$

  where $\beta^{(n)}$ and $\alpha^{(n)}$ for $n \geq 1$ are two positive relaxation factors.

Remark 4.3.
- If $\forall n \geq 1$, $\alpha^{(n)} = 1$; in this case, the third method of relaxation is reduced to the second method.
- If $\forall n \geq 1$, $\beta^{(n)} = 1$; in this case the third method of relaxation is reduced to the first method.
- If $\forall n \geq 1$, $\alpha^{(n)} = 1$ and $\beta^{(n)} = 1$, the standard iterative method is obtained.
- Some results studies ony been made concerning the choice of the relaxation parameters and the convergence of the proposed algorithm were developed by the same authors.

4.4.7. *Variant of the KMF Algorithm.* Following numerical simulations carried out to solve the data completion problem for Laplace's equation, we observed that the measurement of the inaccessible part influences the results. Thus, the smaller the measurement, the better the results [**?**]. Hence, The main idea of the proposed variant of the KMF algorithm is based in completing the missing data in alternative way to the two sub-parts of the inaccessible boundary. Then; the inaccessible part is subdivided in two parts, and the KMF standard algorithm is used to complete the data in the first part, then to complete the data in the second part in an alternative way [98, 99, 100, 101].

For this, we consider $\Gamma_1 = \Gamma_{1,1} \cup \Gamma_{1,2}$ such that $\Gamma_{1,1} \cap \Gamma_{1,2} = \emptyset$ and $mes(\Gamma_{1,1}) = mes(\Gamma_{1,2})$.

The algorithm consists of the following steps:

`Step 1:` Specify an initial guess u_0 on Γ_1 and solve:

$$\begin{cases} -\Delta u^{(0)} = 0 & in & \Omega \\ u^{(0)} = u_0 & on & \Gamma_{1,1} \cup \Gamma_{1,2} \\ \partial_n u^{(0)} = g & on & \Gamma_0 \end{cases} \tag{35}$$

$$\text{to obtain} \quad v_{1,0} = \partial_n u^{(0)}_{/\Gamma_{1,1}} \quad \text{and} \quad v_{2,0} = \partial_n u^{(0)}_{/\Gamma_{1,2}}$$

`Step 2:` For $n \geq 0$, solve the two well-posed problems:

$$\begin{cases} -\Delta u^{(2n-1)} = 0 & in & \Omega \\ \partial_n u^{(2n-1)} = v_{1,n-1} & on & \Gamma_{1,1} \\ \partial_n u^{(2n-1)} = v_{2,n-1} & on & \Gamma_{1,2} \\ u^{(2n-1)} = f & on & \Gamma_0 \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u^{(2n-1)} = 0 & in & \Omega \\ u^{(2n-1)} = u_{1,n} & on & \Gamma_{1,1} \\ \partial_n u^{(2n-1)} = v_{2,n-1} & on & \Gamma_{1,2} \\ u^{(2n-1)} = f & on & \Gamma_0 \end{cases} \tag{36}$$

$$\text{to obtain } u_{1,n} = u^{(2n-1)}_{/\Gamma_{1,1}} \qquad \text{to obtain } u_{2,n} = u^{(2n-1)}_{/\Gamma_{1,2}}$$

`Step 3:` For $n \geq 0$, solve the two well-posed problems:

$$\begin{cases} -\Delta u^{(2n)} = 0 & in & \Omega \\ u^{(2n)} = u_{1,n} & on & \Gamma_{1,1} \\ u^{(2n)} = u_{2,n} & on & \Gamma_{1,2} \\ \partial_n u^{(2n)} = f & on & \Gamma_0 \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u^{(2n)} = 0 & in & \Omega \\ \partial_n u^{(2n)} = v_{1,n} & on & \Gamma_{1,1} \\ u^{(2n)} = u_{2,n} & on & \Gamma_{1,2} \\ \partial_n u^{(2n)} = g & on & \Gamma_0 \end{cases} \tag{37}$$

$$\text{to obtain} \quad v_{1,n} = \partial_n u^{(2n)}_{/\Gamma_{1,1}} \qquad \text{to obtain} \quad v_{2,n} = \partial_n u^{(2n)}_{/\Gamma_{1,2}}$$

`Step 4:` Repeat the step 3 and 4 until a prescribed stopping criterion is satisfed.

Remark 4.4. It should be noted that:

- The KMF developed algorithm can be seen as two parallel problems of the KMF standard algorithm. These two problems are initialized with the same initial data; each problem allows to obtain approximation on each subpart $\Gamma_{1,i}$ where $i = 1, 2$ (for the approximation on $\Gamma_{1,1}$ the two first well-posed problems in (36) and (37), for the approximation on $\Gamma_{1,2}$ the two second well-posed problems in (36) and (37).
- Each solved problem allows an approximation in one of the inaccessible sub-parts that can be introduced in the other well-posed problems.

4.5. Other methods. The data completion problem is of great importance and its resolution remains a chalenge to address several problems arising from different scientific fields. Hence, the investigation of several researchers to develop different methods to remedy its mal-posed aspect. Thus, in addition to already mentioned methods, other interesting methods have been developed to solve this famous inverse problem and even compared them.

- In [102] the author presents a new method called inverse method of order one, which is an evolution of the data completion technique developed in [66], later called method of order zero. It significantly improves the convergence and the reconstruction of solutions; in particular, the normal derivative when the boundary of the domain provides angular points.
- The method presented by Ben Belgacem and al. [103, 104] is to solve a problem of Steklov-Poincaré after formulating the problem and demonstrate that its resolution amounts to finding the unique traces defined on the unknown part of the boundary to complete. Indeed; the solution u of the Cauchy problem is duplicated in a couple of functions (v, w) where both functions v and w satisfy the Laplace equation in Ω; they differ in the state they are subject on Γ_0: v has the Dirichlet condition and w has the Neumann condition. And to ensure that $u = v = w$, the traces and the normal derivative on Γ_i of functions v and w must coincide, i.e. $(v, \partial_n v)_{/\Gamma_i} = (\omega, \partial_n \omega)_{/\Gamma_i}$.
  Moreover, in [104, 105] the Cauchy problem has been studied theoretically and numerically. A non-standard variational approach was presented, it transforms the Cauchy problem in a pseudo-differential variational equation type Steklov-Poincaré.
  This equation is placed on the part of the boundary where the values of potential and flux are unknown. To resolve this Steklov-Poincaré problem, the authors considered the least square method, they demonstrated that for incompatible data, the Steklov-Poincaré problem that has no solution generally has at least one solution pseudo consistent which is a minimizing sequence.
  They also demonstrated that the cost functional, of least square corresponding to Steklov Poincaré problem differs from an additive constant of the functional Kohn Vogelius written for this problem. Then in [105], they have solved numerically the variational problem of Steklov-Poincaré by applying Tikhonov regularization scheme using the finite element method.
- Work has been done for the Cauchy problem by Andrieux and al. [97] where a cost-Kohn Vogelius functional was introduced. They also demonstrated that the proposed algorithm by Kozlov and al. can be interpreted as a minimization method of alternating directions on their cost functional.

5. Numerical Examples

In this part, we will present some numerical results obtained by solving the data completion problem by the KMF iterative method and its variant deveoped and presented in 4.4.7., by focusing on the choice of initial data, the calculated errors and the choice of stopping criteria in this case.

Thus, we consider a typical benchmark test example in a non-smooth geometry, such as a square $\Omega = (0, L) \times (0, L)$ where $L = 1$, namely, the analytical harmonic function to be retrieved is given by:

$$u_{ex}(x) = cos(x)cosh(y) + sin(x)sinh(y). \tag{38}$$

where;
$\Gamma_0 = \{0\} \times (0, L)$ as underspecified boundary, $\Gamma_1 = (0, L) \times \{0\}$, $\Gamma_2 = \{L\} \times (0, L)$ and $\Gamma_3 = (0, L) \times \{L\}$ as overspecified boundary.

The known data is given by: $u_{/\Gamma_1} = cos(x)$
$u_{/\Gamma_2} = cos(L)cosh(y) + sin(L)sinh(y)$,
$\partial_n u_{/\Gamma_2} = -sin(L)cosh(y) + cos(L)sinh(y)$
$\partial_n u_{/\Gamma_3} = cos(x)sinh(L) + sin(x)cosh(L)$
and the unknown data on the underspecified boundary Γ_0 is given by:

$$u_{/\Gamma_0} = cosh(y) \quad \text{and} \quad \partial_n u_{/\Gamma_0} = -sinh(y) \tag{39}$$

The following stoping criterion was addopted:

$$E = \|u_{n+1} - u_n\| \leq 10^{-5} \tag{40}$$

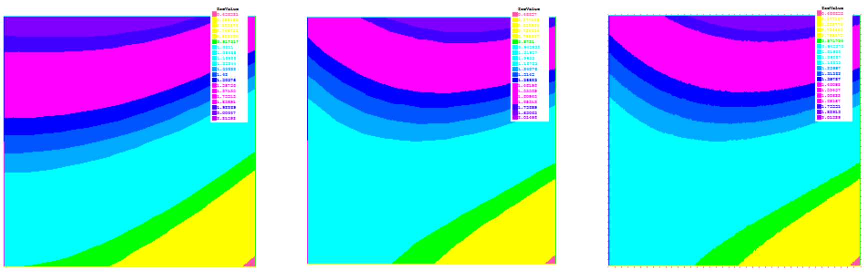
The convergence of the algorithm may be investigated by evaluating at every iteration the error:

$$e_u = \|u_n - u_{ex}\|_{0,\Gamma_1} \quad \text{and} \quad e_v = \|\partial_n u_n - \partial_n u_{ex}\|_{0,\Gamma_1} \tag{41}$$

where u_n is the approximation obtained for the function on the boundary Γ_1 after n iterations and u_{ex} is the exact solution of the problem (1). However, in practical applications the error e_u cannot be evaluated since the analytical solution is not known and therefore the error E has to be used.

For the step 1 of the algorithm, as an initial guess $u_0 \in H^{1/2}(\Gamma_0)$, we have chosen:

$$u_0(y) = 1 + y(-L + sinh(L)) + y^2/2, y \in [0, 1] \tag{42}$$



u_iter 0 u_iter 20 u_exact

FIGURE 3. The reconstruction function u with the new algorithm (u_iter final) in comparison with (u_exact) and u obtained with the initian guess (u_iter 0) .

The figure 4 present the error e_u and e_v obtained with the KMF standard algorithm and with the developed KMF algorithm according to the number of iterations.

6. Conclusion

The purpose of this article is to make a general synthesis of an important class of inverse problem where we present the set of results and work dedicated to the data completion problem for Laplaces's equation. This problem arises in various fields and its importance can be seen by the number of theoretical and numerical studies carried out and from the large number of methods developed to solve it. Hence; areas of applications of this type of problems are presented; in addition, to a review of interesting results regarding the three aspects of his ill-posed nature in the sense of

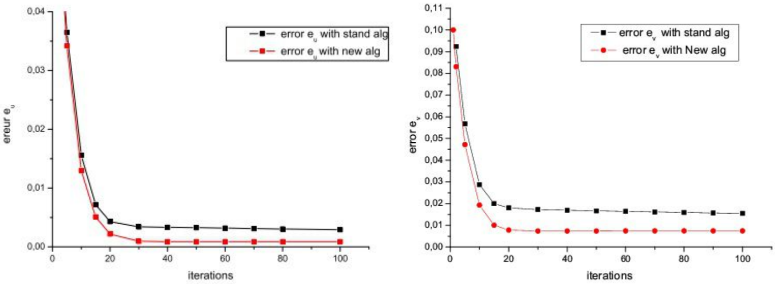


FIGURE 4. The numerical results obtained for e_u and e_v.

Hadamard; namely, existence, unicity and stability. In addition, importance is given to the developed regularizing methods and its variants to solve this inverse problem. To make their methods robust and faster, researchers continue to develop them and to propose new procedures to better approach the solution.

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