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[p,q]-order of solutions of complex differential equations in a sector of the unit disc

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Abstract. In this article, we study the growth of solutions of linear complex differential equations by using the concept of [p,q]-order in a sector of the unit disc instead of the whole unit disc, and we obtain similar results as in the case of the unit disc.

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1. Introduction

In this article, we assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna's theory in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, see [6, 7, 8, 12]. Many authors have investigated the growth of solutions of the linear complex differential equation

$$f^{(k)}(z) + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \tag{1}$$

where $k \geq 2$ and the coefficients A_j $(j = 0, 1, \ldots, k-1)$ are analytic functions in the unit disc Δ, see [2, 3, 7, 9] and references therein. Belaïdi in [2, 3] investigated the growth of solutions of the equation (1) by using the concepts of [p,q]-order in the unit disc Δ. Wu in [15], and Long in [10] have investigated new problem related to linear differential equations with analytic coefficients in a sector of the unit disc

$$\Omega_{\alpha,\beta} = \{z \in \mathbb{C} : \alpha < \arg z < \beta, |z| < 1\},$$

and they obtained different results concerning the growth of their solutions. In this paper, we continue to investigate this new problem and study the growth of solutions of equation (1) when the coefficients A_j $(j = 0, 1, \ldots, k-1)$ are analytic functions of [p,q]-order in the sector $\Omega_{\alpha,\beta}$. Before stating our main results, we give some notations and basic definitions of meromorphic functions in the unit disc Δ and in a sector $\Omega_{\alpha,\beta}$ of the unit disc. The order of a meromorphic function f in Δ is defined by

$$\sigma(f) = \limsup_{r\to 1^-} \frac{\log T(r,f)}{\log \frac{1}{1-r}},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. If f is analytic function in Δ, then

$$\sigma_M(f) = \limsup_{r\to 1^-} \frac{\log M(r,f)}{\log \frac{1}{1-r}},$$

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where $M(r,f)=\max\limits_{\substack{|z|=r\\ z\in\Delta}}|f(z)|$ is the maximum modulus function, it follows by Tsuji [12, page 205] that if f is an analytic function in Δ, then we have the inequalities

$$\sigma(f)\leq\sigma_M(f)\leq\sigma(f)+1,$$

which are the best possible in the sense that there are analytic functions g and h such that $\sigma(g)=\sigma_M(g)$ and $\sigma_M(h)=\sigma(h)+1$, see [4]. Recently has been introduced the concept of [p,q]-order for meromorphic and analytic functions in the unit disc in order to study the growth of solutions of the linear complex differential equations, for that, let us define inductively that for $r\in(0,+\infty)$, $\exp_1 r:=e^r$ and $\exp_{p+1}r:=\exp\left(\exp_p r\right)$, $p\in\mathbb{N}$. We define also for sufficiently large r in $(0,+\infty)$, $\log_1^+ r:=\log^+ r=\max(0,\log r)$, $\log_{p+1}^+ r:=\log^+\left(\log_p^+ r\right)$, $p\in\mathbb{N}$. Moreover, we denote by $\exp_0 r:=r$, $\log_0^+ r:=r$.

Definition 1.1. [2] Let $p\geq q\geq 1$ be integers. Let f be a meromorphic function in Δ, the [p,q]-order of f is defined by

$$\sigma_{[p,q]}(f)=\limsup_{r\to1^-}\frac{\log_p^+ T(r,f)}{\log_q\frac{1}{1-r}}.$$

For an analytic function f in Δ, we also define

$$\sigma_{M,[p,q]}(f)=\limsup_{r\to1^-}\frac{\log_{p+1}^+ M(r,f)}{\log_q\frac{1}{1-r}}.$$

It is easy to see that $0\leq\sigma_{[p,q]}(f)\leq+\infty$. If f is non-admissible, i.e., $T(r,f)=O\left(\log\frac{1}{1-r}\right)$, then $\sigma_{[p,q]}(f)=0$ for any $p\geq q\geq 1$. By Definition 1.1, $\sigma_{[1,1]}(f)=\sigma(f)$ is the order of f in Δ, $\sigma_{[2,1]}(f)=\sigma_2(f)$ is the hyper-order of f in Δ and $\sigma_{[p,1]}(f)=\sigma_p(f)$ is the p-iterated order of f in Δ.

Proposition 1.1. [2] *Let $p\geq q\geq 1$ be integers, and let f be an analytic function in Δ of [p,q]-order. The following two statements hold :*
(i) If $p=q$, then

$$\sigma_{[p,q]}(f)\leq\sigma_{M,[p,q]}(f)\leq\sigma_{[p,q]}(f)+1.$$

(ii) If $p>q$, then

$$\sigma_{[p,q]}(f)=\sigma_{M,[p,q]}(f).$$

In this article, Ω usually denotes the sector $\Omega_{\alpha,\beta}$ $(0\leq\alpha<\beta\leq2\pi)$ of the unit disc, and for any given $\varepsilon\in\left(0,\frac{\beta-\alpha}{2}\right)$, Ω_ε denotes the sector

$$\Omega_{\alpha,\beta,\varepsilon}=\{z\in\mathbb{C}:\alpha+\varepsilon<\arg z<\beta-\varepsilon,|z|<1\}.$$

Wu, in [15], has used the Ahlfors-Shimizu characteristic function to measure the order of growth of a meromorphic function f in Ω, and by same, Long [10] has defined the p-order of a meromorphic function f in Ω. Before defining the [p,q]-order of meromorphic function f in Ω, we recall the definition of the Ahlfors-Shimizu characteristic function, see [5, 12]. Let f be a meromorphic function in Ω, set $\Omega(r)=\Omega\cap\{z\in\mathbb{C}:0<|z|<r<1\}$. Define

$$S(r,\Omega,f)=\frac{1}{\pi}\iint\limits_{\Omega(r)}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2d\sigma,$$

the Ahlfors-Shimizu characteristic function is defined by

$$T_0(r, \Omega, f) = \int_0^r \frac{S(t, \Omega, f)}{t} dt.$$

It follows by Hayman [6, pages 12-13], Goldberg and Ostrovskii [5, page 20] that

$$T_0(r, \mathbb{C}, f) = T(r, f) + O(1).$$

Now, we introduce the concept of [p,q]-order and [p,q]-type of meromorphic functions in a sector Ω.

Definition 1.2. Let $p \geq q \geq 1$ be integers. Let f be a meromorphic function in Ω. Then, the [p,q]-order of f in Ω is defined by

$$\sigma_{[p,q],\Omega}(f) := \limsup_{r \to 1^-} \frac{\log_p^+ T_0(r, \Omega, f)}{\log_q \frac{1}{1-r}}.$$

It is clear that $0 \leq \sigma_{[p,q],\Omega}(f) \leq +\infty$. If f is non-admissible in Ω, i.e., $T_0(r, \Omega, f) = O\left(\log \frac{1}{1-r}\right)$, then $\sigma_{[p,q],\Omega}(f) = 0$ for any $p \geq q \geq 1$. By Definition 1.2, $\sigma_{[1,1],\Omega}(f) = \sigma_\Omega(f)$ is the order of f in Ω, see [15] and $\sigma_{[p,1],\Omega}(f) = \sigma_{p,\Omega}(f)$ is the p-order of f in Ω, see [10].

Definition 1.3. Let $p \geq q \geq 1$ be integers, and f be a meromorphic function in Ω with $[p, q]$-order $0 < \sigma_{[p,q],\Omega}(f) < +\infty$. Then, the $[p, q]$-type of f in Ω is given by

$$\tau_{[p,q],\Omega}(f) := \limsup_{r \to 1^-} \frac{\log_{p-1}^+ T_0(r, \Omega, f)}{\left(\log_{q-1} \frac{1}{1-r}\right)^{\sigma_{[p,q],\Omega}(f)}}.$$

2. Main results

In recent years, Belaïdi in [3], Latreuch and Belaïdi in [9] have investigated the growth of solutions of equation (1) in the unit disc with analytic coefficients of finite [p,q]-order, and they obtained the following results.

Theorem 2.1. [3] *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| = r : z \in H \subseteq \Delta\} > 0$, and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in Δ such that for some real constants $0 \leq \gamma < \eta$, we have*

$$T(r, A_0(z)) \geq \exp_p\left(\eta \log_q\left(\frac{1}{1-r}\right)\right),$$

$$T(r, A_j(z)) \leq \exp_p\left(\gamma \log_q\left(\frac{1}{1-r}\right)\right), \ j = 1, 2, \ldots, k-1$$

as $|z| = r \to 1^-$ for $z \in H$. Then every non-trivial solution f of (1) satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = +\infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geq \eta$.

Theorem 2.2. [3] *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| = r : z \in H \subseteq \Delta\} > 0$, and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in Δ satisfying $\max_{1 \leq j \leq k-1}\left\{\sigma_{[p,q]}(A_j)\right\} \leq \sigma_{[p,q]}(A_0) = \eta$. Suppose that there* *exists a real number γ satisfying $0 \leq \gamma < \eta$ such that for any given ϵ $(0 < \epsilon < \eta - \gamma)$ sufficiently small, we have*

$$T\left(r, A_0(z)\right) \geq \exp_p\left((\eta - \epsilon)\log_q\left(\frac{1}{1-r}\right)\right),$$

$$T\left(r, A_j(z)\right) \leq \exp_p\left(\gamma \log_q\left(\frac{1}{1-r}\right)\right),\ j = 1, 2, \ldots, k-1$$

as $|z| = r \to 1^-$ for $z \in H$. Then every non-trivial solution f of (1) *satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = +\infty$ and*

$$\sigma_{[p,q]}(A_0) \leq \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \leq \max_{0\leq j\leq k-1}\left\{\sigma_{M,[p,q]}(A_j)\right\}.$$

Furthermore, if $p > q$ then

$$\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

Theorem 2.3 ([9])**.** *Let $p \geq q \geq 1$ be integers, and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in Δ satisfying*

$$\max_{1\leq j\leq k-1}\left\{\sigma_{[p,q]}(A_j)\right\} < \sigma_{[p,q]}(A_0).$$

Then every non-trivial solution f of (1) *satisfies $\sigma_{[p,q]}(f) = +\infty$ and*

$$\sigma_{[p,q]}(A_0) \leq \sigma_{[p+1,q]}(f) \leq \max_{0\leq j\leq k-1}\left\{\sigma_{M,[p,q]}(A_j)\right\}.$$

Furthermore, if $p > q$ then

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

The main purpose of this article is to investigate the growth of solutions of (1) in sector Ω by using the concept of [p,q]-order similarly to the case of the unit disc. We need the following definitions : For $E \subset [0, 1)$, the upper and lower densities of E are defined by

$$\overline{\text{dens}}E = \limsup_{r\to 1^-}\frac{m\left(E \cap [0, r)\right)}{m\left([0, r)\right)},\quad \underline{\text{dens}}E = \liminf_{r\to 1^-}\frac{m\left(E \cap [0, r)\right)}{m\left([0, r)\right)}$$

respectively, where $m(G) = \int_G \frac{dr}{1-r}$ for $G \subset [0, 1)$.

We mainly obtain the following results by similar method as in [10, 15].

Theorem 2.4. *Let $p \geq q \geq 1$ be integers and $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$. Let E be a set of complex numbers satisfying $\overline{\text{dens}}\left\{|z| = r : z \in E \subseteq \Omega\right\} > 0$, and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in Ω such that for some real constants η and γ satisfying $0 \leq \gamma < \eta$, we have*

$$T_0\left(r, \Omega_\varepsilon, A_0(z)\right) \geq \exp_p\left(\eta \log_q\left(\frac{1}{1-r}\right)\right), \tag{2}$$

$$T_0\left(r, \Omega, A_j(z)\right) \leq \exp_p\left(\gamma \log_q\left(\frac{1}{1-r}\right)\right),\ j = 1, 2, \ldots, k-1 \tag{3}$$

as $|z| = r \to 1^-$ for $z \in E$. Then every non-trivial solution f of (1) *satisfies $\sigma_{[p,q],\Omega}(f) = +\infty$ and $\sigma_{[p+1,q],\Omega}(f) \geq \eta$.*

Theorem 2.5. *Let $p \geq q \geq 1$ be integers and $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$. Let E be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| = r : z \in E \subseteq \Omega\} > 0$, and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in Ω satisfying $\max_{1\leq j\leq k-1} \left\{\sigma_{[p,q],\Omega}(A_j)\right\} \leq \sigma_{[p,q],\Omega_\varepsilon}(A_0) = \eta$. Suppose that there exists a real number γ satisfying $0 \leq \gamma < \eta$ such that for any given ϵ $(0 < \epsilon < \eta - \gamma)$ sufficiently small, we have*

$$T_0(r, \Omega_\varepsilon, A_0(z)) \geq \exp_p\left((\eta - \epsilon)\log_q\left(\frac{1}{1-r}\right)\right),$$

$$T_0(r, \Omega, A_j(z)) \leq \exp_p\left(\gamma \log_q\left(\frac{1}{1-r}\right)\right), \ j = 1, 2, \ldots, k-1$$

as $|z| = r \to 1^-$ for $z \in E$. Then every non-trivial solution f of (1) *satisfies $\sigma_{[p,q],\Omega}(f) = +\infty$ and*

$$\sigma_{[p,q],\Omega_\varepsilon}(A_0) \leq \sigma_{[p+1,q],\Omega}(f) \ \text{ and } \sigma_{[p+1,q],\Omega_\varepsilon}(f) \leq \sigma_{[p,q],\Omega}(A_0) + 1.$$

Furthermore, if $p > q$, then

$$\sigma_{[p,q],\Omega_\varepsilon}(A_0) \leq \sigma_{[p+1,q],\Omega}(f) \ \text{ and } \ \sigma_{[p+1,q],\Omega_\varepsilon}(f) \leq \sigma_{[p,q],\Omega}(A_0).$$

Theorem 2.6. *Let $p \geq q \geq 1$ be integers and $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$. Let $A_0(z)$, $A_1(z)$, $\ldots$, $A_{k-1}(z)$, be analytic functions in Ω. If*

$$\max_{1\leq j\leq k-1} \left\{\sigma_{[p,q],\Omega}(A_j)\right\} < \sigma_{[p,q],\Omega_\varepsilon}(A_0), \tag{4}$$

then every non-trivial solution of (1) *satisfies $\sigma_{[p,q],\Omega}(f) = +\infty$ and*

$$\sigma_{[p,q],\Omega_\varepsilon}(A_0) \leq \sigma_{[p+1,q],\Omega}(f) \ \text{ and } \sigma_{[p+1,q],\Omega_\varepsilon}(f) \leq \sigma_{[p,q],\Omega}(A_0) + 1.$$

Furthermore, if $p > q$, then

$$\sigma_{[p,q],\Omega_\varepsilon}(A_0) \leq \sigma_{[p+1,q],\Omega}(f) \ \text{ and } \ \sigma_{[p+1,q],\Omega_\varepsilon}(f) \leq \sigma_{[p,q],\Omega}(A_0).$$

If there exist some other coefficients A_j $(j = 1, 2, \ldots, k-1)$ having the same [p,q]-order as A_0, then we have the following result by making use of the concept of [p,q]–type.

Theorem 2.7. *Let $p \geq q \geq 1$ be integers and $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$. Let $A_0(z)$, $A_1(z)$, $\ldots$, $A_{k-1}(z)$, be analytic functions in Ω. Suppose that*

$$\max_{1\leq j\leq k-1} \left\{\sigma_{[p,q],\Omega}(A_j)\right\} \leq \sigma_{[p,q],\Omega_\varepsilon}(A_0) = \sigma \ \ (0 < \sigma < +\infty) \tag{5}$$

and

$$\max_{1\leq j\leq k-1}\{\tau_{[p,q],\Omega}(A_j) : \sigma_{[p,q],\Omega}(A_j) = \sigma_{[p,q],\Omega_\varepsilon}(A_0)\} < \tau_{[p,q],\Omega_\varepsilon}(A_0) = \tau(0 < \tau < +\infty). \tag{6}$$

Then every non-trivial solution of (1) *satisfies $\sigma_{[p,q],\Omega}(f) = +\infty$ and*

$$\sigma_{[p,q],\Omega_\varepsilon}(A_0) \leq \sigma_{[p+1,q],\Omega}(f) \ \text{ and } \sigma_{[p+1,q],\Omega_\varepsilon}(f) \leq \sigma_{[p,q],\Omega}(A_0) + 1.$$

Furthermore, if $p > q$, then

$$\sigma_{[p,q],\Omega_\varepsilon}(A_0) \leq \sigma_{[p+1,q],\Omega}(f) \ \text{ and } \ \sigma_{[p+1,q],\Omega_\varepsilon}(f) \leq \sigma_{[p,q],\Omega}(A_0).$$

3. Some lemmas

Lemma 3.1. [11] *Let*

$$u(z)=\frac{\left(ze^{-i\theta_0}\right)^{\pi/\delta}+2\left(ze^{-i\theta_0}\right)^{\pi/(2\delta)}-1}{\left(ze^{-i\theta_0}\right)^{\pi/\delta}-2\left(ze^{-i\theta_0}\right)^{\pi/(2\delta)}-1}, \tag{7}$$

where $0\le\theta_0=\frac{\alpha+\beta}{2}<2\pi$, $0<\delta=\frac{\beta-\alpha}{2}<\pi$. *Then* $u(z)$ *is a conformal map of angular domain* Ω, $(0\le\beta-\alpha<2\pi)$ *onto the unit disc* Δ. *Moreover, for any positive number* ε *satisfying* $0<\varepsilon<\delta$, *the transformation* (7) *satisfies*

$$u\left(\left\{z:\frac{1}{2}<|z|<r\right\}\cap\{z:|\arg z-\theta_0|<\delta-\varepsilon\}\right)\subset\left\{u:|u|<1-\frac{\varepsilon}{2^{\frac{\pi}{2\delta}+1}\delta}(1-r)\right\},$$

$$u^{-1}\left(\{u:|u|<\varrho\}\right)\subset\left(\left\{z:|z|<1-\frac{\delta}{8\pi}(1-\varrho)\right\}\cap\{z:|\arg z-\theta_0|<\delta\}\right),$$

where $\varrho<1$ *is a constant. The inverse transformation of* (7) *is*

$$z(u)=e^{i\theta_0}\left(\frac{-(1+u)+\sqrt{2(1+u^2)}}{1-u}\right)^{\frac{2\delta}{\pi}}. \tag{8}$$

Lemma 3.2. [15] *Let* f *be a meromorphic function in* Ω, *where* $0\le\alpha<\beta<2\pi$. *For any given* $\varepsilon\in\left(0,\frac{\beta-\alpha}{2}\right)$, *set* $\delta=\frac{\beta-\alpha}{2}$ *and* $b=\frac{\varepsilon}{2^{\pi/(2\delta)+1}\delta}$. *Then the following statements hold*

$$T_0\left(\varrho,\mathbb{C},f\left(z\left(u\right)\right)\right)\le\frac{16\pi}{\delta}T_0\left(1-\frac{\delta}{8\pi}\left(1-\varrho\right),\Omega,f(z)\right)+O(1), \tag{9}$$

$$T_0\left(r,\Omega_\varepsilon,f(z)\right)\le\frac{2}{b}T_0\left(1-b\left(1-r\right),\mathbb{C},f\left(z(u)\right)\right)+O(1), \tag{10}$$

where $z(u)$ *is the inverse transformation of* (7).

Remark 3.1. By applying the formula $T\left(r,f\right)=T_0\left(r,\mathbb{C},f\right)+O(1)$, Lemma 3.2 and the definition of [p,q]-order, we immediately obtain that

$$\sigma_{[p,q],\Omega_\varepsilon}\left(f\left(z\right)\right)\le\sigma_{[p,q]}\left(f\left(z\left(u\right)\right)\right)\le\sigma_{[p,q],\Omega}\left(f\left(z\right)\right).$$

Lemma 3.3. [15] *Let* f *be a meromorphic function in* Ω, *where* $0\le\alpha<\beta<2\pi$ *and* $z(u)$ *be the inverse transformation of* (7). *Set* $F(u)=f\left(z\left(u\right)\right)$ *and* $\psi\left(u\right)=f^{(\ell)}\left(z\left(u\right)\right)$. *Then*

$$\psi\left(u\right)=\sum_{j=1}^{\ell}\alpha_jF^{(j)}(u), \tag{11}$$

where the coefficients α_j *are polynomials (with numerical coefficients) in the variables* $V(u)\left(=\frac{1}{z'(u)}\right), V'(u), V''(u),\ldots$. *Moreover, we have*

$$T\left(\varrho,\alpha_j\right)=O\left(\log\frac{1}{1-\varrho}\right),\quad j=1,2,\ldots,\ell. \tag{12}$$

Using (11) and by simple calculation, we can easily get the following lemma, see [15, Proof of Theorem 1.6].

Lemma 3.4. *Suppose $f \not\equiv 0$ is a solution of (1) in Ω. Then $F(u) = f(z(u))$ is a solution of*

$$F^{(k)}(u) + B_{k-1}(u)F^{(k-1)}(u) + \cdots + B_0(u)F(u) = 0 \tag{13}$$

in Δ, where $B_0(u) = A_0(z(u))$ and for $j = 1, 2, \ldots, k-1$

$$B_j(u) = \frac{\alpha_j}{\alpha_k} + \frac{\alpha_j}{\alpha_k}\sum_{n=j}^{k-1} A_n(z(u)). \tag{14}$$

Remark 3.2. From (12) and (14), we have for $j = 1, 2, \ldots, k-1$,

$$T(\varrho, B_j) \leq \sum_{n=j}^{k-1} T(\varrho, A_n(z(u))) + O\left(\log\frac{1}{1-\varrho}\right). \tag{15}$$

Lemma 3.5. [2] *Let $p \geq q \geq 1$ be integers. If $B_0(u), B_1(u), ..., B_{k-1}(u)$ are analytic functions of [p, q]-order in the unit disc Δ, then every solution $F \not\equiv 0$ of (13) satisfies*

$$\sigma_{[p+1,q]}(F) = \sigma_{M,[p+1,q]}(F) \leq \max_{0\leq j\leq k-1}\left\{\sigma_{M,[p,q]}(B_j)\right\}.$$

Lemma 3.6. *Let $p \geq q \geq 1$ be integers. If $A_0(z), ..., A_{k-1}(z)$ are analytic functions of [p, q]-order in sector Ω satisfying $\max_{0\leq j\leq k-1}\left\{\sigma_{[p,q],\Omega}(A_j)\right\} \leq \eta$, then for any given $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$, every solution $f \not\equiv 0$ of (1) satisfies*

$$\sigma_{[p+1,q],\Omega_\varepsilon}(f) \leq \eta + 1.$$

Furthermore, if $p > q$ then

$$\sigma_{[p+1,q],\Omega_\varepsilon}(f) \leq \eta.$$

Proof. Let $f(z) \not\equiv 0$ be a solution of equation (1). Then by Lemma 3.4, $F(u) = f(z(u))$ is a solution of equation (13) and by Remark 3.1, Remark 3.2 and Lemma 3.5 , we obtain

$$\begin{aligned}
\sigma_{[p+1,q],\Omega^\varepsilon}(f) &\leq \sigma_{[p+1,q]}(F) = \sigma_{M,[p+1,q]}(F) \leq \max_{0\leq j\leq k-1}\left\{\sigma_{M,[p,q]}(B_j)\right\} \\
&\leq \max_{0\leq j\leq k-1}\left\{\sigma_{[p,q]}(B_j)\right\} + 1 \leq \max_{0\leq j\leq k-1}\left\{\sigma_{[p,q],\Omega}(A_j)\right\} + 1 \\
&\leq \eta + 1.
\end{aligned}$$

if $p > q$, we obtain

$$\begin{aligned}
\sigma_{[p+1,q],\Omega^\varepsilon}(f) &\leq \sigma_{[p+1,q]}(F) = \sigma_{M,[p+1,q]}(F) \leq \max_{0\leq j\leq k-1}\left\{\sigma_{M,[p,q]}(B_j)\right\} \\
&= \max_{0\leq j\leq k-1}\left\{\sigma_{[p,q]}(B_j)\right\} \leq \max_{0\leq j\leq k-1}\left\{\sigma_{[p,q],\Omega}(A_j)\right\} \leq \eta.
\end{aligned}$$

□

Lemma 3.7. *Let $p \geq q \geq 1$ be integers, and let f be a meromorphic function in Ω such that $0 < \sigma_{[p,q],\Omega}(f) = \sigma < +\infty$ and $0 < \tau_{[p,q],\Omega}(f) = \tau < +\infty$. Then for any given $\beta < \tau$, there exists a subset $E \subset [0,1)$ that satisfies $\int_E \frac{dr}{1-r} = +\infty$ such that for all $r \in E$ we have*

$$\log^+_{p-1} T_0(r, \Omega, f) > \beta\left(\log_{q-1}\frac{1}{1-r}\right)^\sigma.$$

Proof. By the definition of the $[p,q]-$type in Ω, there exists an increasing sequence $\{r_m\}_{m=1}^{+\infty} \subset [0,1)$ satisfying $\frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m < r_{m+1}$, $(r_m \underset{m\to+\infty}{\longrightarrow} 1^-)$ and

$$\lim_{m\to+\infty} \frac{\log_{p-1}^{+} T_0\left(r_m, \Omega, f\right)}{\left(\log_{q-1} \frac{1}{1-r_m}\right)^{\sigma}} = \tau.$$

Then, there exists a positive integer m_1 such that for all $m \geq m_1$ and for any given ε $(0 < \varepsilon < \tau)$, we have

$$\log_{p-1}^{+} T_0\left(r_m, \Omega, f\right) > (\tau - \varepsilon) \left(\log_{q-1} \frac{1}{1-r_m}\right)^{\sigma}. \tag{16}$$

For $r \in \left[r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m\right]$, we have

$$\lim_{m\to+\infty} \frac{\left(\log_{q-1} \left(1 - \frac{1}{m}\right) \left(\frac{1}{1-r}\right)\right)^{\sigma}}{\left(\log_{q-1} \frac{1}{1-r}\right)^{\sigma}} = 1.$$

Then for any given $0 < \beta < \tau - \varepsilon$, there exists a positive integer m_2 such that for all $m \geq m_2$, and for all $r \in \left[r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m\right]$, we have

$$\frac{\left(\log_{q-1} \left(1 - \frac{1}{m}\right) \left(\frac{1}{1-r}\right)\right)^{\sigma}}{\left(\log_{q-1} \frac{1}{1-r}\right)^{\sigma}} > \frac{\beta}{\tau - \varepsilon}. \tag{17}$$

By (16) and (17), for all $m \geq m_3 = \max\{m_1; m_2\}$ and for all

$$r \in \left[r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m\right],$$

we have

$$\begin{aligned}
\log_{p-1}^{+} T_0\left(r, \Omega, f\right) &\geq \log_{p-1}^{+} T_0\left(r_m, \Omega, f\right) \\
&> (\tau - \varepsilon) \left(\log_{q-1} \frac{1}{1-r_m}\right)^{\sigma} \\
&\geq (\tau - \varepsilon) \left(\log_{q-1} \left(1 - \frac{1}{m}\right) \left(\frac{1}{1-r}\right)\right)^{\sigma} \\
&> \beta \left(\log_{q-1} \frac{1}{1-r}\right)^{\sigma}.
\end{aligned}$$

Set

$$E = \bigcup_{m=m_3}^{+\infty} \left[r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m\right].$$

Then

$$\int_E \frac{dt}{1-t} = \sum_{m=m_3}^{+\infty} \int_{r_m}^{\frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m} \frac{dt}{1-t} = \sum_{m=m_3}^{+\infty} \log \frac{m}{m-1} = +\infty.$$

$\square$

Lemma 3.8. [1] *Let $g:(0,1)\to\mathbb{R}$ and $h:(0,1)\to\mathbb{R}$ be monotone increasing functions such that $g(r)\leq h(r)$ holds outside of an exceptional set $E\subset[0,1)$ for which $\int_E \frac{dr}{1-r}<+\infty$. Then there exists a constant $d\in(0,1)$ such that if $s(r)=1-d(1-r)$, then $g(r)\leq h(s(r))$ for all $r\in[0,1)$.*

4. Proofs of theorems

Proof of Theorem 2.4.

Proof. Suppose that $f\not\equiv 0$ is a solution of (1) in the sector Ω. From Lemma 3.4, the function $F(u)=f(z((u))$ is a solution of (13), where $z(u)$ is defined by (8). Then, by the formula $T(r,f)=T_0(r,\mathbb{C},f)+O(1)$ and (10), we have

$$\begin{aligned} T(\varrho,B_0(u)) &= T(\varrho,A_0(z(u)))=T_0(\varrho,\mathbb{C},A_0(z(u)))+O(1)\\ &\geq \frac{b}{2}T_0\left(1-\frac{1-\varrho}{b},\Omega_\varepsilon,A_0(z)\right). \end{aligned} \tag{18}$$

By (9), (15) and the formula $T(r,f)=T_0(r,\mathbb{C},f)+O(1)$, for $j=1,2,\ldots,k-1$ we obtain

$$\begin{aligned} T(\varrho,B_j(u)) &\leq \sum_{n=j}^{k-1}T(\varrho,A_n(z(u)))+O\left(\log\frac{1}{1-\varrho}\right)\\ &= \sum_{n=j}^{k-1}T_0(\varrho,\mathbb{C},A_n(z(u)))+O\left(\log\frac{1}{1-\varrho}\right)\\ &\leq \frac{16\pi}{\delta}\sum_{n=j}^{k-1}T_0\left(1-\frac{\delta}{8\pi}(1-\varrho),\Omega,A_n(z)\right)+O\left(\log\frac{1}{1-\varrho}\right). \end{aligned} \tag{19}$$

Now, as $|u|=\varrho\to 1^-$, $u\in\tilde{E}$ ($\tilde{E}$ is a set, image of E by the transformation (7) satisfying $\overline{\text{dens}}E_1=\overline{\text{dens}}\left\{|u|=\varrho:u\in\tilde{E}\right\}>0$), it follows from (2), (3), (18) and (19) that

$$\begin{aligned} T(\varrho,B_0) &\geq \frac{b}{2}\exp_p\left(\eta\log_q\left(\frac{b}{1-\varrho}\right)\right)\\ &= O\left(\exp_p\left(\eta\log_q\left(\frac{1}{1-\varrho}\right)\right)\right) \end{aligned} \tag{20}$$

and for $j=1,2,\ldots,k-1$

$$\begin{aligned} T(\varrho,B_j) &\leq \frac{16\pi}{\delta}(k-j)\exp_p\left(\gamma\log_q\left(\frac{8\pi}{\delta(1-\varrho)}\right)\right)+O\left(\log\frac{1}{1-\varrho}\right)\\ &= O\left(\exp_p\left(\gamma\log_q\left(\frac{1}{1-\varrho}\right)\right)+\log\frac{1}{1-\varrho}\right). \end{aligned} \tag{21}$$

By (13), we can write

$$T(\varrho,B_0)=m(\varrho,B_0)\leq\sum_{j=1}^{k-1}m(\varrho,B_j)+\sum_{j=1}^{k}m\left(\varrho,\frac{F^{(j)}}{F}\right)+O(1). \tag{22}$$

It follows by $(20), (21), (22)$ and lemma of logarithmic derivative Tsuji [12, page 213] that

$$\exp_p\left(\eta\log_q\left(\frac{1}{1-\varrho}\right)\right) \leq O\left(\exp_p\left(\gamma\log_q\left(\frac{1}{1-\varrho}\right)\right)+\log\frac{1}{1-\varrho}\right) +O\left(\log^+ T(\varrho,F)\right) \tag{23}$$

holds for all u satisfying $|u|=\varrho\in E_1\backslash H$ as $\varrho\to 1^-$ ($E_1=\left\{|u|=\varrho : u\in\tilde{E}\right\}$ and $H\subset(0,1)$ is a set with $\int_H \frac{d\varrho}{1-\varrho}<+\infty$). By using Lemma 3.8 and (23), for all u satisfying $|u|=\varrho\in E_1$ as $\varrho\to 1^-$, we obtain

$$\exp_p\left(\eta\log_q\left(\frac{1}{1-\varrho}\right)\right) \leq O\left(\exp_p\left(\gamma\log_q\left(\frac{1}{d(1-\varrho)}\right)\right)+\log\frac{1}{d(1-\varrho)}\right) +O\left(\log^+ T(1-d(1-\varrho),F)\right), \tag{24}$$

where $d\in(0,1)$. Thus, from (24) we get $\sigma_{[p,q]}(F)=+\infty$ and $\sigma_{[p+1,q]}(F)\geq\eta$. Then, by Remark 3.1, we get that

$$\sigma_{[p,q],\Omega}(f(z))=+\infty \text{ and } \sigma_{[p+1,q],\Omega}(f(z))\geq\eta.$$

□

Proof of Theorem 2.5.

Proof. Suppose that $f\not\equiv 0$ is a solution of (1) in the sector Ω. Then for any given $\epsilon>0$, by the result of Theorem 2.4, we have $\sigma_{[p,q],\Omega}(f(z))=+\infty$ and

$$\sigma_{[p+1,q],\Omega}(f(z))\geq\eta-\epsilon. \tag{25}$$

Since $\epsilon>0$ is arbitrary we get from (25) that $\sigma_{[p+1,q],\Omega}(f(z))\geq\eta=\sigma_{[p,q],\Omega_\varepsilon}(A_0)$. On the other hand, by Lemma 3.6 we have $\sigma_{[p+1,q],\Omega_\varepsilon}(f(z))\leq\sigma_{[p,q],\Omega}(A_0)+1$, and if $p>q$, we have $\sigma_{[p+1,q],\Omega_\varepsilon}(f(z))\leq\sigma_{[p,q],\Omega}(A_0)$. □

Proof of Theorem 2.6.

Proof. Suppose that $f\not\equiv 0$ is a solution of (1) in the sector Ω. From Lemma 3.4, the function $F(u)=f(z((u))$ is a solution of (13), where $z(u)$ is defined by (8). As in proof of Theorem 2.4, we obtain (18) and (19). It follows from (18) that

$$\sigma_{[p,q]}(B_0)\geq\sigma_{[p,q],\Omega_\varepsilon}(A_0). \tag{26}$$

Then, from (19), (4) and (26) we get

$$\max_{1\leq j\leq k-1}\left\{\sigma_{[p,q]}(B_j)\right\}\leq\max_{1\leq j\leq k-1}\left\{\sigma_{[p,q],\Omega}(A_j)\right\}<\sigma_{[p,q],\Omega_\varepsilon}(A_0)\leq\sigma_{[p,q]}(B_0). \tag{27}$$

From (27), by using Theorem 2.3, we get $\sigma_{[p,q]}(F)=+\infty$, and

$$\sigma_{[p,q]}(B_0)\leq\sigma_{[p+1,q]}(F)\leq\max_{0\leq j\leq k-1}\left\{\sigma_{M,[p,q]}(B_j)\right\}\leq\max_{0\leq j\leq k-1}\left\{\sigma_{[p,q]}(B_j)\right\}+1,$$

then by Remark 3.1, Lemma 3.6 we see that $\sigma_{[p,q],\Omega}(f)=+\infty$ and $\sigma_{[p,q],\Omega_\varepsilon}(A_0)\leq\sigma_{[p+1,q],\Omega}(f)$, $\sigma_{[p+1,q],\Omega_\varepsilon}(f)\leq\sigma_{[p,q],\Omega}(A_0)+1$.
If $p>q$, then we have $\sigma_{[p,q]}(B_0)=\sigma_{[p+1,q]}(F)$ thus $\sigma_{[p,q],\Omega_\varepsilon}(A_0)\leq\sigma_{[p+1,q],\Omega}(f)$ and $\sigma_{[p+1,q],\Omega_l}(f)\leq\sigma_{[p,q],\Omega}(A_0)$. □

Proof of Theorem 2.7.

Proof. Suppose that $f \not\equiv 0$ is a solution of (1) in the sector Ω. From Lemma 3.4, the function $F(u) = f(z((u))$ is a solution of (13), where $z(u)$ is defined by (8). If $\sigma_{[p,q],\Omega}(A_j) < \sigma_{[p,q],\Omega_\varepsilon}(A_0)$ for all $j = 1, \cdots, k-1$, then Theorem 2.7 reduces to Theorem 2.6. Thus, we assume that at least one of A_j $(j = 1, \cdots, k-1)$ satisfies $\sigma_{[p,q],\Omega}(A_j) = \sigma_{[p,q],\Omega_\varepsilon}(A_0) = \sigma$. So, there exists a set $I \subseteq \{1, \cdots, k-1\}$ such that for $j \in I$ we have $\sigma_{[p,q],\Omega}(A_j) = \sigma_{[p,q],\Omega_\varepsilon}(A_0) = \sigma$ and $\tau_{[p,q],\Omega}(A_j) < \tau_{[p,q],\Omega_\varepsilon}(A_0) = \tau$ and for $j \in \{1, \cdots, k-1\} \backslash I$ we have $\sigma_{[p,q],\Omega}(A_j) < \sigma_{[p,q],\Omega_\varepsilon}(A_0) = \sigma$. Hence, we can choose α_1, α_2 satisfying $\tau_{[p,q],\Omega}(A_j) < \alpha_1 < \alpha_2 < \tau$ $(j \in I)$ such that for any given ϵ $\left(0 < \epsilon < \frac{\alpha_2 - \alpha_1}{2}\right)$ and for sufficiently large r, we have

$$T_0(r, \Omega, A_j(z)) \le \exp_{p-1}\left\{\left(\tau_{[p,q],\Omega}(A_j) + \epsilon\right)\left(\log_{q-1}\frac{1}{1-r}\right)^\sigma\right\}$$
$$\le \exp_{p-1}\left\{(\alpha_1 + \epsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^\sigma\right\}, \ j \in I \tag{28}$$

and

$$T_0(r, \Omega, A_j(z)) \le \exp_p\left((\sigma_1 + \epsilon)\log_q\left(\frac{1}{1-r}\right)\right), \ j \in \{1, \cdots, k-1\} \backslash I, \tag{29}$$

where $\sigma_1 + \epsilon < \sigma$. By Lemma 3.7, there exists a subset $G \subset [0, 1)$ that satisfies $\int_G \frac{dr}{1-r} = +\infty$ such that for all $r \in G$ we have

$$T_0(r, \Omega_\varepsilon, A_0(z)) > \exp_{p-1}\left(\alpha_2\left(\log_{q-1}\frac{1}{1-r}\right)^\sigma\right). \tag{30}$$

Now, as $|u| = \varrho \to 1^-$, $\varrho \in \tilde{G}$ ($\tilde{G}$ is a set, image of G by the transformation (7) satisfying $\int_{\tilde{G}} \frac{d\varrho}{1-\varrho} = +\infty$). Then, by (18) and (30)

$$T(\varrho, B_0(u)) = T(\varrho, A_0(z(u))) = T_0(\varrho, \mathbb{C}, A_0(z(u))) + O(1)$$
$$\ge \frac{b}{2}T_0\left(1 - \frac{1-\varrho}{b}, \Omega_\varepsilon, A_0(z)\right) \ge O\left(\exp_{p-1}\left(\alpha_2\left(\log_{q-1}\frac{1}{1-\varrho}\right)^\sigma\right)\right). \tag{31}$$

Also, by (19), (28) and (29) for $j = 1, 2, \ldots, k-1$

$$T(\varrho, B_j) \le \frac{16\pi}{\delta}\sum_{n=j}^{k-1} T_0\left(1 - \frac{\delta}{8\pi}(1-\varrho), \Omega, A_n(z)\right) + O\left(\log\frac{1}{1-\varrho}\right)$$
$$\le \quad O\left(\exp_{p-1}\left\{(\alpha_1 + \epsilon)\left(\log_{q-1}\frac{1}{1-\varrho}\right)^\sigma\right\}\right)$$
$$+O\left(\exp_p\left((\sigma_1 + \epsilon)\log_q\left(\frac{1}{1-\varrho}\right)\right)\right) + O\left(\log\frac{1}{1-\varrho}\right)$$
$$= O\left(\exp_{p-1}\left\{(\alpha_1 + \epsilon)\left(\log_{q-1}\frac{1}{1-\varrho}\right)^\sigma\right\} + \log\frac{1}{1-\varrho}\right). \tag{32}$$

By (13), we can write

$$T(\varrho, B_0) = m(\varrho, B_0) \le \sum_{j=1}^{k-1} m(\varrho, B_j) + \sum_{j=1}^{k} m\left(\varrho, \frac{F^{(j)}}{F}\right) + O(1). \tag{33}$$

It follows by $(31), (32), (33)$ and lemma of logarithmic derivative Tsuji [12, page 213] that

$$\exp_{p-1}\left(\alpha_2\left(\log_{q-1}\frac{1}{1-\varrho}\right)^{\sigma}\right) \leq O\left(\exp_{p-1}\left\{(\alpha_1+\epsilon)\left(\log_{q-1}\frac{1}{1-\varrho}\right)^{\sigma}\right\}\right)$$
$$O\left(\log\frac{1}{1-\varrho}\right)+O\left(\log^{+}T(\varrho,F)\right) \tag{34}$$

holds for all u satisfying $|u| = \varrho \in \tilde{G}\backslash H$ as $\varrho \to 1^{-}$, where $H \subset (0,1)$ is a set with $\int_H \frac{d\varrho}{1-\varrho} < +\infty$. By using Lemma 3.8 and (34), for all u satisfying $|u| = \varrho \in \tilde{G}$ as $\varrho \to 1^{-}$, we obtain

$$\exp_{p-1}\left(\alpha_2\left(\log_{q-1}\frac{1}{1-\varrho}\right)^{\sigma}\right) \leq O\left(\exp_{p-1}\left\{(\alpha_1+\epsilon)\left(\log_{q-1}\frac{1}{d(1-\varrho)}\right)^{\sigma}\right\}\right)$$
$$+O\left(\log\frac{1}{d(1-\varrho)}\right)+O\left(\log^{+}T(1-d(1-\varrho),F)\right), \tag{35}$$

where $d \in (0,1)$. Thus, from (35) we get $\sigma_{[p,q]}(F) = +\infty$ and $\sigma_{[p+1,q]}(F) \geq \sigma$. Then, by Remark 3.1, we get that

$$\sigma_{[p,q],\Omega}(f(z)) = +\infty \text{ and } \sigma_{[p+1,q],\Omega}(f(z)) \geq \sigma.$$

On the other hand, by Lemma 3.6 we have $\sigma_{[p+1,q],\Omega_\varepsilon}(f(z)) \leq \sigma_{[p,q],\Omega}(A_0)+1$, and if $p > q$, we have $\sigma_{[p+1,q],\Omega_\varepsilon}(f(z)) \leq \sigma_{[p,q],\Omega}(A_0)$. □

5. Conclusion

Throughout this article, we have investigated the properties of growth of solutions of linear complex differential equations by using the concept of [p,q]-order in a sector of the unit disc instead of the whole unit disc. We have obtained similar results as in the case of the unit disc. Recently, several authors [2, 3, 9, 13, 14] have studied the growth of solutions of linear complex differential equations with analytic coefficients of [p,q]-order in the unit disc. So, it is interesting to investigate the growth of solutions when the coefficients of linear complex differential equations are meromorphic of [p,q]-order in a sector of the unit disc.

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