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On the existence of positive solutions for boundary value problems with sign- changing weight and Caffarelli-Kohn-Nirenberg exponents

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Abstract. In this paper we consider the existence of positive solutions to the singular infinite semipositone problems with sign-changing weight. We use the method of sub-supersolution to establish our existence result.

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1. Introduction

The study of positive solutions of singular partial differential equations or systems has been an extremely active research topic during the past few years. Such singular nonlinear problems arise naturally and they occupy a central role in the interdisciplinary research between analysis, geometry, biology, elasticity, mathematical physics, etc.

In this paper, we are concerned with the existence of positive solutions to the boundary value problem

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) = \lambda |x|^{-(\alpha+1)p+\beta} g(x)(f(u) - \frac{1}{u^\gamma}), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ with $0 \in \Omega$ with smooth boundary, $1 < p < N$, $0 \leq \alpha < \frac{N-p}{p}$, $\gamma \in (0,1)$, λ, β are positive constants, $g(x)$ is a C^1 sign-changing function that maybe negative near the boundary and be positive in the interior and $f : (0,\infty) \to (0,\infty)$ is a C^1 nondecreasing function. Elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u)$, where motivated by the following Caffarelli, Kohn and Nirenberg's inequality (see [3, 15]). The study of this type of problems motivated by it's various applications, for example, in fluid mechanics, in newtonian fluids, in flow through porous media and in glaciology (see [1, 4]). So the study of singular elliptic problems has more practical meaning. We refer to ([11, 6, 2, 7]) for additional result on elliptic problem, we study problem (1) in the semipositone case. See [10], where the authors discussed the problem (1) when $g \sim 1$, $\alpha = 0$ and $\beta = p = 2$. In [9], the authors extended the study of [10] to the case when $p > 1$. In [12], the

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prpblem in [9] was studid with weight function $g(x)$. Here we focus on futher extending the study in [12] for quasilinear elliptic problem involving singularity. Due to this singularity in the weights, the extensions are challenging and nontrivial. Our approach is based on the method of sub-supper solution (see[5, 8, 13]). In this paper, we denote $W_0^{1,p}(\Omega, |x|^{-\alpha p})$, the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = (\int_\Omega |x|^{-\alpha p}|\nabla u|^p dx)^{\frac{1}{p}}$. To precisely state our existence result, we consider the eigenvalue problem

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla\phi|^{p-2}\nabla\phi) = \lambda|x|^{-(\alpha+1)p+\beta}|\phi|^{p-2}\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Let ϕ be the eigenfunction corresponding to the first eigenvalue λ_1 of (2) such that $\phi(x) > 0$ in Ω, and $\|\phi\|_\infty = 1$.
Let $m, \sigma, \delta > 0$ be such that

$$\sigma \le \phi \le 1, \qquad x \in \Omega - \overline{\Omega}_\delta, \tag{3}$$

$$|x|^{-\alpha p}\Big(1 - \frac{\gamma p}{p-1+\gamma}\Big)|\nabla\phi|^p \ge m, \qquad x \in \overline{\Omega}_\delta, \tag{4}$$

where $\overline{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \le \delta\}$. This is possible since $|\nabla\phi| \ne 0$ on $\partial\Omega$ while $\phi = 0$ on $\partial\Omega$. We will also consider the unique solution $e \in W_0^{1,p}(\Omega, |x|^{-\alpha p})$ of the boundary value problem

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla e|^{p-2}\nabla e) = |x|^{-(\alpha+1)p+\beta}, & x \in \Omega, \\ e = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is known that $e > 0$ in Ω and $\dfrac{\partial e}{\partial n} < 0$ on $\partial\Omega$.

Here we assume that the weight function $g(x)$ takes negative values in $\overline{\Omega}_\delta$, but require $g(x)$ be strictly positive in $\Omega - \overline{\Omega}_\delta$.

To be precise we assume that there exist positive constants a, b such that $g(x) \ge -a$, on $\overline{\Omega}_\delta$ and $g(x) \ge b$ on $\Omega - \overline{\Omega}_\delta$.

2. Existence result

A non-negative function ψ is said to be a subsolution of problem (1), if it satisfy $\psi \ge 0$ on $\partial\Omega$ and

$$\int_\Omega |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi \cdot \nabla w dx \le \int_\Omega \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f(\psi) - \frac{1}{\psi^\gamma}\Big]w dx \quad \forall w \in W,$$

where $W = \{w \in C_0^\infty(\Omega) : \ w \ge 0 \text{ for all } x \in \Omega\}$ (see [14]).
A function z is said supersolution of (1), if it satisfy $z \ge 0$ on $\partial\Omega$, and

$$\int_\Omega |x|^{-\alpha p}|\nabla z|^{p-2}\nabla z \cdot \nabla w dx \ge \int_\Omega \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f(z) - \frac{1}{z^\gamma}\Big]w dx, \quad \forall w \in W.$$

Then the following result holds:

Lemma 2.1. *(see [8]). If there exist a sub-solution ψ and supersolution z such that $\psi \le z$ in Ω then (1) has a weak-solution u such that $\psi \le u \le z$.*
We make the following assumptions:
(H_1) $f : (0, \infty) \to (0, \infty)$ *is C^1 nondecreasing function.*

(H_2) $\lim\limits_{s\to\infty} \dfrac{f(s)}{s^{p-1}} = 0.$

(H_3) *suppose that there exist* $\epsilon > 0$ *such that*

i) $f\Big(\dfrac{\epsilon^{\frac{1}{p-1}}(p-1+\gamma)\sigma}{p}\Big) > \Big(\dfrac{p}{\epsilon^{\frac{1}{p-1}}\sigma(p-1+\gamma)}\Big)^{\gamma}$,

ii) $\dfrac{\epsilon^{\frac{\gamma+p-1}{p-1}}\lambda_1(p-1+\gamma)^{\gamma}}{ap^{\gamma}} < \dfrac{m\epsilon}{af(\epsilon^{\frac{1}{p-1}})}$,

iii) $\dfrac{\epsilon\lambda_1}{Nb} < \dfrac{m\epsilon}{af\Big(\epsilon^{\frac{1}{p-1}}\Big)}$, *where* $N = f\Big(\dfrac{\epsilon^{\frac{1}{p-1}}(p-1+\gamma)\sigma}{p}\Big) - \Big(\dfrac{p}{\epsilon^{\frac{1}{p-1}}\sigma(p-1+\gamma)}\Big)^{\gamma}$.

iv) Let $\eta > 0$ *be such that* $\eta \geq \max |x|^{-(\alpha+1)p+\beta}$, *in* $\overline{\Omega}_\delta$.

We are now ready to give our existence result.

Theorem 2.2. *Let* $(H_1) - (H_3)$ *hold, then there exists positive weak solution of (1) for every* $\lambda \in [\lambda_*(\epsilon), \lambda^*(\epsilon)]$, *where*

$\lambda^* = \dfrac{m\epsilon}{\eta af\Big(\epsilon^{\frac{1}{p-1}}\Big)}$ *and* $\lambda_* = \max\left\{\dfrac{\epsilon^{\frac{\gamma+p-1}{p-1}}\lambda_1(p-1+\gamma)^{\gamma}}{ap^{\gamma}}, \dfrac{\epsilon\lambda_1}{Nb}\right\}$.

Remark 2.1. Note that (H_3) implies $\lambda_* < \lambda^*$.

Proof. Now we construct a positive sub-solution of (1). For this, we let

$$\psi = \frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}.$$

Let $w \in W$. Since $\nabla\psi = \epsilon^{\frac{1}{p-1}}\phi^{\frac{1-\gamma}{p-1+\gamma}}\nabla\phi$, then a calculation shows that

$$\begin{aligned}
&\int_\Omega |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w dx = \epsilon\int_\Omega |x|^{-\alpha p}\phi^{\frac{(p-1)(1-\gamma)}{p-1+\gamma}}|\nabla\phi|^{p-2}\nabla\phi\cdot\nabla w dx\\
&= \epsilon\int_\Omega |x|^{-\alpha p}|\nabla\phi|^{p-2}\nabla\phi\Big[\nabla(\phi^{1-\frac{\gamma p}{p-1+\gamma}}w) - \nabla(\phi^{1-\frac{\gamma p}{p-1+\gamma}})w\Big]dx\\
&= \epsilon\int_\Omega |x|^{-\alpha p}|\nabla\phi|^{p-2}\nabla\phi\cdot\nabla(\phi^{1-\frac{\gamma p}{p-1+\gamma}}w)dx - \epsilon\int_\Omega |x|^{-\alpha p}|\nabla\phi|^{p-2}\nabla\phi\cdot\nabla(\phi^{1-\frac{\gamma p}{p-1+\gamma}})w dx\\
&= \epsilon\int_\Omega |x|^{-(\alpha+1)p+\beta}\lambda_1\phi^{\frac{-\gamma p}{p-1+\gamma}}\phi^p w dx - \epsilon\int_\Omega |x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)\phi^{-\frac{\gamma p}{p-1+\gamma}}|\nabla\phi|^p w dx\\
&= \epsilon\int_\Omega \Big[|x|^{-(\alpha+1)p+\beta}\lambda_1\phi^{p-\frac{\gamma p}{p-1+\gamma}} - |x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)\phi^{-\frac{\gamma p}{p-1+\gamma}}|\nabla\phi|^p\Big] w dx.
\end{aligned}$$

First we consider the case when $x \in \overline{\Omega}_\delta$. We have $|x|^{-\alpha p}\Big(1 - \dfrac{\gamma p}{p-1+\gamma}\Big)|\nabla\phi|^p \geq m$ and $g(x) \geq -a$. Hence since $\lambda \leq \lambda^* = \dfrac{m\epsilon}{\eta af(\epsilon^{\frac{1}{p-1}})}$, we have

$$\begin{aligned}
&-|x|^{-\alpha p}\epsilon\Big(1-\tfrac{\gamma p}{p-1+\gamma}\Big)\phi^{\frac{-\gamma p}{p-1+\gamma}}|\nabla\phi|^p \leq -m\epsilon\phi^{\frac{-\gamma p}{p-1+\gamma}} \leq -m\epsilon\\
&\quad \leq -\lambda a\eta f(\epsilon^{\frac{1}{p-1}}) \leq -\lambda a|x|^{-(\alpha+1)p+\beta} f\Big(\tfrac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big),
\end{aligned} \tag{5}$$

and since $\lambda \geq \lambda_* = \dfrac{\epsilon^{\frac{\gamma+p-1}{p-1}}\lambda_1(p-1+\gamma)^\gamma}{ap^\gamma}$, we have

$$|x|^{-(\alpha+1)p+\beta}\epsilon\phi^{-\frac{\gamma p}{p-1+\gamma}}\lambda_1\phi^p \leq \frac{\lambda|x|^{-(\alpha+1)p+\beta}ap^\gamma}{\epsilon^{\frac{\gamma}{p-1}}(p-1+\gamma)^\gamma} \leq \frac{|x|^{-(\alpha+1)p+\beta}\lambda a}{\left(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\right)^\gamma}. \tag{6}$$

By combining (5) and (6) we see that

$$\begin{aligned}&\epsilon\Big[|x|^{-(\alpha+1)p+\beta}\phi^{p-\frac{\gamma p}{p-1+\gamma}}\lambda_1 - |x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)\phi^{-\frac{\gamma p}{p-1+\gamma}}|\nabla\psi|^p\Big]\\ &\quad\leq \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f\Big(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big) - \frac{1}{\left(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\right)^\gamma}\Big].\end{aligned}$$

On the other hand, on $\Omega - \overline{\Omega}_\delta$, we have $g(x) \geq b$ and $\sigma \leq \phi^{\overline{\frac{p}{p-1+\gamma}}} \leq 1$. Thus for $\lambda \geq \lambda_* \geq \dfrac{\epsilon\lambda_1}{Nb}$, we have

$$\begin{aligned}&\epsilon\Big(|x|^{-(\alpha+1)p+\beta}\phi^{-\frac{\gamma p}{p-1+\gamma}}\lambda_1\phi^p - |x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)\phi^{-\frac{\gamma p}{p-1+\gamma}}|\nabla\phi|^p\Big)\\ &\quad\leq |x|^{-(\alpha+1)p+\beta}\epsilon\lambda_1\phi^{p-\frac{\gamma p}{p-1+\gamma}} \leq |x|^{-(\alpha+1)p+\beta}\epsilon\lambda_1 \leq |x|^{-(\alpha+1)p+\beta}\lambda bN\\ &\quad\leq |x|^{-(\alpha+1)p+\beta}\lambda b\Big[f\Big(\frac{p-1+\gamma}{p}\Big)\sigma\epsilon^{\frac{1}{p-1}} - \frac{1}{\left((\frac{p-1+\gamma}{p})\sigma\epsilon^{\frac{1}{p-1}}\right)^\gamma}\Big]\\ &\quad\leq |x|^{-(\alpha+1)p+\beta}\lambda g(x)\Big[f\Big(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big) - \frac{1}{\left((\frac{p-1+\gamma}{p})\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\right)^\gamma}\Big]\\ &\quad= \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f(\psi) - \frac{1}{\psi^\gamma}\Big].\end{aligned}$$

Hence

$$\begin{aligned}&\int_\Omega |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w dx\\ &\quad\leq \epsilon\int_\Omega \Big[|x|^{-(\alpha+1)p+\beta}\lambda_1(\phi^{p-\frac{\gamma p}{p-1+\gamma}}) - |x|^{-\alpha p}\Big(1-\frac{\gamma p}{p-1+\gamma}\Big)\phi^{-\frac{\gamma p}{p-1+\gamma}}|\nabla\phi|^p\Big]wdx\\ &\quad\leq \int_\Omega \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f\Big(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\Big) - \frac{1}{\left(\frac{p-1+\gamma}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\gamma}}\right)^\gamma}\Big]wdx\\ &\quad= \int_\Omega \lambda|x|^{-(\alpha+1)p+\beta}g(x)\Big[f(\psi)-\frac{1}{\psi^\gamma}\Big]wdx.\end{aligned}$$

So ψ is a sub- solution of (1) for $\lambda \in [\lambda_*, \lambda^*]$.
Now we will construct a supersolution of (1). For this, we let $z := ce$ and $w \in W$. Since $\nabla z = c\nabla e$ then a calculation shows that

$$\begin{aligned}&\int_\Omega |x|^{-\alpha p}|\nabla z|^{p-2}\nabla z\cdot\nabla w dx = c^{p-1}\int_\Omega |x|^{-\alpha p}|\nabla e|^{p-2}\nabla e\cdot\nabla w dx\\ &\quad= -c^{p-1}\textstyle\int_\Omega div(|x|^{-\alpha p}|\nabla e|^{p-2}\nabla e)wdx \ = c^{p-1}\displaystyle\int_\Omega |x|^{-(\alpha+1)p+\beta}wdx.\end{aligned}$$

By (H_2) we can choose c large enough so that

$$(c\|e\|_\infty)^{p-1}(\lambda\|g(x)\|_\infty\|e\|_\infty)^{-1} \geq f(c\|e\|_\infty).$$

Hence

$$c^{p-1} \geq \lambda \|g(x)\|_\infty f(c\|e\|_\infty) \geq \lambda g(x) f(ce) \geq \lambda g(x)\Big[f(ce) - \frac{1}{(ce)^\gamma}\Big] = \lambda g(x)\Big[f(z) - \frac{1}{z^\gamma}\Big].$$

Thus we have

$$\int_\Omega |x|^{-\alpha p} |\nabla z|^{p-2} \nabla z \cdot \nabla w dx = c^{p-1} \int_\Omega |x|^{-(\alpha+1)p+\beta} w dx$$
$$\geq \int_\Omega |x|^{-(\alpha+1)p+\beta} \lambda g(x)\Big[f(ce) - \tfrac{1}{(ce)^\gamma}\Big] w dx = \int_\Omega |x|^{-(\alpha+1)p+\beta} \lambda g(x)\Big[f(z) - \frac{1}{z^\gamma}\Big] w dx.$$

So z is a supersolution of (1) with $z \geq \psi$ for c large. Thus, there exist a positive weak solution u of (1) such that $\psi \leq u \leq z$. This completes the proof of Theorem 2.2. □

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