Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 100–114
ISSN: 1223-6934

Three critical solutions for variational - hemivariational inequalities involving $p(x)$-Kirchhoff type equation

M. Alimohammady and F. Fattahi

Abstract. In this paper, we study the existence of three solutions to the p(x)-Kirchhoff type equations in $\mathbb{R}^N$. By means of nonsmooth three critical points theorem and the theory of the variable exponent Sobolev spaces, we establish the existence of three critical points for the problem. Moreover, we study the existence of three radially symmetric solutions for a class of quasilinear elliptic inclusion problem with discontinuous nonlinearities in $\mathbb{R}^N$. Our approach is based on critical point theory for locally Lipschitz functionals due to Iannizzotto.

2010 Mathematics Subject Classification. Primary 49J40; Secondary 47J20, 47J30, 35J35.
Key words and phrases. Nonsmooth critical point theory, principle of symmetric criticality, variational-hemivariational inequality, $p(x)-$Kirchhoff type operator.

1. Introduction

In this paper, we are concerned with the following nonlinear elliptic differential inclusion with $p(x)-$Kirchhoff-type problem

$$\begin{cases} M\Big(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} - |u|^{p(x)})dx\Big)[\Delta_{p(x)}u - |u|^{p(x)-2}u] \\ \qquad\qquad\qquad\qquad \in -\lambda\partial F(x,u) - \mu\partial G(x,u) \quad in\,\mathbb{R}^N \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad on\,\mathbb{R}^N, \end{cases} \tag{1}$$

where $p(x) \in C(\mathbb{R}^N)$ is continuous function satisfying

$$1 < p^- = \inf_{x\in\mathbb{R}^N} p(x) \le p(x) \le p^+ = \sup_{x\in\mathbb{R}^N} p(x) < +\infty,$$

and $\lambda, \mu > 0$. $F, G : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a function in which $F(\cdot, u)$ is measurable for every $u \in \mathbb{R}$ and $F(x, \cdot)$ is locally Lipschitz for a.e. $x \in \mathbb{R}^N$. $\partial F(x, u)$ and $\partial G(x, u)$ denotes the generalized Clarke gradient of $F(x, u)$ and $G(x, u)$ at $u \in \mathbb{R}$.

Let X be real Banach space. We assume that it is also given a functional $\chi : X \to \mathbb{R} \cup \{+\infty\}$ which is convex, lower semicontinuous, proper whose effective domain $dom(\chi) = \{x \in X : \chi(x) < +\infty\}$ is a (nonempty, closed, convex) cone in X. Our aim is to study the following variational-hemivariational inequality problem: Find $u \in \mathcal{B}$ (it is called a weak solution of problem (1)) if for all $v \in \mathcal{B}$,

$$M\Big(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} - |u(x)|^{p(x)})dx\Big) \int_{\mathbb{R}^N} (|\nabla u|^{p(x)-2}\nabla u\nabla v - |u|^{p(x)-2}uv)dx$$

Received March 25, 2016.

$$-\lambda \int_{\mathbb{R}^N} F^0(x,u;v)dx - \mu \int_{\mathbb{R}^N} G^0(x,u;v)dx \geq 0, \tag{2}$$

where $\mathcal{B}$ is a closed convex subset of $X = W_0^{1,p(\cdot)}(\mathbb{R}^N)$, and F^0, G^0 are the generalized directional derivatives of the locally Lipschitz functions F, G.
The operator $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is the so-called $p(x)-$Laplacian, which becomes $p-$Laplacian when $p(x) \equiv p$ is a constant. More recently, the study of $p(x)-$Laplacian problems has attracted more and more attention (cf. [2]).
The problem (1) is a generalization of an equation introduced by Kirchhoff (cf. [20]). The study of Kirchhoff model has already been extended to the case involving the p-Laplacian (cf. [8], [10]) and $p(x)-$Laplacian (cf. [6], [15]).

Applications of problems involving the p(x)-Laplace operator is applied to the modeling of various phenomena such as elastic mechanics, thermorheological and electrorheological fluids, mathematical mathematical biology and plasma physics (cf. [10], [30], [31]). In recent years, differential equations and variational problems have been studied in many papers, we refer to some interesting works (cf. [27], [28]).

Many authors investigated variational methods to a class of non-differentiable functionals to prove some existence theorems for PDE with discontinuous nonlinearities. In [33] author studied a priori bounds for a class of variational inequalities involving general elliptic operators of second-order and terms of generalized directional derivatives; in [4], authors studied variational-hemivariational inequalities involving the p-Laplace operator and a nonlinear Neumann boundary condition; in [1], authors studied variational-hemivariational inequality by using the mountain pass theorem.

However, authors appeared some technical difficulties for studying problem on unbounded domains (cf. [3]). Therefore, to resolve this issue the space of radially symmetric functions was introduced. For instance, the existence of radially symmetric solutions for a class of differential inclusion problems was considered by many authors. In [32] author studied infinitely many radially symmetric solutions for a class of hemivariational inequalities with the Cerami compactness condition and the principle of symmetric criticality for locally Lipschitz functions; in [24] author studied the existence of infinitely many radial respective non-radial solutions for a class of hemivariational inequalities; in [18] authors studied the existence of infinitely many radially symmetric solutions for a class of perturbed elliptic equations with discontinuous nonlinearities under some hypotheses on the behavior of the potential.

More recently, the study of the three-critical-points for nonsmooth functionals was investigated. In [23] authors studied the existence of three critical points which extends the variational principle of Ricceri [29] to nonsmooth functionals. In [19] author studied three-critical-points theorem based on a minimax inequality and on a truncation argument which extended to Motreanu-Panagiotopoulos functionals. In [34], authors studied the existence of at least three critical points for a p(x)-Laplacian differential inclusion based on the nonsmooth analysis.

The purpose of this paper is to prove the existence of at least three solutions for a variational-hemivariational inequality depending on two parameters in $W_0^{1,p(x)}(\mathbb{R}^N)$. In fact, the existence result for $p(x)-$Kirchhoff-type problem with locally Lipschitz functions under special hypotheses on F and G is investigated. Also, for the second part under further additional assumptions, the quasilinear elliptic inclusion problem is considered. A major problem is that the compact embedding for $W_0^{1,p(x)}(\mathbb{R}^N)$ into

$L^\infty(\mathbb{R}^N)$ is required. Hence, we overcome this gap by using the subspace of radially symmetric functions of $W_0^{1,p(x)}(\mathbb{R}^N)$, denoted by $W_{0,r}^{1,p(x)}(\mathbb{R}^N)$, can be embedded compactly into $L^\infty(\mathbb{R}^N)$.

The paper is organized as follows. We prepare the basic definitions and properties in the framework of the generalized Lebesgue and Sobolev spaces. Besides, some basic notions about generalized directional derivative and hypotheses on F, G are given. Next, we give the main results about the existence of three solutions in theorem 3.7. The final part of this paper is concerned with the existence of three radially symmetric solutions in theorem 4.5.

2. Preliminaries

We recall some basic facts about the variable exponent Lebesgue-Sobolev (cf. [11],[13],[16]).
The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\mathbb{R}^N) = \{u : \mathbb{R}^N \longrightarrow \mathbb{R} : \int_{\mathbb{R}^N} |u(x)|^{p(x)} dx < \infty\}$$

and is endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \{\ \lambda > 0 : \int_{\mathbb{R}^N} |\frac{u(x)}{\lambda}|^{p(x)} dx\} \leq 1\}.$$

Note that, when $p \equiv Const.$, the Luxemburg norm $\|\cdot\|_{p(\cdot)}$ coincides with the standard norm $\|\cdot\|_p$ of the Lebesgue space $L^p(\mathbb{R}^N)$.

The generalized Lebesgue-Sobolev space $W^{L,p(\cdot)}(\mathbb{R}^N)$ for $L = 1, 2, ...$ is defined as

$$W^{L,p(\cdot)}(\mathbb{R}^N) = \{u \in L^{p(\cdot)}(\mathbb{R}^N) : D^\alpha u \in L^{p(\cdot)}(\mathbb{R}^N), |\alpha| \leq L\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$ with $\alpha = (\alpha_1, \alpha_2, \cdot\cdot\cdot, \alpha_N)$ is a multi-index and $|\alpha| = \Sigma_{i=1}^N \alpha_i$.
The space $W^{L,p(\cdot)}(\mathbb{R}^N)$ with the norm

$$\|u\|_{W^{L,p(\cdot)}(\mathbb{R}^N)} = \sum_{|\alpha| \leq L} \|D^\alpha u\|_{p(\cdot)},$$

is a separable reflexive Banach space(cf. [12]).
The space $W_0^{L,p(\cdot)}(\mathbb{R}^N)$ denotes the closure in $W^{L,p(\cdot)}(\mathbb{R}^N)$ of the set of all $W^{L,p(\cdot)}(\mathbb{R}^N)$ -functions with compact support. Hence, an equivalent norm for the space $W_0^{L,p(\cdot)}(\mathbb{R}^N)$ is given by

$$\|u\|_{W_0^{L,p(\cdot)}(\Omega)} = \sum_{|\alpha| = L} \|D^\alpha u\|_{p(\cdot)}.$$

If $\Omega \subset \mathbb{R}^N$ is open bounded domain, let p_L^* denote the critical variable exponent related to p, defined for all $x \in \bar{\Omega}$ by the pointwise relation

$$p_L^*(x) = \begin{cases} \frac{Np(x)}{N - Lp(x)} & Lp(x) < N, \\ \\ +\infty & Lp(x) \geq N. \end{cases} \tag{3}$$

For every $u \in W_0^{L,p(\cdot)}(\Omega)$ the Poincaré inequality holds, where $C_p > 0$ is a constant

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C_p \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

(see (cf. [17])).

Proposition 2.1. *(cf.* [16]*) Let p' be the function obtained by conjugating the exponent p pointwise, that is $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \bar{\Omega}$, then p' belongs to $C_+(\overline{\Omega})$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the following Hölder type inequality valid,*

$$\int_\Omega |u(x)v(x)|dx \leq (\frac{1}{p^-} + \frac{1}{p'^-})\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

where $L^{\acute{p}(\cdot)}(\Omega)$ is the conjugate space of $L^{p(\cdot)}(\Omega)$.

Proposition 2.2. *For $\phi(u) = \int_{\mathbb{R}^N} [|\nabla u|^{p(x)} - |u(x)|^{p(x)}]dx$, and $u, u_n \in X$, we have*

(i) $\|u\| < (=; >)1 \Leftrightarrow \phi(u) < (=; >)1$,

(ii) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \phi(u) \leq \|u\|^{p^-}$,

(iii) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \phi(u) \leq \|u\|^{p^+}$,

(iv) $\|u_n\| \to 0 \Leftrightarrow \phi(u_n) \to 0$,

(v) $\|u_n\| \to \infty \Leftrightarrow \phi(u_n) \to \infty$.

Proof is similar to that in (cf. [16]).

Proposition 2.3. *(cf.* [16],[21]*) For $p, q \in C_+(\overline{\Omega})$ in which $q(x) \leq p_L^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{L,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

Remark 2.1. (i) By the proposition (2.3) there is a continuous and compact embedding of $W_0^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$, where $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.
(ii) Denote by

$$\|u\| = \inf\{\lambda > 0 \ : \ \int_{\mathbb{R}^N} [|\frac{\nabla u}{\lambda}|^{p(x)} - |\frac{u}{\lambda}|^{p(x)}]dx \leq 1\},$$

which is a norm on $W_0^{1,p(\cdot)}(\mathbb{R}^N)$.

Here, we recall some definitions and basic notions of the theory of generalized differentiation for locally Lipschitz functions. We refer the reader to (cf. [5], [7], [25], [26]).
Let X be a Banach space and $X^\star$ its topological dual. By $\|\cdot\|$ we will denote the norm in X and by $< \cdot, \cdot >_X$ the duality brackets for the pair $(X, X^\star)$.
A function $h : X \to \mathbb{R}$ is said to be locally Lipschitz continuous, when to every $x \in X$ there correspond a neighborhood V_x of x and a constant $L_x \geq 0$ such that

$$|h(z) - h(w)| \leq L_x\|z - w\|, \ \forall z, w \in V_x.$$

For a locally *Lipschitz* function $h : X \to \mathbb{R}$, the generalized directional derivative of h at $u \in X$ in the direction $\gamma \in X$ is defined by

$$h^0(u; \gamma) = \limsup_{w \to u, t \to 0^+} \frac{h(w + t\gamma) - h(w)}{t}.$$

The generalized gradient of h at $u \in X$ is

$$\partial h(u) = \{x^\star \in X^\star \ : < x^\star, \gamma >_X \leq h^0(u;\gamma),\ \forall \gamma \in X\},$$

which is non-empty, convex and $w^\star-$compact subset of $X^\star$, where $< \cdot, \cdot >_X$ is the duality pairing between $X^\star$ and X.

Proposition 2.4. *(cf. [7]) Let $h, g : X \to \mathbb{R}$ be locally Lipschitz functionals. Then, for any $u, v \in X$ the following hold:*
(1) *$h^0(u;\cdot)$ is subadditive, positively homogeneous;*
(2) *∂h is convex and weak* compact;*
(3) $(-h)^0(u;v) = h^0(u;-v)$;
(4) *the set-valued mapping $h : X \to 2^{X^*}$ is weak* u.s.c.;*
(5) $h^0(u;v) = \max_{u^* \in \partial h(u)} < u^*, v >$;
(6) *$\partial(\lambda h)(u) = \lambda \partial h(u)$ for every $\lambda \in \mathbb{R}$;*
(7) $(h+g)^0(u;v) \leq h^0(u;v) + g^0(u;v)$;
(8) *the function $m(u) = \min_{\nu \in \partial h(u)} \nu_{X^*}$ exists and is lower semicontinuous; i.e., $\liminf_{u \to u_0} m(u) \geq m(u_0)$;*
(9) $h^0(u;v) = \max_{u^* \in \partial h(u)} \langle u^*, v \rangle \leq L\|v\|$.

Proposition 2.5. *(cf. [7])(Lebourg's mean value theorem) Let $h : X \to \mathbb{R}$ be a locally Lipschitz functional. Then, for every $u, v \in X$ there exists $w \in [u,v]$, $w^* \in \partial h(u)$ such that $h(u) - h(v) = \langle w^*, u - v \rangle$.*

Definition 2.1. (cf. [26]) Let X be a Banach space, $\mathcal{I} : X \to (-\infty, +\infty]$ is called a Motreanu-Panagiotopoulos-type functional, if $\mathcal{I} = h + \chi$, where $h : X \to \mathbb{R}$ is locally Lipschitz and $\chi : X \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous.

Definition 2.2. (cf. [19]) An element $u \in X$ is called a critical point for $\mathcal{I} = h + \chi$ if

$$h^0(u; v-u) + \chi(v) - \chi(u) \geq 0, \quad \forall v \in X.$$

The Euler-Lagrange functional associated to problem (1) is given by

$$\mathcal{I}(u) = \widehat{M}\Big(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} - |u|^{p(x)})dx\Big) - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} G(x,u)dx,$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$ and $M(t)$ is supposed to verify the following assumptions:
(M_1) There exist m_1 and m_0 in which $m_1 \geq m_0 > 0$ and for all $t \in \mathbb{R}^+$, $m_0 \leq M(t) \leq m_1$;
(M_2) For all $t \in \mathbb{R}^+$, $\widehat{M}(t) \geq M(t)t$.
Denote $\Phi : W_0^{1,p(\cdot)}(\mathbb{R}^N) \to \mathbb{R}$, as follows

$$\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}[|\nabla u|^{p(x)} - |u|^{p(x)}]dx.$$

The next lemma characterizes some properties of Φ (cf. [14]).

Proposition 2.6. *Let $\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}[|\nabla u|^{p(x)} - |u|^{p(x)}]dx$. Then*
(i) $\Phi : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous.
(ii) Φ' is of (S_+) type.
(iii) Φ' is a homeomorphism.

Proposition 2.7. *(cf. [7]) Let $F, G : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be locally Lipschitz function and set $\mathcal{F}(u) = \int_{\mathbb{R}^N} F(x, u(x))dx$, $\mathcal{G}(u) = \int_{\mathbb{R}^N} G(x, u(x))dx$. Then $\mathcal{F}$, $\mathcal{G}$ are well-defined and*

$$\mathcal{F}^0(u; v) \leq \int_{\mathbb{R}^N} F^0(u(x); v(x))dx, \quad \mathcal{G}^0(u; v) \leq \int_{\mathbb{R}^N} G^0(u(x); v(x))dx, \forall u, v \in X.$$

3. Three solutions for a differential inclusion problem

For the reader's convenience, we recall the nonsmooth three critical points theorem.

Theorem 3.1. [19] *Let X be a separable and reflexive Banach space, Λ a real interval and $\mathcal{B}$ a nonempty, closed, convex subset of X. $\Phi \in C^1(X, \mathbb{R})$ a sequentially weakly l.s.c. functional and bounded on any bounded subset of X such that Φ' is of type $(S)_+$, suppose that $\mathcal{F} : X \to \mathbb{R}$ is a locally Lipschitz functional with compact gradient. Assume that:*
(i) $lim_{\|u\| \to +\infty}[\Phi - \lambda \mathcal{F}] = +\infty, \quad \forall \lambda \in \Lambda,$
(ii) There exists $\rho_0 \in \mathbb{R}$ such that

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} [\Phi + \lambda(\rho_0 - \mathcal{F}(u))] < \inf_{u \in X} \sup_{\lambda \in \Lambda} [\Phi + \lambda(\rho_0 - \mathcal{F}(u))].$$

Then, there exist $\lambda_1, \lambda_2 \in \Lambda$ $(\lambda_1 < \lambda_2)$ and $\sigma > 0$ such that for every $\lambda \in [\lambda_1, \lambda_2]$ and every locally Lipschitz functional $\mathcal{G} : X \to \mathbb{R}$ with compact derivative, there exists $\mu_1 > 0$ such that for every $\mu \in]0, \mu_1[$ the functional $\Phi - \lambda \mathcal{F} + \mu \mathcal{G}$ has at least three critical points whose norms are less than σ.

Let us introduce the following conditions of our problem.
We assume that $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which is locally Lipschitz in the second variable and satisfies the following properties:
(F_1) $|\xi| \leq K(|s|^{t(x)-1} + |s|^{z(x)-1})$ for all $\xi \in \partial F(x, s)$ with $(x, s) \in \mathbb{R}^N \times \mathbb{R}$ $(1 \leq p^- \leq p(x) \leq p^+ < z^- \leq z(x) \leq z^+ < t^- \leq t(x) \leq t^+ < p^*(x))$;
(F_2) $|F(x, s)| \leq H(|s|^{\alpha(x)} + |s|^{\beta(x)})$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$ $(H > 0,\ 1 \leq \alpha^- \leq \alpha(x) \leq \alpha^+ < \beta^- \leq \beta(x) \leq \beta^+ < p^- \leq p(x) \leq p^+ < p^*(x))$;
(F_3) $F(x, 0) = 0$ for a.e. $x \in \mathbb{R}^N$ and there exists $\hat{u} \in W_0^{1,p(\cdot)}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} F(x, \hat{u})dx > 0$ for a.e. $x \in \mathbb{R}^N$;
(G) $|\xi| \leq K'(1 + |s|^{r(x)-1})$ for all $\xi \in \partial G(x, s)$ with $(x, s) \in \mathbb{R}^N \times \mathbb{R}$ $(1 \leq p^- \leq p(x) \leq p^+ < r^- \leq r(x) \leq r^+ < p^*(x))$.

We need the following lemmas in the proof of our main result.

Lemma 3.2. *If (F_1) holds, then $\mathcal{F} : X \to \mathbb{R}$ is locally Lipschitz functional with compact gradient.*

Proof. First we prove that $\mathcal{F}$ is Lipschitz continuous on each bounded subset of X. Let $u, v \in B(0, M)$ $(M > 0)$ and $\|u\|, \|v\| \leq 1$. From proposition 2.5, the Hölder inequality and the embedding of X in $L^{t(x)}(\mathbb{R}^N)$ and $L^{z(x)}(\mathbb{R}^N)$

$$\begin{aligned}|\mathcal{F}(u) - \mathcal{F}(v)| &\leq \int_{\mathbb{R}^N} |F(x, u(x)) - F(x, v(x))|dx \\ &\leq \int_{\mathbb{R}^N} K(|u(x)|^{t(x)-1} + |v(x)|^{t(x)-1} + |u(x)|^{z(x)-1} + |v(x)|^{z(x)-1})|u(x) - v(x)|dx\end{aligned}$$

$$\leq K(\||u(x)|^{t(x)-1}+|v(x)|^{t(x)-1}\|)_{L^{t'(x)}(\mathbb{R}^N)}\|u-v\|_{L^{t(x)}(\mathbb{R}^N)}$$
$$+K(\||u(x)|^{z(x)-1}+|v(x)|^{z(x)-1}\|)_{L^{z'(x)}(\mathbb{R}^N)}\|u-v\|_{L^{z(x)}(\mathbb{R}^N)}$$
$$\leq 2K(c_1M^{z^- -1}+c_2M^{t^- -1})\|u-v\|,$$

where $c_1,\ c_2$ are positive constants.
We prove that $\partial\mathcal{F}$ is compact. Let $\{u_n\}$ be a sequence in X such that $\|u_n\|\leq M$ and choose $u_n^*\in\partial\mathcal{F}(u_n)$ for any $n\in\mathbb{N}$. From (F_1) it follows that for any $n\in\mathbb{N},\ v\in X$,

$$\begin{aligned} < u_n^*, v > &\leq \int_{\mathbb{R}^N}|u_n^*(x)||v(x)|dx \leq \int_{\mathbb{R}^N}K(|u(x)|^{t(x)-1}+|u(x)|^{z(x)-1})|v(x)|dx \\ &\leq (c_3M^{t^- -1}+c_4M^{z^- -1})\|v\|, \end{aligned}$$

where $c_3,\ c_4$ are positive constants.
Consequently,

$$\|u_n^*\|_{X^*}\leq (c_3M^{t^- -1}+c_4M^{z^- -1}).$$

The sequence $\{u_n^*\}$ is bounded and hence, up to a subsequence, $u_n^*\rightharpoonup u^*$.
Suppose on the contrary; we assume that there exists $\epsilon>0$ for which $\|u_n^*-u^*\|_{X^*}>\epsilon$ (choose a subsequence if necessary). For every $n\in\mathbb{N}$, we can find $\{v_n\}\in X$ with $\|v_n\|<1$ and

$$\langle u_n^*-u^*, v_n\rangle>\epsilon. \tag{4}$$

Then, $\{v_n\}$ is a bounded sequence and up to a subsequence, $v_n\rightharpoonup v,\ \|v_n-v\|_{L^{t(x)}(\Omega)}\to 0$ and $\|v_n-v\|_{L^{z(x)}(\Omega)}\to 0$. Hence,

$$|\langle u_n^*-u^*, v\rangle|<\frac{\epsilon}{4},\quad |\langle u^*, v_n-v\rangle|<\frac{\epsilon}{4},$$
$$\|v_n-v\|_{L^{t(x)}}<\frac{\epsilon}{4Kc_3M^{t^- -1}},\quad \|v_n-v\|_{L^{z(x)}}<\frac{\epsilon}{4Kc_4M^{z^- -1}}.$$

It follows that,

$$\begin{aligned} \langle u_n^*-u^*, v_n\rangle &\leq \langle u_n^*, v_n-v\rangle+\langle u_n^*-u^*, v\rangle+\langle u^*, v-v_n\rangle \\ &\leq \int_{\mathbb{R}^N}|u_n^*(x)||v_n(x)-v(x)|dx+\langle u_n^*-u^*, v\rangle+\langle u^*, v-v_n\rangle \\ &\leq K(c_3M^{t^- -1}\|v_n-v\|_{L^{t(x)}}+c_4M^{z^- -1}\|v_n-v\|_{L^{z(x)}}) \\ &\qquad +\langle u_n^*-u^*, v\rangle+\langle u^*, v-v_n\rangle\to 0, \end{aligned}$$

which contradicts (15). □

Lemma 3.3. *Let G be satisfied. Then $\mathcal{G}$ is a locally Lipschitz functional with compact gradient.*

The proof is similar to lemma (3.2).

The next lemma points out the relationship between the critical points of $\mathcal{I}(u)$ and solutions of Problem (2).

Lemma 3.4. *Every critical point of the functional $\mathcal{I}$ is a solution of Problem* (1).

Proof. Let $u\in X$ be a critical point of $\mathcal{I}(u)=\Phi(u)-\lambda\mathcal{F}(u)-\mu\mathcal{G}(u)+\chi(u)$. Then $u\in\mathcal{B}$ and by definition 2.2

$$\langle\Phi' u, v-u\rangle+\lambda(-\mathcal{F})^0(u;v-u)+\mu(-\mathcal{G})^0\langle u;v-u\rangle\geq 0,\quad \forall v\in X.$$

Using proposition 2.7 and proposition 2.4, we obtain the desired inequality. □

Lemma 3.5. *(cf. [19]) Let (F_1) and (F_3) be satisfied. Then, there exists $\hat{u} \in \mathcal{B}$ such that $\mathcal{F}(\hat{u}) > 0$.*

Lemma 3.6. *If (F_2) holds, then for any $\lambda \in (0, +\infty)$, the function $\Phi - \lambda\mathcal{F}$ is coercive.*

Proof. For $u \in X$ such that $\|u\| \geq 1$

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} F(x,u)dx \leq \int_{\mathbb{R}^N} H(|u|^{\alpha(x)} + |u|^{\beta(x)})dx \leq H(\|u\|^{\alpha^+}_{L^{\alpha(x)}(\mathbb{R}^N)} + \|u\|^{\beta^+}_{L^{\beta(x)}(\mathbb{R}^N)}).$$

By the embedding theorem for suitable positive constant c_5, c_6 it implies that

$$\mathcal{F}(u) \leq H(c_5\|u\|_X^{\alpha^+} + c_6\|u\|_X^{\beta^+}).$$

Consequently, by using proposition 2.2, for any $\lambda > 0$,

$$\Phi(u) - \lambda\mathcal{F}(u) \geq \frac{1}{p^+}\|u\|_X^{p^-} - H(c_5\|u\|_X^{\alpha^+} + c_6\|u\|_X^{\beta^+}).$$

Since $p^- > \min\{\alpha^+, \beta^+\}$, it follows that

$$\lim_{\|u\| \to +\infty} [\Phi - \lambda\mathcal{F}] = +\infty, \quad \forall u \in X, \ \lambda \in (0, +\infty).$$

□

Theorem 3.7. *Let F_1, F_2, F_3 are satisfied. Then there exist $\lambda_1, \lambda_2 > 0 (\lambda_1 < \lambda_2)$ and $\sigma > 0$ such that for every $\lambda \in [\lambda_1, \lambda_2]$ and every $\mathcal{G}$ satisfying G, there exists $\mu_1 > 0$ such that for every $\mu \in]0, \mu_1[$ problem (1) admits at least three solutions whose norms are less than σ.*

Proof. Due to Lemma 3.4, we are going to prove the existence of a critical point of functional $\mathcal{I}$. First, we check if $\mathcal{I}$ satisfies the conditions of the nonsmooth three critical points theorem 3.1. It is clear that Lemma 2.6 shows that Φ satisfies the weakly sequentially lower semicontinuous property and Φ' is of type (S_+). Moreover, according to Lemma 3.2, the functional $\mathcal{F}$ is weakly sequentially semicontinuous.
Since Lemma 3.6, implies that $\Phi - \lambda\mathcal{F}$ is coercive on X for all $\lambda \in \Lambda =]0, +\infty[$, so, the assumption (i) of theorem 3.1, satisfies.

Case 1. Let us assume that $\|u\| \leq 1$.
Set for every $r > 0$,

$$\theta_1(r) = \sup\{\mathcal{F}(u); u \in X, \frac{m_1}{p^-}\|u\|^{p^-} \leq r\},$$

we indicate that

$$\lim_{r \to 0^+} \frac{\theta_1(r)}{r} = 0. \tag{5}$$

From (F_1), it is follows that for every $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that for every $x \in \Omega, u \in \mathbb{R}$ and $\xi \in \partial F(x,u)$

$$|\xi| \leq \epsilon|u|^{t(x)-1} + c(\epsilon)|u|^{z(x)-1}. \tag{6}$$

Applying Lebourgs mean value theorem and using the Sobolev embedding theorem for every $u \in X$, there exist suitable positive constants c_7 and c_8

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} F(x,u)dx \leq \int_{\mathbb{R}^N} K(|u|^{t(x)} + |u|^{z(x)})dx \leq K(\|u\|^{t^+}_{L^{t(x)}(\mathbb{R}^N)} + \|u\|^{z^+}_{L^{z(x)}(\mathbb{R}^N)})$$
$$\leq Kc_7(\|u\|_X^{t^+} + \|u\|_X^{z^+}) \leq Kc_8(r^{\frac{t^+}{p^-}} + r^{\frac{z^+}{p^-}}).$$

It follows from $\min\{t^+, z^+\} > p^-$ that

$$\lim_{r\to 0^+} \frac{\theta_1(r)}{r} = 0.$$

From Lemma (3.5), $\hat{u} \neq 0$. Hence, in view of (5), there is $r \in \mathbb{R}$ in which
$0 < r < \frac{m_1}{p^-}\|\hat{u}\|^{p^-}$, $\qquad 0 < \frac{\theta_1(r)}{r} < \frac{\mathcal{F}(\hat{u})}{\frac{m_1}{p^-}\|\hat{u}\|^{p^-}}$.
Choose $\rho_0 > 0$ such that

$$\theta_1(r) < \rho_0 < \frac{r\mathcal{F}(\hat{u})}{\frac{m_1}{p^-}\|\hat{u}\|^{p^-}}, \tag{7}$$

especially, $\rho_0 < \mathcal{F}(\hat{u})$.
We claim that

$$\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{B}}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] < r. \tag{8}$$

It is obvious that the mapping

$$\lambda \mapsto \sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{B}}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))]$$

is upper semicontinuous on Λ and

$$\lim_{\lambda\to+\infty}\inf_{u\in\mathcal{B}}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] \leq \lim_{\lambda\to+\infty}[\frac{m_1}{p^-}\|\hat{u}\|^{p^-} + \lambda(\rho_0 - \mathcal{F}(\hat{u}))] = -\infty.$$

Therefore, there exists $\bar{\lambda} \in \Lambda$ in which

$$\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{B}}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = \inf_{u\in\mathcal{B}}[\frac{m_1}{p^-}\|u\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))].$$

We consider two cases:
(I) If $\bar{\lambda}\rho_0 < r$, we obtain

$$\inf_{u\in\mathcal{B}}[\frac{m_1}{p^-}\|u\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \bar{\lambda}\rho_0 < r.$$

(II) If $\bar{\lambda}\rho_0 \geq r$, from (7) we obtain

$$\inf_{u\in\mathcal{B}}[\frac{m_1}{p^-}\|u\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \frac{m_1}{p^-}\|\hat{u}\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(\hat{u})) \leq$$

$$\leq \frac{m_1}{p^-}\|\hat{u}\|^{p^-} + \frac{r}{\rho_0}(\rho_0 - \mathcal{F}(\hat{u})) \leq r.$$

We claim that

$$\inf_{u\in\mathcal{B}}\sup_{\lambda\in\Lambda}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] \geq r. \tag{9}$$

Infact, for every $u \in \mathcal{B}$ there are two cases:
(I) If $\mathcal{F}(u) < \rho_0$,

$$\sup_{\lambda\in\Lambda}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = +\infty.$$

(II) If $\mathcal{F}(u) \geq \rho_0$, by (7)

$$\sup_{\lambda\in\Lambda}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = \Phi(u) \geq \frac{m_0}{p^+}\|u\|^{p^+} \geq r.$$

From (8), (9) and the assumption (ii) of Theorem 3.1, this case verified.

Case 2. Assume that $\|u\| \geq 1$.
Similar to case 1:
Set for every $r > 0$

$$\theta_2(r) = \sup\{\mathcal{F}(u); u \in X, \frac{m_1}{p^-}\|u\|^{p^+} \leq r\}.$$

We claim that

$$\lim_{r \to 0^+} \frac{\theta_2(r)}{r} = 0. \tag{10}$$

In order to Proposition 2.3, for every $u \in X$ by continuous and compact embedding, it implies the existence of c_9 and c_{10} such that

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} F(x,u)dx \leq \int_{\mathbb{R}^N} K(|u|^{t(x)} + |u|^{z(x)})dx \leq K(\|u\|^{t^+}_{L^{t(x)}(\mathbb{R}^N)} + \|u\|^{z^+}_{L^{z(x)}(\mathbb{R}^N)})$$

$$\leq Kc_9(\|u\|^{t^+}_X + \|u\|^{z^+}_X) \leq Kc_{10}(r^{\frac{t^+}{p^+}} + r^{\frac{z^+}{p^+}}).$$

It follows from $\min\{t^+, z^+\} > p^+$ that

$$\lim_{r \to 0^+} \frac{\theta_2(r)}{r} = 0.$$

Using Lemma 3.5 $\hat{u} \neq 0$, therefore, due to (10), there is some $r \in \mathbb{R}$ such that

$$0 < r < \frac{m_1}{p^-}\|\hat{u}\|^{p^+}, \qquad 0 < \frac{\theta_2(r)}{r} < \frac{\mathcal{F}(\hat{u})}{\frac{m_1}{p^-}\|\hat{u}\|^{p^+}}.$$

Let $\rho_0 > 0$ such that

$$\theta_2(r) < \rho_0 < \frac{r\mathcal{F}(\hat{u})}{\frac{m_1}{p^-}\|\hat{u}\|^{p^+}}. \tag{11}$$

We claim that

$$\sup_{\lambda \in \Lambda} \inf_{u \in \mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] < r. \tag{12}$$

Because of the mapping

$$\lambda \mapsto \sup_{\lambda \in \Lambda} \inf_{u \in \mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))]$$

is upper semicontinuous on Λ, so

$$\lim_{\lambda \to +\infty} \inf_{u \in \mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] \leq \lim_{\lambda \to +\infty} [\frac{m_1}{p^-}\|\hat{u}\|^{p^+} + \lambda(\rho_0 - \mathcal{F}(\hat{u}))] = -\infty.$$

Therefore, there exists $\bar{\lambda} \in \Lambda$

$$\sup_{\lambda \in \Lambda} \inf_{u \in \mathcal{B}} [\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = \inf_{u \in \mathcal{B}} [\frac{m_1}{p^-}\|u\|^{p^+} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))].$$

We consider two cases:
(I) If $\bar{\lambda}\rho_0 < r$, we obtain

$$\inf_{u \in \mathcal{B}} [\frac{m_1}{p^-}\|u\|^{p^+} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \bar{\lambda}\rho_0 < r.$$

(II) If $\bar{\lambda}\rho_0 \geq r$, from (11) we obtain

$$\inf_{u \in \mathcal{B}} [\frac{m_1}{p^-}\|u\|^{p^+} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \frac{m_1}{p^-}\|\hat{u}\|^{p^+} + \bar{\lambda}(\rho_0 - \mathcal{F}(\hat{u})) \leq$$

$$\leq \frac{1}{p^-}\|\hat{u}\|^{p^+} + \frac{r}{\rho_0}(\rho_0 - \mathcal{F}(\hat{u})) \leq r.$$

Next, we claim that

$$\inf_{u\in\mathcal{B}} \sup_{\lambda\in\Lambda}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] \geq r. \tag{13}$$

For every $u \in \mathcal{B}$ two cases can occur:
(I) If $\mathcal{F}(u) < \rho_0$ we have

$$\sup_{\lambda\in\Lambda}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = +\infty.$$

(II) If $\mathcal{F}(u) \geq \rho_0$ we have by (11)

$$\sup_{\lambda\in\Lambda}[\Phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = \Phi(u) \geq \frac{m_0}{p^+}\|u\|^{p^-} \geq r.$$

For function $\mathcal{G}$ which satisfies (G), it follows from Lemma 3.3, that the functional $\mathcal{G} : X \to \mathbb{R}$ is locally Lipschitz with weakly sequentially semicontinuous. From Theorem 3.1 there exist $\lambda_1, \lambda_2 \in \Lambda$ (without loss of generality we may assume $0 < \lambda_1 < \lambda_2$) and $\sigma > 0$ with the following property that, for $\lambda \in [\lambda_1, \lambda_2]$ there exists $\mu_1 > 0$ in which: for every $\mu_1 \in]0, \mu[$, the functional $\Phi - \lambda\mathcal{F} - \mu\mathcal{G}$ admits at least three critical points $u_0, u_1, u_2 \in \mathcal{B}$ with $\|u_i\| < \sigma$. So by Lemma 3.4 u_0, u_1, u_2 are three solutions of the problem (1). $\square$

4. Three radially symmetric solutions for a differential inclusion problem

In this part we apply Theorem 3.1 to show the existence of at least three radially symmetric solutions for a variational-hemivariational inequality. The main difficulty in studying our problem is that there is no compact embedding of $W_0^{1,p(x)}(\Omega)$ to $L^\infty(\mathbb{R}^N)$. However, the subspace of radially symmetric functions of $W_0^{1,p(x)}(\mathbb{R}^N)$, denoted by $W_{0,r}^{1,p(x)}(\mathbb{R}^N)$ can be embedded compactly into $L^\infty(\mathbb{R}^N)$ whenever $N < p^- \leq p^+ < +\infty$.

Choosing $X = W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N)$ and applying the nonsmooth version of the principle of symmetric criticality we consider the differential inclusion problem

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u \in \lambda\partial a(x)F(x,u) + \mu\partial b(x)G(x,u) & \text{on } \mathbb{R}^N \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{14}$$

where λ, μ are positive parameters and $F, G : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functions. $a, b \in L^\infty(\mathbb{R}^N)$, are radially symmetric and $a, b \geq 0$.

Let $O(N)$ be the group of orthogonal linear transformations in $\mathbb{R}^N$. We say that a function $l : \mathbb{R}^N \to \mathbb{R}$ is radially symmetric if $l(gx) = l(x)$ for every $g \in O(N)$ and $x \in \mathbb{R}^N$. The action of the group $O(N)$ on $W_0^{1,p(\cdot)}(\mathbb{R}^N)$ can be defined by $(gu)(x) := u(g^{-1}x)$, for every $g \in O(N)$ and $u \in W_0^{1,p(\cdot)}(\mathbb{R}^N)$. We can define the subspace of radially symmetric functions of $W_0^{1,p(\cdot)}(\mathbb{R}^N)$ by

$$W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N) = \{u \in W_0^{1,p(\cdot)}(\mathbb{R}^N) : gu = u, \forall g \in O(N)\}.$$

Proposition 4.1. [9] *The embedding* $W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, *is compact whenever* $N < p^- \leq p^+ < +\infty$.

The energy functional $\tilde{\mathcal{I}} : W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N) \to \mathbb{R}$ associated to problem (14) is given by

$$\tilde{\mathcal{I}} = \Phi(u) - \lambda\tilde{\mathcal{F}}(u) - \mu\tilde{\mathcal{G}}(u) + \chi(u)$$

such that

$$\tilde{\mathcal{F}}(u) = \int_{\mathbb{R}^N} a(x)F(x,u)dx, \qquad \tilde{\mathcal{G}}(u) = \int_{\mathbb{R}^N} b(x)G(x,u)dx, \qquad \forall u \in W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N),$$

where $\chi(u)$ is the indicator function of the set $\mathcal{B}$.
By the principle of symmetric criticality of Krawcewicz and Marzantowicz (cf. [22]), u is a critical point of $\mathcal{I}$ if and only if u is a critical point of $\tilde{\mathcal{I}}^r = \mathcal{I}|_{W_{0,r}^{1,p(\cdot)}(\mathbb{R}^N)}$.

Lemma 4.2. *Assuming (F_1) satisfies, $\mathcal{F} : X \to \mathbb{R}$ will be locally Lipschitz functional and sequentially weakly semicontinuous.*

Proof. By similar argument of Lemma 3.2 we show that $\mathcal{F}$ is Lipschitz continuous on each bounded subset of X. Let $u, v \in B(0,M)$ $(M > 0)$, and $\|u\|, \|u\| \leq 1$. From proposition 2.5 and thanks to proposition 2.3

$$\begin{aligned}
|\mathcal{F}(u) - \mathcal{F}(v)| &\leq \int_{\mathbb{R}^N} |a(x)(F(x,u(x)) - F(x,v(x)))|dx \\
&\leq \int_{\mathbb{R}^N} Ka(x)(|u(x)|^{t(x)-1} + |v(x)|^{t(x)-1} + |u(x)|^{z(x)-1} + |v(x)|^{z(x)-1}) \\
&\qquad\qquad \times|u(x) - v(x)|dx \\
&\leq K\|a\|_\infty\|u - v\|_\infty \int_{\mathbb{R}^N} |u(x)|^{t(x)-1}dx + \int_{\mathbb{R}^N} |v(x)|^{t(x)-1}dx \\
&\qquad + \int_{\mathbb{R}^N} |u(x)|^{z(x)-1}dx + \int_{\mathbb{R}^N} |v(x)|^{z(x)-1}dx \\
&\leq K\|a\|_\infty\|u - v\|_\infty \Big(\|u\|_{L^{t(x)}}^{t^- -1} + \|v\|_{L^{t(x)}}^{t^- -1} + \|u\|_{L^{z(x)}}^{z^- -1} + \|v\|_{L^{z(x)}}^{z^- -1}\Big) \\
&\leq K\|a\|_\infty\|u - v\|_X \Big(\|u\|_X^{t^- -1} + \|v\|_X^{t^- -1} + \|u\|_X^{z^- -1} + \|v\|_X^{z^- -1}\Big) \\
&\leq 2K\|u - v\|_X(c_{11}M^{t^- -1} + c_{12}M^{z^- -1})
\end{aligned}$$

where $c_{11},\ c_{12}$ are positive constants.
We show $\partial\mathcal{F}$ is compact. Let $\{u_n\}$ be a sequence in X such that $\|u_n\| \leq M$ and choose $u_n^* \in \partial\mathcal{F}(u_n) \subseteq \int_{\mathbb{R}^N} a(x)\partial F(x,u_n(x))dx$ for any $n \in \mathbb{N}$. From (F_1) it follows that for any $n \in \mathbb{N},\ v \in X$,

$$\begin{aligned}
< u_n^*, v > \quad &\leq \quad \int_{\mathbb{R}^N} |u_n^*(x)||v(x)|dx \leq \int_{\mathbb{R}^N} K|a(x)|(|u(x)|^{t(x)-1} + |u(x)|^{z(x)-1})|v(x)|dx \\
&\leq \quad K\|a\|_{L^\infty}(c_{13}M^{t^- -1} + c_{14}M^{z^- -1})\|v\|,
\end{aligned}$$

where $c_{13},\ c_{14}$ are positive constants.
Therefore,

$$\|u_n^*\|_{X^*} \leq K\|a\|_{L^\infty}(c_{13}M^{t^- -1} + c_{14}M^{z^- -1}).$$

The sequence $\{u_n^*\}$ is bounded and hence, up to a subsequence, $u_n^* \rightharpoonup u^*$.
Suppose on the contrary; there exists $\epsilon > 0$ for which $\|u_n^* - u^*\|_{X^*} > \epsilon$ (choose a subsequence if necessary). For every $n \in \mathbb{N}$, we can find $v_n \in X$ with $\|v_n\| < 1$ and

$$\langle u_n^* - u^*, v_n\rangle > \epsilon. \tag{15}$$

Then, $\{v_n\}$ is a bounded sequence and up to a subsequence, $\{v_n\}$ be a sequence in $W^{1,p(\cdot)}_{r,0}(\Omega)$ which converges weakly to $v \in W^{1,p(\cdot)}_{r,0}(\Omega)$. By proposition 4.1, $v_n \to v$ strongly in $L^\infty(\Omega)$. Therefore,

$$|\langle u_n^* - u^*, v\rangle| < \frac{\epsilon}{4}, \ \ |\langle u^*, v_n - v\rangle| < \frac{\epsilon}{4}, \ \ \|v_n - v\|_{L^\infty} < \frac{\epsilon}{2K\|a\|_{L^\infty}(c_3 M^{t^- -1} + c_4 M^{z^- -1})}.$$

It follows that,

$$\begin{aligned}\langle u_n^* - u^*, v_n\rangle &\leq \langle u_n^*, v_n - v\rangle + \langle u_n^* - u^*, v\rangle + \langle u^*, v - v_n\rangle \\ &\leq \int_{\mathbb{R}^N} |u_n^*(x)||v_n(x) - v(x)|dx + \langle u_n^* - u^*, v\rangle + \langle u^*, v - v_n\rangle \\ &\leq K\|a\|_{L^\infty}(c_{13}M^{t^- -1} + c_{14}M^{z^- -1})\|v_n - v\|_{L^\infty} \\ &\qquad + \langle u_n^* - u^*, v\rangle + \langle u^*, v - v_n\rangle \to 0,\end{aligned}$$

which contradicts (15). □

Lemma 4.3. *If G satisfies, then $\mathcal{G}$ is a locally Lipschitz functional with compact gradient.*

The proof is similar to Lemma (4.2).

Lemma 4.4. *If (F_2) holds, then for any $\lambda \in (0, +\infty)$, the function $\Phi - \lambda\mathcal{F}$ is coercive.*

Proof. For $u \in X$ such that $\|u\| \geq 1$

$$\begin{aligned}\mathcal{F}(u) &= \int_{\mathbb{R}^N} a(x)F(x,u)dx \leq \int_{\mathbb{R}^N} H|a(x)|(|u|^{\alpha(x)} + |u|^{\beta(x)})dx \\ &\leq H\|a\|_{L^\infty}(\|u\|^{\alpha^+}_{L^{\alpha(x)}(\mathbb{R}^N)} + \|u\|^{\beta^+}_{L^{\beta(x)}(\mathbb{R}^N)}).\end{aligned}$$

By the embedding theorem for suitable positive constant c_{15}, c_{16}

$$\mathcal{F}(u) \leq H\|a\|_{L^\infty}(c_{15}\|u\|_X^{\alpha^+} + c_{16}\|u\|_X^{\beta^+}).$$

Hence, from Proposition 2.2, for any $\lambda > 0$,

$$\Phi(u) - \lambda\mathcal{F}(u) \geq \frac{1}{p^+}\|u\|_X^{p^-} - H\|a\|_{L^\infty}(c_{15}\|u\|_X^{\alpha^+} + c_{16}\|u\|_X^{\beta^+}).$$

Since $p^- > \min\{\alpha^+, \beta^+\}$, it implies that

$$\lim_{\|u\|\to+\infty}[\Phi - \lambda\mathcal{F}] = +\infty, \quad \forall u \in X, \ \lambda \in (0, +\infty).$$

□

Theorem 4.5. *Let $a, b \in L^\infty(\Omega)$ be two radial functions and F_1, F_2, F_3 are satisfied. Then there exist $\lambda_1, \lambda_2 > 0 (\lambda_1 < \lambda_2)$ and $\tilde{\sigma} > 0$ such that for every $\lambda \in [\lambda_1, \lambda_2]$ and every $\mathcal{G}$ satisfying G, there exists $\mu_1 > 0$ such that for every $\mu \in]0, \mu_1[$ problem (14) admits at least three distinct, radially symmetric solutions whose norms are less than $\tilde{\sigma}$.*

Proof. **Case 1.** Let us assume that $\|u\| < 1$.
Put for every $r > 0$,

$$\theta_1(r) = \sup\{\mathcal{F}(u); u \in X, \frac{m_1}{p^-}\|u\|^{p^-} \leq r\},$$

we prove that

$$\lim_{r\to 0^+} \frac{\theta_1(r)}{r} = 0. \tag{16}$$

In view of (F_1), it is follows that for every $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that for every $x \in \mathbb{R}^N, u \in \mathbb{R}$ and $\xi \in \partial F(x,u)$

$$|\xi| \leq \epsilon |u|^{t(x)-1} + c(\epsilon)|u|^{z(x)-1}. \tag{17}$$

Applying Lebourgs mean value theorem and using the Sobolev embedding theorem for every $u \in X$, there exist suitable positive constants c_{17} and c_{18}

$$\begin{aligned}
\mathcal{F}(u) &= \int_{\mathbb{R}^N} F(x,u)dx \leq \int_{\mathbb{R}^N} Ka(x)(|u|^{t(x)} + |u|^{z(x)})dx \\
&\leq K\|a\|_{L^\infty}(\|u\|^{t^+}_{L^{t(x)}(\mathbb{R}^N)} + \|u\|^{z^+}_{L^{z(x)}(\mathbb{R}^N)}) \\
&\leq K\|a\|_{L^\infty} c_{17}(\|u\|^{t^+}_X + \|u\|^{z^+}_X) \leq K\|a\|_{L^\infty} c_{18}(r^{\frac{t^+}{p^-}} + r^{\frac{z^+}{p^-}}).
\end{aligned}$$

By using $\min\{t^+, z^+\} > p^-$ we conclude that

$$\lim_{r\to 0^+} \frac{\theta_1(r)}{r} = 0.$$

The remainder proof for the existence of three radially symmetric solutions of problem (14) is similarly to Theorem 3.7. □

References

[1] M. Alimohammady, F. Fattahi, Existence of solutions to hemivaritional inequalities involving the p(x)-biharmonic operator, *Electron. J. Diff. Equ.* **2015** (2015), 1–12.

[2] M. Allaoui, Existence of Solutions For A Robin Problem Involving The $p(x)-$Laplacian, *Applied Mathematics E-Notes* **14** (2014), 107–115.

[3] T. Bartsch, M. Willem, Infinitely many nonradial solutions of a Euclidean scalar field equation, *J. Funct. Anal.* **117** (1993), 447–460.

[4] G. Bonannoa, P. Winkert, Multiplicity results to a class of variational-hemivariational inequalities, *Topological methods in nonlinear analysis* **43**(2) (2014), 493–516.

[5] S. Carl, V.K. Le, D. Motreanu, *Nonsmooth variational problems and their inequalities*, Springer Monographs in Mathematics, Springer, New York, 2007.

[6] N.T. Chung, On some $p(x)-$Kirchhoff type equations with weights, *Rocky Mountain J. Appl. Math. Informatics* **32** (2014), 113–128.

[7] F.H. Clarke, *Optimization and nonsmooth analysis*, Wiley, 1983.

[8] F.J.S.A. Corrêa, G.M. Figueiredo, On an elliptic equation of p-Kirchhoff type via variational methods, *Bull. Aust. Math. Soc.* **74** (2006), 263–277

[9] G.W. Dai, Infinitely many solutions for a $p(x)-$Laplacian equation in $\mathbb{R}^N$, *Nonlinear Anal.* **71** (2009), 1133–1139.

[10] S. Dhanalakshmi, R. Murugesu, Existence of fractional order mixed type functional integro-differential equations with nonlocal conditions, *Int. J. Adv. Appl. Maths. and Mech.* **1**(3) (2014), 11–21.

[11] L. Diening, Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Math. Nachr.* **268** (2004), 31–43.

[12] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents,* Lecture Notes, Vol. 2017, Springer-Verlag, Berlin, 2011.

[13] D.E. Edmunds, J. Rákosník, Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. R. Soc. A* **437** (1992), 229–236.

[14] A.R. El Amrouss, A. Ourraoui, Existence of solutions for a boundary problem involving $p(x)-$biharmonic operator, *Bol. Soc. Paran. Mat.* **31**(1) (2013), 179–192.

[15] L. Eugenio Cabanillas, LL. Adrian, G. Aliaga, M. Willy Barahona, V. Gabriel Rodriguez, Existence of solutions for a class of p(x)-Kirchhoff type equation via topological methods, *Rocky Mountain Int. J. Adv. Appl. Math. and Mech.* **2**(4) (2015), 64–72.

[16] X.L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), 424–446.

[17] X.L. Fan, Q.H. Zhang, Existence of solutions for $p(x)-$Laplacian Dirichlet problems, *Nonlinear Anal.* **52** (2003), 1843–1852.

[18] SH. Heidarkhani, F. Gharehgazlouei, A. Solimaninia, Existence of infinitely many symmetric solutions to perturbed elliptic equations with discontinuous nonlinearities in $\mathbb{R}^N$, *Electronic Journal of Differential Equations* 2015 (2015), 1–17.

[19] A. Iannizzotto, Three critical points for perturbed nonsmooth functionals and applications, *Nonlinear Analysis TMA* **72** (2010), 1319–1338.

[20] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[21] O. Kováčik , J. Rákosnínk , On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41** (1991), 592–618.

[22] W. Krawcewicz, W. Marzantowicz, Some remarks on the Ljusternik-Schnirelman method for non-differentiable functionals invariant with respect to a finite group action, *Rocky Mountain J. Math.* **20** (1990), 1041–1049.

[23] A. Kristály, W. Marzantowicz, C. Varga, A nonsmooth three critical points theorem with applications in differential inclusions, *J. Glob. Optim.* (2009).

[24] A. Kristály, Infinitely many radial and non-radial solutions for a class of hemivariational inequalities, *Rocky Mountain Journal of Mathematics* **35**(4) (2005), 1173–1190.

[25] D. Motreanu, P. D. Panagiotopoulos, *Minimax theorems and qualitative properties of the solutions of hemivariational inequalities*, Kluwer Academic Publishers, Dordrecht, 1999.

[26] D. Motreanu, V. Rădulescu, *Variational and non-variational methods in nonlinear analysis and boundary value problems*, Kluwer Academic Publishers, Boston-Dordrecht-London, 2003.

[27] V. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Analysis TMA* **121** (2015), 336–369.

[28] V. Rădulescu, D. Repovš, *Partial differential equations with variable exponents*, Variational Methods and Qualitative Analysis, CRC Press, Taylor & Francis Group, Boca Raton, 2015.

[29] B. Ricceri, Minimax theorems for limits of parametrized functions having at most one local minimum lying in a certain set, *Topol. Appl.* **153** (2006), 3308–3312.

[30] M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2002.

[31] R. Temam, A nonlinear eigenvalue problem: the shape at equilibrium of a confined plasma, *Arch. Ration. Mech. Anal.* **60**(1) (1975) 51–73.

[32] C. Varga, Existence and infinitely many solutions for an abstract class of hemivariational inequalities, *Journal of Inequalities and Applications* **2005** (2005), 89–105.

[33] P. Winkert, On the Boundedness of Solutions to Elliptic Variational Inequalities, *Set-Valued and Var. Anal.* **22**(4) (2014), 763–781.

[34] Z. Yuan, L. Huang, Nonsmooth extention of a three critical points theorem by Ricceri with an application to $p(x)-$laplacian differential inclusions, *Electron. J. Diff. Equ.* **2015** (2015), 1–16.

(M. Alimohammady, F. Fattahi) Department of Mathematics, University of Mazandaran Babolsar, IRAN

E-mail address: `Amohsen@umz.ac.ir, F.Fattahi@stu.umz.ac.ir`