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On a nonlocal problem involving the generalized anisotropic $\overrightarrow{p}(\cdot)$-Laplace operator

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Abstract. In this paper, we study an anisotropic nonlocal problem which is a stationary counterpart of the Kirchhoff equation settled in the variable exponent Sobolev spaces $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. By using the variational approach and applying the Mountain-Pass theorem along with the Fountain theorem, we obtain the existence and multiplicity of nontrivial weak solutions.

Key words and phrases. $\overrightarrow{p}(\cdot)$-Laplace operator, Leray-Lions type operator, variational approach, nonlocal problem, anisotropic variable exponent Sobolev spaces, Mountain-Pass theorem, Fountain theorem.

1. Introduction

We study the anisotropic nonlocal problem

$$\begin{cases} -M\left(\int_\Omega \sum_{i=1}^N A_i\left(x,\partial_{x_i}u\right)dx\right)\sum_{i=1}^N \partial_{x_i} a_i\left(x,\partial_{x_i}u\right) = \lambda f(x,u), \ x\in\Omega, \\ u=0, \ x\in\partial\Omega, \end{cases} \qquad (\mathbf{P})$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary, $a_i:\Omega\times\mathbb{R}\to\mathbb{R}$ is Carathéodory such that $a_i(x,\xi)$ is the continuous derivative with respect to ξ of the mapping $A_i:\Omega\times\mathbb{R}\to\mathbb{R}$, $A_i=A_i(x,\xi)$, i.e., $a_i(x,\xi)=\frac{\partial}{\partial\xi}A_i(x,\xi)$, $p_i\in C\left(\overline{\Omega}\right)$ such that $2\leq p_i(x)<N$ for any $x\in\overline{\Omega}$ and $i\in\{1,...,N\}$; λ is a positive parameter; Kirchhoff function M is continuous and f is a Carathéodory function.

The operator $\sum_{i=1}^N \partial_{x_i} a_i\left(x,\partial_{x_i}u\right)$ that appears in $(\mathbf{P})$ is called the anisotropic $\overrightarrow{p}(\cdot)$-Laplace operator which is a generalization of the well-known $p(\cdot)$-Laplace operator given by $\sum_{i=1}^N \partial_{x_i}\left(|\partial_{x_i}u|^{p(x)-2}\partial_{x_i}u\right)$, for the case $p_i=p$ for each $i\in\{1,...,N\}$. For the papers involving the $p(\cdot)$-Laplace operator see, e.g., [5, 6, 16, 24, 27]. The nonlinear differential equations involving the $p(\cdot)$-Laplace operator has been very popular for the last decade, since they can be used to model dynamical phenomena which arise from the study of electrorheological fluids or elastic mechanics. Moreover, differential equations with variable exponent growth are used for the modelling of many physical processes such as stationary thermo-rheological viscous flows of non-Newtonian fluids and the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium (see, e.g, [2, 7, 30]).

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The equation $(\mathbf{P})$ is a generalization of the Kirchhoff equation [21]

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where ρ, P_0, h, E, L are constants. Equation (1.1) contains a nonlocal coefficient $\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ depending on the average $\frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ of the kinetic energy $\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2$ on $[0, L]$. We refer the reader to [4, 8, 13, 15] for Kirchhoff-type equations involving the $p(\cdot)$-Laplace operator.

The operator $\sum_{i=1}^N \partial_{x_i} a_i \left(x, \partial_{x_i} u\right)$ used in this paper is more general than the operator $\sum_{i=1}^N \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right)$ which is already dealt by many authors [1, 9, 10, 17, 25]. The main advantage of this sort of operators is that they could be used to model some processes requiring distinct behavior of partial differential derivatives in various directions. There are already studies which dealt with problem $(\mathbf{P})$ for the case $M(t) = 1$ under the similar conditions assumed in the present paper. For example, in [11], the authors investigated an anisotropic Neumann problem of the following type

$$\begin{cases} -\sum_{i=1}^N \partial_{x_i} a_i \left(x, \partial_{x_i} u\right) = f(x,u), & \text{in } \Omega, \\ u \geq 0, & \text{in } \Omega, \\ \sum_{i=1}^N a_i \left(x, \partial_{x_i} u\right) v_i = g(x,u), & \text{on } \partial\Omega. \end{cases}$$

By showing some compact boundary trace embeddings, they obtained the existence and the uniqueness of solutions.

In [12], the author investigated an anisotropic Dirichlet problem of the following type

$$\begin{cases} -\sum_{i=1}^N \partial_{x_i} a_i \left(x, \partial_{x_i} u\right) = f(x,u), & \text{in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases}$$

Using the symmetric Mountain-Pass theorem of Ambrosetti and Rabinowitz, the author obtained the existence of an unbounded sequence of weak solutions to problem.

In [9], the authors studied a nonhomogeneous anisotropic Kirchhoff problem of the following type

$$\begin{cases} -M \left(\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx \right) \sum_{i=1}^N \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right) = \lambda f(x,u), \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \end{cases} \tag{$\mathbf{P}^*$}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, λ is a positive parameter, $p_i \in C\left(\overline{\Omega}\right)$ such that $2 \leq p_i(x) < N$ for any $x \in \overline{\Omega}$ and $i \in \{1, ..., N\}$, M and f continuous functions which obey some specific conditions. Applying the Mountain Pass Theorem of Ambrosetti and Rabinowitz, the existence of a nontrivial weak solution is obtained in the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$, provided that the positive parameter λ that multiplies the nonlinearity f is small enough.

The goal of the present paper is to generalize the results of [9]. To this end, the Laplace type operator $\sum_{i=1}^N \partial_{x_i} (|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u)$ appeared in $(\mathbf{P}^*)$ is replaced by a Leray-Lions type operator $\sum_{i=1}^N \partial_{x_i} a_i(x, \partial_{x_i} u)$ that appears in problem $(\mathbf{P})$, which is a more general operator. This caused some difficulties in calculations and required more general conditions. Moreover, thanks to Fountain theorem, we show not only the existence of a nontrivial weak solution, but also the multiplicity of nontrivial weak solutions in the present paper. To our best knowledge, the present papers results are not covered in the literature.

2. Preliminaries

Set

$$C_+\left(\overline{\Omega}\right) = \left\{ p \in C\left(\overline{\Omega}\right) : \min_{x\in\overline{\Omega}} p\left(x\right) \ > 1 \right\}.$$

For $p \in C_+\left(\overline{\Omega}\right)$, we use the notations

$$p^- := \inf_{x\in\Omega} p\left(x\right) \quad \text{and} \quad p^+ := \sup_{x\in\Omega} p\left(x\right).$$

The variable exponent Lebesgue space is given by

$$L^{p(\cdot)}\left(\Omega\right) = \left\{ u \mid \text{the map } u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega \left|u\left(x\right)\right|^{p(x)} dx < \infty \right\},$$

where $p \in C_+\left(\overline{\Omega}\right)$. The regular norm given on this pace is

$$|u|_{p(\cdot)} = \inf\left\{ \mu > 0 : \int_\Omega \left|\frac{u\left(x\right)}{\mu}\right|^{p(x)} dx \leq 1 \right\},$$

which is called the Luxemburg norm. The pair $\left(L^{p(\cdot)}\left(\Omega\right), |\cdot|_{p(\cdot)}\right)$ defines a separable and reflexive Banach space [[23], Theorem 2.5, Corollary 2.7]. When $0 < |\Omega| < \infty$ and $p_1, p_2 \in C_+\left(\overline{\Omega}\right)$ such that $p_1 \leq p_2$ in Ω, then the embedding $L^{p_2(\cdot)}\left(\Omega\right) \hookrightarrow L^{p_1(\cdot)}\left(\Omega\right)$ is continuous [[23], Theorem 2.8].

For any $u \in L^{p(\cdot)}\left(\Omega\right)$ and $v \in L^{p'(\cdot)}\left(\Omega\right)$ the following Hölder-type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(\cdot)} |v|_{p'(\cdot)} \leq 2 |u|_{p(\cdot)} |v|_{p'(\cdot)} \tag{2.1}$$

holds[[23], Theorem 2.1], where $L^{p'(\cdot)}\left(\Omega\right)$ denotes the conjugate space of $L^{p(\cdot)}\left(\Omega\right)$, and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ [[23], Corollary 2.7].

We introduce the convex functional $\rho_{p(\cdot)} : L^{p(\cdot)}\left(\Omega\right) \to \mathbb{R}$ by

$$\rho_{p(\cdot)}\left(u\right) = \int_\Omega \left|u\left(x\right)\right|^{p(x)} dx,$$

which is called the $p\left(\cdot\right)$−modular of the $L^{p(\cdot)}\left(\Omega\right)$ spaces. The followings are some important properties of $p\left(\cdot\right)$−modular [23],

$$|u|_{p(\cdot)} < 1\left(=1; >1\right) \Leftrightarrow \rho_{p(\cdot)}\left(u\right) < 1\left(=1; >1\right), \tag{2.2}$$

$$|u|_{p(\cdot)} > 1 \implies |u|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}\left(u\right) \leq |u|_{p(\cdot)}^{p^+}, \tag{2.3}$$

$$|u|_{p(\cdot)} < 1 \implies |u|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}\left(u\right) \leq |u|_{p(\cdot)}^{p^-}, \tag{2.4}$$

$$|u_n|_{p(\cdot)} \to 0 \ \left(\to \infty\right) \Leftrightarrow \rho_{p(\cdot)}\left(u_n\right) \to 0 \ \left(\to \infty\right), \tag{2.5}$$

$$|u_n - u|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0. \tag{2.6}$$

provided that $u \in L^{p(\cdot)}\left(\Omega\right)$, $\left(u_n\right) \subset L^{p(\cdot)}\left(\Omega\right)$ and $p^+ < \infty$.

The variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{1,p(\cdot)} = |\nabla u|_{p(\cdot)}.$$

We remark that this norm is equivalent to the norm

$$\|u\|_{p(\cdot)} = \textstyle\sum_{i=1}^N |\partial_{x_i} u|_{p(\cdot)},$$

if $p(x) \geq 2$, $\forall x \in \overline{\Omega}$ (see [26]). As a result, $W_0^{1,p(\cdot)}(\Omega)$ becomes a separable and reflexive Banach space. Furthermore, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous provided that $s \in C_+\left(\overline{\Omega}\right)$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = +\infty$ if $p(x) \geq N$. We refer to the papers [14, 19, 20, 23, 27] for further reading related to the variable exponent Lebesgue-Sobolev spaces.

Let us denote by $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$ the vectorial function $\overrightarrow{p}(\cdot) = (p_1(\cdot), ..., p_N(\cdot))$ with $p_i \in C_+\left(\overline{\Omega}\right)$, $i \in \{1, ..., N\}$. We define $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, the anisotropic variable exponent Sobolev space a natural generalization of the variable exponent space $W_0^{1,p(\cdot)}(\Omega)$, as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \textstyle\sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}.$$

We know that $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a separable and reflexive Banach space [17, 25]. It is clear that in the case when $p_i \in C_+\left(\overline{\Omega}\right)$ are constant functions, the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ turns into the space $W_0^{1,\overrightarrow{p}}(\Omega)$, where $\overrightarrow{p}$ is the constant vector $(p_1, ..., p_N)$. This kind of spaces studied in [18, 28, 29, 31, 32].

On the other hand, in order to facilitate the manipulation of the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we introduce $\overrightarrow{P}_+, \overrightarrow{P}_- \in \mathbb{R}^N$ as

$$\overrightarrow{P}_+ = \left(p_1^+, ..., p_N^+\right), \quad \overrightarrow{P}_- = \left(p_1^-, ..., p_N^-\right),$$

and $P_+^+, P_-^+, P_-^- \in \mathbb{R}^+$ as

$$P_+^+ = \max\left\{p_1^+, ..., p_N^+\right\}, \ P_-^+ = \max\left\{p_1^-, ..., p_N^-\right\}, \ P_-^- = \min\left\{p_1^-, ..., p_N^-\right\}.$$

Throughout the paper, for the exponent $p_1(\cdot), ..., p_N(\cdot)$, we assume that $p_i \in C_+\left(\overline{\Omega}\right)$, $i \in \{1, ..., N\}$ such that

$$2 \leq p_i(x) < N, \quad \textstyle\sum_{i=1}^N \dfrac{1}{p_i^-} > 1, \tag{2.7}$$

and define $P_-^* \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1}, \quad P_{-,\infty} = \max\left\{P_-^+, P_-^*\right\}.$$

Proposition 2.1. [[25], *Theorem 1*] *Suppose that $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary and relation* (2.7) *is fulfilled. For any $q \in C\left(\overline{\Omega}\right)$ verifying*

$$1 < q(x) < P_{-,\infty} \text{ for all } x \in \overline{\Omega}, \tag{2.8}$$

the embedding

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is continuous and compact.

Definition 2.1. Let X be a Banach space and $J : X \to \mathbb{R}$ be a C^1-functional. We say that J satisfies the Palais-Smale ((**PS**) for short) condition, if every sequence $\{u_n\} \subset X$ such that $|J(u_n)| \leq c$ and $J'(u_n) \to 0$ as $n \to \infty$, contains a convergent subsequence in the norm of X.

Definition 2.2. It is said that $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a weak solution to (**P**) if

$$M\left(\int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\, dx\right) \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\, \partial_{x_i}\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx = 0,$$

where $\varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

The functional $J_\lambda : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \mathbb{R}$ associated to the problem (**P**) is

$$J_\lambda(u) = \widehat{M}\left(\int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\, dx\right) - \lambda \int_\Omega F(x,u)\, dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$ and $F(x,t) = \int_0^t f(x,s)\, ds$ for $t \in \mathbb{R}$ and $x \in \Omega$. Since problem (**P**) is in variational setting, it is well known that weak solutions of (**P**) correspond to the critical points of functional J_λ.

3. Main results

In the present paper, we assume that M, f, A_i and a_i, $i \in \{1, ..., N\}$, fulfil the following conditions:

($\mathbf{M}_1$) Assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and satisfies the growth condition

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1} \quad \text{for all } t > 0, \tag{3.1}$$

where m_1, m_2 and α are real numbers such that $1 < m_1 \leq m_2$ with $\alpha > 1$.

($\mathbf{f}_1$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition such that

$$|f(x,t)| \leq c_1 + c_2 |t|^{q(x)-1} \ \forall (x,t) \in \Omega \times \mathbb{R},$$

where $c_1, c_2 > 0$ and $q \in C(\overline{\Omega})$ with $2 \leq P_-^- \leq P_+^+ < \alpha P_+^+ < q^- \leq q^+ < P_-^*$ for all $x \in \Omega$;

($\mathbf{f}_2$) $f(x,t) = o\left(|t|^{\alpha P_+^+ - 1}\right)$, $t \to 0$ uniformly for $x \in \Omega$;

($\mathbf{f}_3$) $f(x,-t) = -f(x,t) \ \forall (x,t) \in \Omega \times \mathbb{R}$*;*

(**AR**) Ambrosetti-Rabinowitz's condition holds, i.e., there exists $\theta > \frac{m_2}{m_1}\alpha P_+^+$ and $K > 0$ such that

$$0 < \theta F(x,t) \leq f(x,t)t \text{ for } |t| \geq K \text{ and for all } x \in \Omega;$$

($\mathbf{a_1}$) The following inequalities hold true

$$|a_i(x,t)| \leq c_3(h_i(x) + |t|^{p_i(x)-1}) \ \forall (x,t) \in \Omega \times \mathbb{R},$$

where $c_3 > 0$ and $h_i \in L^{p'(x)}(\Omega)$ are nonnegative measurable functions.

($\mathbf{a}_2$) The following inequalities hold true

$$|t|^{p_i(x)} \leq a_i(x,t)t \leq p_i(x) A_i(x,t) \ \forall (x,t) \in \Omega \times \mathbb{R}.$$

As a corollary of ($\mathbf{a}_2$), A_i are $p_i(x)$-homogeneous, i.e.,

$$A_i(x,t\xi) \leq A_i(x,\xi) t^{p_i(x)}, \tag{3.2}$$

t, $\xi \in \mathbb{R}$ with $t \geq 1$ and $x \in \Omega$.

Indeed, if we set $g(t) = A_i(x,t\xi)$, then by ($\mathbf{a}_2$), we get

$$\begin{aligned} g'(t) &= a_i(x,t\xi)\xi = \frac{1}{t} a_i(x,t\xi) t\xi \leq \frac{p_i(x)}{t} A_i(x,t\xi) = \frac{p_i(x)}{t} g(t) \\ \frac{g'(t)}{g(t)} &\leq \frac{p_i(x)}{t}, \end{aligned}$$

and integrating both side of the last inequality over $(1,t)$, we conclude that

$$A_i(x,t\xi) \leq A(x,\xi) t^{p_i(x)}.$$

($\mathbf{A}$) $A_i(x,-\xi) = A_i(x,\xi)$ for all $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$.

As we mentioned before the operator $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$ that appears in problem ($\mathbf{P}$) is a Leray-Lions type operator and it can be particularized to some well-known operators. For example, when we take

$$a_i(x,t) = |t|^{p_i(x)-2} t \ \text{ for all } i \in \{1,...,N\},$$

we have

$$A_i(x,t) = 1/p_i(x)\, |t|^{p_i(x)} \quad \text{for all } i \in \{1,...,N\},$$

and we get $\overrightarrow{p}(\cdot)$ −Laplace operator

$$\textstyle\sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right).$$

On the other hand, if we let

$$a_i(x,t) = (1+|t|^2)^{(p_i(x)-2)/2} t \ \text{ for all } i \in \{1,...,N\},$$

we have

$$A_i(x,t) = 1/p_i(x)[(1+|t|^2)^{p_i(x)/2} - 1] \ \text{ for all } i \in \{1,...,N\},$$

then we obtain the anisotropic variable mean curvature operator

$$\textstyle\sum_{i=1}^{N} \partial_{x_i} [(1+|\partial_{x_i} u|^2)^{(p_i(x)-2)/2} \partial_{x_i} u)].$$

Theorem 3.1. *Assume that* ($\mathbf{M}_1$), ($\mathbf{f}_1$), ($\mathbf{f}_2$), ($\mathbf{AR}$), ($\mathbf{a}_1$) *and* ($\mathbf{a}_2$) *hold. Then there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0,\lambda^*)$ *problem* ($\mathbf{P}$)*has a nontrivial weak solution.*

To obtain the result of Theorem 3.1, we will apply Mountain-Pass theorem (see [33]). Therefore, we have to verify Lemma 3.3 and Lemma 3.4.

First of all, we must show that J_λ satisfies some basic properties.

Lemma 3.2. *The functional* J_λ *is well-defined on* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *and Fréchet differentiable, i.e.,* $J_\lambda \in C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ *whose derivative is*

$$\langle J'_\lambda(u), \varphi \rangle = M\left(\int_\Omega \textstyle\sum_{i=1}^{N} A_i(x,\partial_{x_i} u)\, dx \right) \displaystyle\int_\Omega \textstyle\sum_{i=1}^{N} a_i(x,\partial_{x_i} u)\, \partial_{x_i} \varphi dx - \lambda \displaystyle\int_\Omega f(x,u)\varphi dx,$$

for all $u, \varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proof. For simplicity, we denote by $\Lambda, K : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \mathbb{R}$,

$$\Lambda(u) := \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\, dx,$$

and

$$K(u) := \widehat{M}(\Lambda(u)),$$

for all $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Then, we write

$$J_\lambda(u) = K(u) - \lambda \int_\Omega F(x,u)\, dx.$$

From assumption $(\mathbf{f}_1)$ and Proposition 2.1, it is easy to see that $\int_\Omega F(x,u)\,dx$ is well-defined on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\int_\Omega F(x,u)\,dx \in C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$. Therefore, showing that K is well-defined on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $K \in C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ is equivalent to saying that $J_\lambda \in C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$.

Using $(\mathbf{a_1})$ and some well-known results, the authors showed in [22] that the functional Λ is well-defined and is of class $C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ and its derivative $\Lambda' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is

$$\langle \Lambda'(u), \varphi \rangle = \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\, \partial_{x_i}\varphi dx,$$

for all $u, \varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Moreover, since M is a continuous function and satisfies growth condition $(\mathbf{M}_1)$, it is easy to see that the composition functional $K(u) = \widehat{M}(\Lambda(u))$ is well-defined and of class $C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ and its derivative $K' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is

$$\langle K'(u), \varphi \rangle = M\left(\int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\, dx\right) \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\, \partial_{x_i}\varphi dx,$$

for all $u, \varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. The all pieces of information mentioned above implies that J_λ is of class $C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ and its derivative is

$$\langle J_\lambda'(u), \varphi \rangle = M\left(\int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\, dx\right) \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\, \partial_{x_i}\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx,$$

for all $u, \varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. □

Lemma 3.3. *Assume that* $(\mathbf{M}_1)$, $(\mathbf{f}_1)$, $(\mathbf{f}_2)$,$(\mathbf{AR})$ *and* $(\mathbf{a_1}), (\mathbf{a}_2)$ *hold. Then the following statements hold true:*

(i) There are two real numbers $\gamma > 0$ *and* $\tau > 0$ *such that* $J_\lambda(u) \geq \tau > 0$, $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *with* $\|u\|_{\overrightarrow{p}(\cdot)} = \gamma$,

(ii) There is $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *such that* $\|u\|_{\overrightarrow{p}(\cdot)} > \gamma$, $J_\lambda(u) < 0$.

Proof. (i) From $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$, there exist $\varepsilon, C_\varepsilon > 0$ such that for all $x \in \Omega$ and $t \in \mathbb{R}$, we obtain $F(x,t) \leq \varepsilon |t|^{\alpha P_+^+} + C_\varepsilon |t|^{q(x)}$. Then, from $(\mathbf{M}_1)$ and $(\mathbf{a}_2)$, we have

$$\begin{aligned}
J_\lambda(u) &= \widehat{M}(\Lambda(u)) - \lambda \int_\Omega F(x,u)\,dx \\
&\geq \frac{m_1}{\alpha}\left(\int_\Omega \textstyle\sum_{i=1}^N A_i(x,\partial_{x_i}u)\,dx\right)^\alpha - \lambda\varepsilon \int_\Omega |u|^{\alpha P_+^+}\,dx - \lambda C_\varepsilon \int_\Omega |u|^{q(x)}\,dx \\
&\geq \frac{m_1}{\alpha\left(P_+^+\right)^\alpha}\left(\int_\Omega \textstyle\sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)}dx\right)^\alpha \qquad (3.3)\\
&\quad -\lambda(\varepsilon |u|_{\alpha P_+^+}^{\alpha P_+^+} + C_\varepsilon(|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}))
\end{aligned}$$

for any $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Since we have the continuous embeddings $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{\alpha P_+^+}(\Omega)$ and $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q^+}(\Omega) \hookrightarrow L^{q^-}(\Omega)$ (see Proposition 2.1), there are constants $c_4, c_5, c_6 > 0$ such that for all $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$c_4 \|u\|_{\overrightarrow{p}(\cdot)} \geq |u|_{\alpha P_+^+},\ c_5 \|u\|_{\overrightarrow{p}(\cdot)} \geq |u|_{q^+} \text{ and } c_6 \|u\|_{\overrightarrow{p}(\cdot)} \geq |u|_{q^-}. \qquad (3.4)$$

Let $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $\|u\|_{\overrightarrow{p}(\cdot)} < 1$. Then $|\partial_{x_i}u|_{p_i(\cdot)} < 1$, and from (2.4), it follows

$$\begin{aligned}
\int_\Omega \textstyle\sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)}dx &\geq \textstyle\sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}^{p_i^+} \geq \sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}^{P_+^+} \\
&\geq N\left(\frac{\sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}}{N}\right)^{P_+^+} = \frac{\|u\|_{\overrightarrow{p}(\cdot)}^{P_+^+}}{N^{P_+^+-1}}. \qquad (3.5)
\end{aligned}$$

By taking into account (2.4) and (3.3) – (3.5), we get

$$\begin{aligned}
J_\lambda(u) &\geq \frac{m_1}{\alpha\left(P_+^+N^{P_+^+-1}\right)^\alpha}\|u\|_{\overrightarrow{p}(\cdot)}^{\alpha P_+^+} - \lambda(c_7\|u\|_{\overrightarrow{p}(\cdot)}^{\alpha P_+^+} + c_8\|u\|_{\overrightarrow{p}(\cdot)}^{q^+} + c_9\|u\|_{\overrightarrow{p}(\cdot)}^{q^+}) \\
&\geq \frac{m_1}{\alpha\left(P_+^+N^{P_+^+-1}\right)^\alpha}\|u\|_{\overrightarrow{p}(\cdot)}^{\alpha P_+^+} - \lambda c_7\|u\|_{\overrightarrow{p}(\cdot)}^{\alpha P_+^+} - 2\max\{c_8,c_9\}\lambda\|u\|_{\overrightarrow{p}(\cdot)}^{q^-} \\
&= \left(\frac{m_1}{\alpha\left(P_+^+N^{P_+^+-1}\right)^\alpha} - \lambda c_7 - \lambda c_{10}\|u\|_{\overrightarrow{p}(\cdot)}^{q^- - \alpha P_+^+}\right)\|u\|_{\overrightarrow{p}(\cdot)}^{\alpha P_+^+}.
\end{aligned}$$

Let us define the function $\Psi(t) : [0,1] \to \mathbb{R}$ by

$$\Psi(t) = \frac{m_1}{\alpha\left(P_+^+N^{P_+^+-1}\right)^\alpha} - \lambda c_7 - \lambda c_{10} t^{q^- - \alpha P_+^+}.$$

Then, if we let $\lambda^* = \frac{m_1}{2c_7\alpha\left(P_+^+N^{P_+^+-1}\right)^\alpha}$, then for every $\lambda \in (0,\lambda^*)$ and $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $\|u\|_{\overrightarrow{p}(\cdot)} < 1$, the function Ψ would be positive in a neighborhood of the origin. Overall, the statement (i) holds.

(*ii*) Thanks to (**AR**), one can easily obtain the inequality $F(x,t) \geq c_{11}|t|^{\theta} - c_{12}$ $\forall x \in \Omega$, $|t| \geq K$. Moreover, from ($\mathbf{a_1}$) we have $A_i(x,t) \leq c_3 h_i(x)|t| + \frac{c_3}{p_i(x)}|t|^{p_i(x)}$. Then for $\omega \in W_0^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}$ and $t > 1$, by (3.2), we get

$$\begin{aligned}
J_\lambda(t\omega) &= \widehat{M}(\Lambda(t\omega)) - \lambda \int_\Omega F(x,t\omega)\,dx \\
&\leq \frac{m_2}{\alpha}\left(\int_\Omega \sum_{i=1}^N A_i(x,\partial_{x_i}\omega)\,t^{p_i(x)}dx\right)^{\alpha} - \lambda c_{11}t^{\theta}\int_\Omega |\omega|^{\theta} + c_{13} \\
&\leq \frac{m_2 t^{\alpha P_+^+}}{\alpha}\left(\int_\Omega \sum_{i=1}^N \left(c_3 h_i(x)|\partial_{x_i}\omega| + \frac{c_3}{p_i(x)}|\partial_{x_i}\omega|^{p_i(x)}\right)dx\right)^{\alpha} \\
&\quad -\lambda c_{11}t^{\theta}\int_\Omega |\omega|^{\theta} + c_{13}.
\end{aligned}$$

Since $\theta > \alpha P_+^+$, we conclude that $J_\lambda(t\omega) \to -\infty$ as $t \to +\infty$. □

Lemma 3.4. *If* ($\mathbf{M}_1$), ($\mathbf{f}_1$), (**AR**) *and* ($\mathbf{a}_2$) *hold,* J_λ *satisfies* (**PS**) *condition.*

Proof. We already know from Lemma 3.3 that the functional J_λ has the Mountain-Pass geometry. Hence, according to the Mountain-Pass theorem (see [3]), we obtain the a sequence $\{u_n\} \subset W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$|J_\lambda(u_n)| \leq c \text{ and } J'_\lambda(u_n) \to 0. \tag{3.6}$$

We shall establish first that $\{u_n\}$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. To this end, assume by way of contradiction that (extracting a subsequence if necessary) we have $\|u_n\|_{\vec{p}(\cdot)} \to \infty$ as $n \to \infty$. Therefore we can consider that $\|u_n\|_{\vec{p}(\cdot)} > 1$ for any n. Using (2.3), (3.6) and the assumptions of Lemma 3.4, we have

$$\begin{aligned}
c + \|u_n\|_{\vec{p}(\cdot)} &\geq J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle \\
&= \widehat{M}(\Lambda(u_n)) - \frac{1}{\theta}M(\Lambda(u_n))\int_\Omega \sum_{i=1}^N a_i(x,\partial_{x_i}u_n)\,\partial_{x_i}u_n dx \\
&\quad +\lambda\int_\Omega\left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx \\
&\geq \frac{m_1}{\alpha}\left(\int_\Omega \sum_{i=1}^N A_i(x,\partial_{x_i}u_n)\,dx\right)^{\alpha} - \frac{m_2}{\theta}\left(\int_\Omega \sum_{i=1}^N A_i(x,\partial_{x_i}u_n)\,dx\right)^{\alpha-1} \\
&\quad \times \int_\Omega \sum_{i=1}^N p_i(x)\,A_i(x,\partial_{x_i}u_n)\,dx \\
&\geq \left(\frac{m_1}{\alpha} - \frac{m_2 P_+^+}{\theta}\right)\left(\int_\Omega \sum_{i=1}^N A_i(x,\partial_{x_i}u_n)\,dx\right)^{\alpha} \\
&\geq \left(\frac{m_1}{\alpha} - \frac{m_2 P_+^+}{\theta}\right)\frac{1}{(P_+^+)^{\alpha}}\left(\int_\Omega \sum_{i=1}^N |\partial_{x_i}u_n|^{p_i(x)}dx\right)^{\alpha}.
\end{aligned}\tag{3.7}$$

For every n, let us denote by $\mp_{n_1}, \mp_{n_2}$ the indices sets

$$\mp_{n_1} = \left\{ i \in \{1,...,N\} : |\partial_{x_i}u|_{p_i(\cdot)} \leq 1 \right\},$$

and

$$\mp_{n_2} = \left\{ i \in \{1, ..., N\} : |\partial_{x_i} u|_{p_i(\cdot)} > 1 \right\}.$$

Then using (2.2) – (2.4) and (3.7), we conclude that

$$\begin{aligned} c + \|u_n\|_{\vec{p}(\cdot)} &\geq \frac{1}{\left(P_+^+\right)^\alpha} \left(\frac{m_1}{\alpha} - \frac{m_2 P_+^+}{\theta} \right) \left(\sum_{i \in \mp_{n_1}} |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_+^+} + \sum_{i \in \mp_{n_2}} |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} \right)^\alpha \\ &\geq \frac{1}{\left(P_+^+\right)^\alpha} \left(\frac{m_1}{\alpha} - \frac{m_2 P_+^+}{\theta} \right) \left(\sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} - \sum_{i \in \mp_{n_1}} |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} \right)^\alpha \\ &\geq \frac{1}{\left(P_+^+\right)^\alpha} \left(\frac{m_1}{\alpha} - \frac{m_2 P_+^+}{\theta} \right) \left(\sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} - N \right)^\alpha. \end{aligned}$$

Applying the Jensen inequality to the convex function $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$, $\sigma(t) = t^{P_-^-}$, $P_-^- \geq 2$, we obtain

$$c + \|u_n\|_{\vec{p}(\cdot)} \geq \frac{1}{\left(P_+^+\right)^\alpha} \left(\frac{m_1}{\alpha} - \frac{m_2 P_+^+}{\theta} \right) \left(\frac{\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^- - 1}} - N \right)^\alpha.$$

When we divide the last inequality by $\|u_n\|_{\vec{p}(\cdot)}^{\alpha P_-^-}$, and pass to the limit as $n \to \infty$, we obtain a contradiction. Thus, $\{\|u_n\|_{\vec{p}(\cdot)}\}$ must be bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Then according to a subsequence, there exists $u_0 \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, such that $u_n \rightharpoonup u_0$. These pieces of information along with Proposition 2.1 mean

$$\begin{aligned} u_n &\rightharpoonup u_0 \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega), \\ u_n &\to u_0 \text{ in } L^{q(\cdot)}(\Omega), \\ u_n(x) &\to u_0(x) \text{ a.e. in } \Omega. \end{aligned} \tag{3.8}$$

From (3.6), $\langle J_\lambda'(u_n), u_n - u_0 \rangle \to 0$. Therefore

$$\begin{aligned} \langle J_\lambda'(u_n), u_n - u_0 \rangle = M(\Lambda(u_n)) \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0)\, dx \\ -\lambda \int_\Omega f(x, u_n)(u_n - u_0)\, dx \to 0. \end{aligned}$$

By deploying $(\mathbf{f}_1)$, (2.1) and Proposition 2.1, it reads

$$\begin{aligned} \left| \int_\Omega f(x, u_n)(u_n - u_0)\, dx \right| &\leq c_1 \left| \int_\Omega |u_n|^{q(x)-2} u_n (u_n - u_0)\, dx \right| + c_2 \left| \int_\Omega (u_n - u_0)\, dx \right| \\ &\leq c_1 \left| |u_n|^{q(x)-1} \right|_{q'(\cdot)} |u_n - u_0|_{q(\cdot)} + c_2 \int_\Omega |u_n - u_0|\, dx. \end{aligned}$$

Taking into account the relations given in (3.8), we obtain

$$\int_\Omega f(x, u_n)(u_n - u_0)\, dx \to 0.$$

Hence,

$$M(\Lambda(u_n)) \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0)\, dx \to 0.$$

Due to $(\mathbf{M}_1)$, we must have

$$\int_{\Omega} \sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u_n\right)\left(\partial_{x_i} u_n - \partial_{x_i} u_0\right) dx \to 0. \tag{3.9}$$

By Lemma 2 in [12], the operator Λ' is of type (S_+) on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, that is, if $\{u_n\} \subset W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is weakly convergent to $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and

$$\limsup_{n\to\infty} \langle \Lambda'(u_n), u_n - u\rangle \le 0,$$

then $\{u_n\}$ converges strongly to u in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Therefore, from (3.9), we obtain that $u_n \to u_0$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, namely J_λ satisfies $(\mathbf{PS})$ condition. □

Proof of Theorem 3.1 is completed. According to information that we gather from Lemma 3.3, Lemma 3.4 and the fact that $J_\lambda(0) = 0$, J_λ satisfies the Mountain-Pass theorem. Therefore, u_0 is a nontrivial critical point of J_λ, that is, u_0 is a nontrivial weak solution to $(\mathbf{P})$. □

In the rest of the paper, we will obtain the existence of infinitely many nontrivial weak solutions of problem $(\mathbf{P})$. The proof is based on the Fountain theorem.

Theorem 3.5. *Assume that* $(\mathbf{M}_1)$, $(\mathbf{f}_1)-(\mathbf{f}_3)$, $(\mathbf{AR})$, $(\mathbf{a_1}), (\mathbf{a}_2)$ *and* $(\mathbf{A})$ *hold. Then for any* $\lambda \in (0, \lambda^*)$, λ^* *obtained in Lemma 3.3,* J_λ *has a sequence of critical points* $\{u_n\}$ *such that* $J_\lambda(u_n) \to +\infty$ *and* $(\mathbf{P})$ *has infinitely many pairs of solutions.*

Since $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a separable and reflexive Banach space, there exist $\{e_j\} \subseteq W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\{e_j^*\} \subseteq (W_0^{1,\overrightarrow{p}(\cdot)}(\Omega))^*$ such that $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{span\{e_j | j = 1, 2, ...\}}$, $(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega))^* = \overline{span\{e_j^* | j = 1, 2, ...\}}$ and

$$\langle e_i, e_j^*\rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \ne j. \end{cases}$$

For convenience, we denote $X = W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and write $X_j = span\{e_j\}$, $Y_k = \oplus_{j=1}^{k} X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$.

Lemma 3.6. [[33], *Fountain Theorem*] *Assume that X is a separable Banach space, $I \in C^1(X, \mathbb{R})$ is an even functional satisfying the* $(\mathbf{PS})$ *condition. Moreover, for each $k = 1, 2, ...$, there exist $\rho_k > r_k > 0$ such that*

(i) $\inf_{\left\{u\in Z_k : \|u\|_{\overrightarrow{p}(\cdot)} = r_k\right\}} I(u) \to +\infty$ *as* $k \to +\infty$;

(ii) $\max_{\left\{u\in Y_k : \|u\|_{\overrightarrow{p}(\cdot)} = \rho_k\right\}} I(u) \le 0.$

Then I has a sequence of critical values which tends to $+\infty$.

Proof of Theorem 8. It is enough to show that J has an unbounded sequence of critical points. According to the assumptions on the nonlinearity f, Lemma 3.3 and Lemma 3.4, J is an even functional and satisfies the $(\mathbf{PS})$ condition. We will only show that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (i) and (ii) hold.

Before proceeding to the proof, we want to note that, if we denote

$$\beta_k := \sup_{u\in Z_k, \|u\|_{\overrightarrow{p}(\cdot)}=1} |u|_{q(x)} \text{ and } \vartheta_k := \sup_{u\in Z_k, \|u\|_{\overrightarrow{p}(\cdot)}=1} |u|_{\alpha P_+^+},$$

then $\beta_k \to 0$ and $\vartheta_k \to 0$ as $k \longrightarrow \infty$ (see [16]).

(i) For any $u \in Z_k$ such that $\|u\|_{\vec{p}(\cdot)} \geq 1$, we have

$$\begin{aligned}
J_\lambda(u) &= \widehat{M}\left(\Lambda\left(u\right)\right) - \lambda \int_\Omega F\left(x,u\right)dx \\
&\geq \frac{m_1}{\alpha\left(P_+^+\right)^\alpha}\left(\int_\Omega \sum_{i=1}^N \left|\partial_{x_i}u\right|^{p_i(x)}dx\right)^\alpha - \lambda\varepsilon\int_\Omega |u|^{\alpha P_+^+}dx - \lambda C_\varepsilon \int_\Omega |u|^{q(x)}dx \\
&\geq \frac{m_1}{\alpha\left(P_+^+\right)^\alpha}\left(\frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^- -1}} - N\right)^\alpha - \lambda c_{14}\,|u|_{\alpha P_+^+}^{\alpha P_+^+} - \lambda C_\varepsilon\,|u|_{q(x)}^{q^+} \\
&\geq \frac{m_1}{\alpha\left(P_+^+\right)^\alpha}\left(\frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^- -1}} - N\right)^\alpha - \lambda c_{15}\,\vartheta_k^{\alpha P_+^+}\,\|u\|_{\vec{p}(\cdot)}^{\alpha P_+^+} - \lambda c_{15}\beta_k^{q^+}\,\|u\|_{\vec{p}(\cdot)}^{q^+} \\
&\geq \frac{m_1 N^\alpha}{\alpha\left(P_+^+\right)^\alpha}\left(\left(\frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^-}} - 2\right) + 1\right)^\alpha - \lambda c_{15}\vartheta_k^{\alpha P_+^+}\,\|u\|_{\vec{p}(\cdot)}^{\alpha P_+^+} - \lambda c_{15}\beta_k^{q^+}\,\|u\|_{\vec{p}(\cdot)}^{q^+},
\end{aligned}$$

where $c_{15} = \max\{c_{14}, C_\varepsilon\}$. Let us fix r_k and set $\|u\|_{\vec{p}(\cdot)} = r_k := (\lambda c_{15} q^+ \beta_k^{q^+})^{\frac{1}{P_-^- - q^+}}$. Since $r_k \to +\infty$ as $k \to +\infty$, for every positive integer N there is a positive integer k_0 such that $k \geq k_0$ implies $\|u\|_{\vec{p}(\cdot)} = r_k > N$. Thus, for sufficiently large k, we can apply the Bernoulli inequality to the term $\left(\left(\frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^-}} - 2\right) + 1\right)^\alpha$. Therefore, for such k, we have

$$\begin{aligned}
J_\lambda(u) &\geq \frac{m_1}{\alpha}\left(\alpha\left(\frac{(\lambda c_{15} q^+ \beta_k^{q^+})^{\frac{P_-^-}{P_-^- - q^+}}}{N^{P_-^-}} - 2\right) + 1\right) - \lambda c_{15}\vartheta_k^{\alpha P_+^+}(\lambda c_{15} q^+ \beta_k^{q^+})^{\frac{\alpha P_+^+}{P_-^- - q^+}} \\
&\quad - \lambda c_{15}\beta_k^{q^+}(\lambda c_{15} q^+ \beta_k^{q^+})^{\frac{q^+}{P_-^- - q^+}} \\
&\geq \frac{m_1}{N^{P_-^-}}(\lambda c_{15} q^+ \beta_k^{q^+})^{\frac{P_-^-}{P_-^- - q^+}} - \lambda c_{15}\vartheta_k^{\alpha P_+^+}(\lambda c_{15} q^+ \beta_k^{q^+})^{\frac{\alpha P_+^+}{P_-^- - q^+}} \\
&\quad - \lambda c_{15}\beta_k^{q^+}(\lambda c_{15} q^+ \beta_k^{q^+})^{\frac{q^+}{P_-^- - q^+}} - c_{16}.
\end{aligned}$$

For sufficiently large k, we have $\lambda c_{15}\vartheta_k^{\alpha P_+^+} < \frac{m_1}{2N^{P_-^-}}$, thus

$$\begin{aligned}
J_\lambda(u) &\geq \frac{m_1}{2N^{P_-^-}}(\lambda c_{15} q^+ \beta_k^{q^+})^{\frac{P_-^-}{P_-^- - q^+}} - \lambda c_{15}\beta_k^{q^+}(\lambda c_{15} q^+ \beta_k^{q^+})^{\frac{q^+}{P_-^- - q^+}} - c_{16} \\
&\geq \frac{m_1}{4N^{P_-^-}}(\lambda c_{15} q^+ \beta_k^{q^+})^{\frac{P_-^-}{P_-^- - q^+}} - c_{16},
\end{aligned}$$

which implies

$$\inf_{u\in Z_k,\|u\|_{\overrightarrow{p}(\cdot)}=r_k} J_\lambda(u)\to+\infty \text{ as } k\to+\infty.$$

The statement of (i) is satisfied.

(ii) From (**AR**), we have the inequality $F(x,t)\geq c_{11}|t|^{\theta}-c_{12}$. Then, for any $\omega\in Y_k\setminus\{0\}$ with $\|\omega\|_{\overrightarrow{p}(\cdot)}=1$ and $1<t=\rho_k$, we get

$$\begin{aligned} J_\lambda(t\omega) &= \widehat{M}\left(\Lambda\left(t\omega\right)\right)-\lambda\int_\Omega F\left(x,t\omega\right)dx \\ &\leq \frac{m_2}{\alpha}\left(\int_\Omega \textstyle\sum_{i=1}^N A_i\left(x,\partial_{x_i}\omega\right)t^{p_i(x)}dx\right)^{\alpha}-\lambda c_{11}t^{\theta}\displaystyle\int_\Omega|\omega|^{\theta}+c_{13} \\ &\leq \frac{m_2 t^{\alpha P_+^+}}{\alpha}\left(\int_\Omega \textstyle\sum_{i=1}^N(c_3h_i\left(x\right)|\partial_{x_i}\omega|+\dfrac{c_3}{p_i\left(x\right)}|\partial_{x_i}\omega|^{p_i(x)})dx\right)^{\alpha} \\ &\quad -\lambda c_{11}t^{\theta}\int_\Omega|\omega|^{\theta}+c_{13}. \end{aligned}$$

Since $\theta>\alpha P_+^+$ and all norms on the finite dimensional vector space Y_k are equivalent, setting $u=t\omega$, we obtain that $J_\lambda(u)\to-\infty$ as $\|u\|_{\overrightarrow{p}(\cdot)}\to+\infty$ for any $u\in Y_k$. This implies

$$\max_{u\in Y_k,\|u\|_{\overrightarrow{p}(\cdot)}=\rho_k} J_\lambda(u)\leq 0.$$

The proof is completed. □

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