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Łukasiewicz Implication Prealgebras

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Abstract. In this paper we revise the Łukasiewicz implication prealgebras which we will call Łukasiewicz $I-$prealgebras to sum up. They were used by Antonio Jesús Rodríguez Salas on his doctoral thesis under the name of Sales prealgebras. These structures are a natural generalization of the notion of $I-$prealgebras, introduced by A. Monteiro in 1968 aiming to study using algebraic techniques the $\{\rightarrow\}$-fragment of the three-valued Łukasiewicz propositional calculus. The importance of Łukasiewicz $I-$prealgebras focuses on the fact that from these structures we can directly prove that Lindembaun-Tarski algebra in the $\{\rightarrow\}$-fragment of the infinite-valued Łukasiewicz implication propositional calculus is a Łukasiewicz residuation BCK-algebra in the sense of Berman and Blok [1]. This last result is indicated without a proof on Komori's paper ([8]) and it is suggested on his general lines on the Rodriguez Salas thesis.

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1. Introduction and preliminaries

In 1982, A. Iorgulescu said that she came up with the idea of the $I-$prealgebras after reading about preboolean sets in [11, 12] and about Nelson algebras and Łukasiewicz algebras in [10], on one side, and about $I-$algebras [13]. For details please go to [7].

On the other hand, in 1980, A. Monteiro introduced a particular class of I-prealgebras. In this paper, we will use Monteiro therminology.

In 1930 Łukasiewicz considered the matrix $Ł_{n+1} = \langle C_{n+1}, \rightarrow, \sim, D\rangle$ and $Ł = \langle [0,1], \rightarrow, \sim, D\rangle$, where:

(i) $C_{n+1} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$, n is a positive integer and $[0,1]$ is the real interval;
(ii) If $p, q \in C_{n+1}$ or $p, q \in [0,1]$, then the implication, $\rightarrow$, is defined by the formula $p \rightarrow q = min\,\{1, 1-p+q\}$, the negation, $\sim$, by $\sim p = 1-p$; and
(iii) $D = \{1\}$ is the set of designated elements.

For the ones interested in focusing on the many algebraization of the Łukasiewicz propositional calculus, we recommend reading the important book [2] indicated in the references section.

In the following, we will denote with $(n+1)-$IŁ, $n \geq 1$, and with $\omega-$IŁ to the propositional calculus determined by the implicative parts of $Ł_{n+1}$ and Ł respectively.

In 1956, Rose [16] indicated an axiomatization of the $\omega-$IŁ, where he proved the substitution rules, the modus ponens and the axioms:

(C1) $p \rightarrow (q \rightarrow p)$,

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(C2) $(p \to q) \to ((q \to r) \to (p \to r))$,
(C3) $((p \to q) \to q) \to ((q \to p) \to p)$,
(C4) $((p \to q) \to (q \to p)) \to (q \to p)$,
are sufficient. On the same article adding the axiom scheme:
(C5) $((x \to_n y) \to x) \to x$,
determined an axiomatization of the $(n+1)-$IŁ, where
(Ab1) $p \to_0 q = q$ and $p \to_{n+1} q = p \to (p \to_n q)$, for $n = 0, 1, 2 \ldots$.

In [9] Monteiro, with the purpose of studying the 3−IŁ with algebraic techniques, introduced the concepts of I_3–prealgebras and 3-valued Łukasiewicz implication algebra. The results obtained by this author were exposed in 1968 in a course given at Universidad Nacional del Sur but they have not been published yet.

On this work, we take our research based on the algebraization method proposed by Monteiro, who has shown his excellent studies on many propositional calculus. To begin with, we consider the I–prealgebras and then the I_{n+1}–prealgebras, as generalizations of the I_3–prealgebras of Monteiro and we redo some proofs of the properties needed for the rest of the work exposed here, indicated by Monteiro in [9]. In particular, we concentrate on those properties in which the axiom referring to the n-valence, of the Definition 4.1, does not take place here.

2. Łukasiewicz $I-$prealgebras

Definition 2.1. The system $\langle A, \to, D\rangle$ is a Łukasiewicz implication prealgebra (or Łukasiewicz $I-$prealgebra) if we verify:
(i) $\langle A, \to\rangle$ is an algebra of type 2,
(ii) D is a non-empty subset of A such that for every $p, q, r \in D$ the conditions are verified:
(R1) $p \to (q \to p) \in D$,
(R2) $(p \to q) \to ((q \to r) \to (p \to r)) \in D$,
(R3) $(p \vee q) \to (q \vee p) \in D$,
(R4) $(p \to q) \vee (q \to p) \in D$, where (Ab2) $p \vee q = (p \to q) \to q$.
And the modus ponens rule:
(MP) $\dfrac{p \in D, p \to q \in D}{q}$,

Example 2.1.
(i) If $\langle A, \to\rangle$ is an algebra of type 2, then $\langle A, \to, A\rangle$ is an Łukasiewicz $I-$prealgebra.
(ii) The matrix $\langle C_{n+1}, \to, \{1\}\rangle$, $n \geq 1$, and $\langle [0,1], \to, \{1\}\rangle$ are Łukasiewicz $I-$ prealgebras.
(iii) If $\langle For(G), \to\rangle$ is an algebra of type 2 absolutely free and $\mathcal{T}$ is the set of the thesis of the $\omega-$IŁ, then $\langle For(G), \to, \mathcal{T}\rangle$ is an Łukasiewicz $I-$prealgebra.

Throughout this section A is the underlying set of the I–prealgebra $\langle A, \to, D\rangle$.

Definition 2.2. Let $\langle A, \to, D\rangle$ be a Łukasiewicz I–prealgebra. Let $p, q \in A$, we say that $p \preceq q$ if $p \to q \in D$.

Lemma 2.1. *Let $\langle A, \to, D\rangle$ be a Łukasiewicz I–prealgebra. Then the following properties are verified:*

(I1) *If* $p \preceq q$ *and* $q \preceq r$ *then* $p \preceq r$.
(I2) *If* $d \in D$ *and* $p \in A$, *then* $p \preceq d$.
(I3) $(q \to p) \to r \preceq p \to r$.
(I4) $p \preceq q \vee p$.
(I5) $q \preceq q \vee p$.
(I6) $(q \vee r) \to s \preceq q \to s$.
(I7) $(q \vee r) \to (p \to r) \preceq q \to (p \to r)$.
(I8) $p \to (q \to r) \preceq (q \vee r) \to (p \to r)$.
(I9) $p \to (q \to r) \preceq q \to (p \to r)$.
(I10) $p \to (q \to p) \preceq q \to (p \to p)$.
(I11) $q \preceq (p \to p)$.
(I12) $p \preceq p$.
(I13) $q \to r \preceq (p \to q) \to (p \to r)$.
(I14) *If* $q \preceq r$, *then* $p \to q \preceq p \to r$.

Proof. (I1):
(1) $p \to q \in D$, [hip.]
(2) $q \to r \in D$, [hip.]
(3) $p \to r \in D$, [(1), (2), R2, MP]
(4) $p \preceq r$. [(3), Definition 2.2]
(I2):
(1) $d \in D$ and $p \in A$, [hip.]
(2) $d \to (p \to d) \in D$, [R1]
(3) $p \to d \in D$. [(1), (2), MP]
(4) $p \preceq d$. [(3), Definition 2.2]
(I3):
(1) $p \to (q \to p) \in D$, [R1]
(2) $(p \to (q \to p)) \to (((q \to p) \to r) \to (p \to r)) \in D$, [R2]
(3) $((q \to p) \to r) \to (p \to r) \in D$. [(1), (2), MP]
(4) $(q \to p) \to r \preceq p \to r$. [(3), Definition 2.2]
(I4):
(1) $p \to ((q \to p) \to p) \in D$, [R1]
(2) $p \to (q \vee p) \in D$, [(1), Ab2]
(3) $p \preceq q \vee p$. [(2), Definition 2.2]
(I5):
(1) $p \preceq q \vee p$, [(I4)]
(2) $q \vee p \preceq p \vee q$, [R3, Definition 2.2]
(3) $p \preceq p \vee q$. [(1), (2), (I1)]
(I6):
(1) $(q \to (q \vee r)) \to (((q \vee r) \to s) \to (q \to s)) \in D$, [R2, Definition 2.2]
(2) $q \to (q \vee r) \in D$, [(I5), Definition 2.2]
(3) $((q \vee r) \to s) \to (q \to s) \in D$, [(2), (1), Definition 2.2, R2, MP]
(4) $(q \vee r) \to s \preceq q \to s$. [(3), Definition 2.2]
(I7):
We obtained it replacing s by $p \to r$ in (I6).
(I8):
(1) $(p \to (q \to r)) \to (((q \to r) \to r) \to (p \to r)) \in D$, [R2]
(2) $p \to (q \to r) \preceq (q \vee r) \to (p \to r)$. [(1), Ab2, Definition 2.2]

(I9):
(1) $p \to (q \to r) \preceq ((q \vee r)) \to (p \to r)$, [(I8)]
(2) $(q \vee r) \to (p \to r) \preceq q \to (p \to r)$, [(I7)]
(3) $p \to (q \to r) \preceq q \to (p \to r)$. [(1), (2), (I1)]
(I10):
We get this result substituting r for p in (I9).
(I11):
(1) $(p \to (q \to p)) \to (q \to (p \to p)) \in D$, [Definition 2.2, (I10)]
(2) $q \to (p \to p) \in D$. [(1), R1, MP]
(I12):
(1) $(p \to (q \to p)) \to (p \to p) \in D$, [(I11)]
(2) $p \to p \in D$. [(1), R1, MP]
(I13):
(1) $((p \to q) \to ((q \to r) \to (p \to r))) \to ((q \to r) \to ((p \to q) \to (p \to r))) \in D$, (I9)]
(2) $(p \to q) \to ((q \to r) \to (p \to r)) \in D$, [R2]
(3) $(q \to r) \to ((p \to q) \to (p \to r)) \in D$. [(2), (1), MP]
(I14):
(1) $q \to r \in D$, [hip.]
(2) $(p \to q) \to (p \to r) \in D$, [(1), (I13), MP]
(3) $p \to q \preceq p \to r$. [(2), Definition 2.2]
□

Theorem 2.2. *$\langle A, \preceq \rangle$ is a quasiorder set.*

Proof. The proof is followed by (I1) and (I12). □

Definition 2.3. Let $\langle A, \to, D \rangle$ be a Łukasiewicz I–prealgebra. Let $p, q \in A$. We will say that $p \equiv q$ if $p \preceq q$ and $q \preceq p$.

Theorem 2.3. *The relation $\equiv$ has the following properties:*
(i) *$p \preceq q$ and $q \equiv r$ imply $p \preceq r$,*
(ii) *$p \preceq q$ and $p \equiv s$ imply $s \preceq q$,*
(iii) *$p \preceq q$, $p \equiv s$ and $q \equiv r$ imply $s \preceq r$.*

Proof. The proof is followed by (I1). □

Theorem 2.4. *The relation $\equiv$ is compatible with the operation $\to$.*

(i) If $p \equiv q$ then $p \to r \equiv q \to r$:
(1) $p \to q \in D$, [hip.]
(2) $(p \to q) \to ((q \to r) \to (p \to r)) \in D$, [R2]
(3) $(q \to r) \to (p \to r) \in D$, [(1), (2), MP]
(4) $q \to r \preceq p \to r$, [(3), Definition 2.2]
In an analogous way, we can prove that:
(5) $p \to r \preceq q \to r$.
(ii) If $p \equiv q$ then $r \to p \equiv r \to q$:
(1) $p \to q \in D$, [hip.]
(2) $(p \to q) \to ((r \to p) \to (r \to q)) \in D$, [(I13)]
(3) $(r \to p) \to (r \to q) \in D$, [(1), (2), MP]
(4) $r \to p \preceq r \to q$, [(3), Definition 2.2]

Similarly, we show that:

(5) $r \to q \preceq r \to p$.

Theorem 2.5. *If $t \in D$ then $[t] = D$, where $[t] = \{p \in A : p \equiv t\}$.*

(i) $D \subseteq [t]$:

Indeed, let d be an element of D, then

(1) $t \in D$, [hip.]
(2) $d \in D$, [hip.]
(3) $d \to t \in D$, [(1), (I2)]
(4) $t \to d \in D$, [(2), (I2)]
(5) $d \preceq t$, [(3), Definition 2.2]
(6) $t \preceq d$, [(4), Definition 2.2]
(7) $d \equiv t$, [(5), (6), Definition 2.3]
(8) $d \in [t]$, [(7)]
(9) $D \subseteq [t]$, [(2), (8)]

(ii) $[t] \subseteq D$:

Indeed:

(1) $p \in [t]$, [hip.]
(2) $p \equiv t$, [(1)]
(3) $t \to p \equiv t \to t$, [(2), Theorem 2.4 (ii)]
(4) $t \to p \in D$, [(3), (I12)]
(5) $t \in D$, [hip.]
(6) $p \in D$. [(5), (4), MP]
(7) $[t] \subseteq D$. [(1), (6)]

3. Łukasiewicz $I-$prealgebras of the Lindenbaum-Tarski algebras

As a consequence of Theorem 2.4 we can explain the quotient set. If $[p] \to [q] = [p \to q]$ and $D = \mathbf{1}$, then $\langle A/\equiv, \to, \mathbf{1}\rangle$ is an algebra of type $(2, 0)$.

Definition 3.1. The algebra $\langle A/\equiv, \to, \mathbf{1}\rangle$ is called the Lindenbaum-Tarski algebra of the Łukasiewicz $I-$prealgebra $\langle A, \to, D\rangle$.

With the intention of indicating important properties of the Lindenbaum-Tarski algebra we previously noted a list of additional properties valid in every Łukasiewicz $I-$prealgebra:

Lemma 3.1. *Let $\langle A, \to, D\rangle$ be a Łukasiewicz $I-$prealgebra. Then the following properties can be verified:*

(I15) $q \vee q \equiv q$
(I16) $(q \to r) \to (p \to r)) \preceq (p \vee r) \to (q \vee r)$.
(I17) $p \to q \preceq (p \vee r) \to (q \vee r)$.
(I18) $p \to r \preceq (p \vee q) \to (r \vee q)$.
(I19) $p \to q \preceq (r \vee p) \to (r \vee q)$.
(I20) $q \to r \preceq (r \vee q) \to (r \vee r)$.

Proof. (I15):

(1) $q \to ((q \to q) \to q) \in D$, [R1]
(2) $q \to (q \vee q) \in D$, [(1), Ab2]

(3) $(q \to q) \to ((q \to q) \vee q) \in D$, [(I5), Definition 2.3]
(4) $(q \to q) \to ((q \to q) \to ((q \to q) \vee q)) \in D$, [(2), (I2)]
(5) $(q \to q) \vee q \in D$, [(4), (I12), MP]
(6) $((q \to q) \to q) \to q \in D$, [(5), Ab2]
(7) $(q \vee q) \to q \in D$, [(6), Ab2]
(8) $q \vee q \equiv q$. [(2), (7), Definition 2.3]

(I16):
(1) $((q \to r) \to (p \to r)) \to (((p \to r) \to r) \to ((q \to r) \to r)) \in D$, [R2]
(2) $((q \to r) \to (p \to r)) \to ((p \vee r) \to (q \vee r)) \in D$. [(1), Ab2]
(3) $(q \to r) \to (p \to r)) \preceq (p \vee r) \to (q \vee r)$. [(2), Definition 2.2]

(I17):
(1) $p \to q \preceq (q \to r) \to (p \to r)$, [R2, Definition 2.2]
(2) $(p \vee r) \to (q \vee r) \in D$, [R3]
(3) $(q \to r) \to (p \to r) \preceq (p \vee r) \to (q \vee r)$, [(2), (I2), Definition 2.2]
(4) $p \to q \preceq (p \vee r) \to (q \vee r)$. [(1), (3), (I1)]

(I18):
Comes from (I3) replacing q by r and r by q.

(I19):
(1) $p \vee r \equiv r \vee p$, [R3, Definition 2.2, Definition 2.3]
(2) $q \vee r \equiv r \vee q$, [R3, Definition 2.2, Definition 2.3]
(3) $p \to q \preceq (q \to r) \to (p \to r)$, [R2, Definition 2.2]
(4) $(q \to r) \to (p \to r) \preceq ((p \vee r) \to r) \to ((q \to r) \to r)$, [R2, Definition 2.2]
(5) $p \to q \preceq (p \vee r) \to (q \vee r)$, [(3), (4), (I1), Ab2]
(6) $(p \vee r) \to (q \vee r) \equiv (r \vee p) \to (q \vee r)$ [(1), Theorem 2.4]
$\equiv (r \vee p) \to (r \vee q)$, [(2), Theorem 2.4]
(7) $p \to q \preceq (r \vee p) \to (r \vee q)$. [(5), (6), Theorem 2.3]

(I20):
Results from (I19) substituting p for q and q for r. □

Now we can analyze the order given by the algebra $A/\equiv$. For that we give some results.

Definition 3.2. Let $p, q \in A$; $[p] \leq [q]$ if and only if $p \preceq q$.

The pair $\langle A/\equiv, \leq\rangle$ is an ordered set which has the properties mentioned in Theorem 3.2.

Theorem 3.2. *$\langle A/\equiv, \leq\rangle$ is an ordered set with a last element $\mathbf{1}$. Besides, it is a join-lattice where the greatest of the elements $[p]$ and $[q]$ is $[p] \vee [q] = [p \vee q]$.*

Proof. (i) $\leq$ is an order: It is a consequence of Theorem 2.2 and the Definition 2.3.
(ii) $[p] \leq \mathbf{1}$, for every $p \in A$: Let $p \in A$, then:
(1) It exists $t \in D$, [Definition 2.1]
(2) $p \to t \in D$, [(I2), Definition 2.2]
(3) $p \to (p \to t) \in D$, [(2), (I2), Definition 2.2]
(4) $[p \to t] = \mathbf{1}$ [(2), Theorem 2.5]
(5) $[p] \leq \mathbf{1}$. [(3), (4), Definition 2.2, Definition 3.2]
(iii) $[p \vee q]$ is the supremum of $[p]$ and $[q]$: Indeed, we can verify:
(s1) $p \preceq p \vee q$, [(I5)]
(s2) $q \preceq p \vee q$, [(I6)]

(s3) $p \preceq r$ and $q \preceq r$ imply $p \vee q \preceq r$:

(1) $p \preceq r$, [hip.]
(2) $q \preceq r$, [hip.]
(3) $p \vee q \preceq r \vee q$, [(1), (I18), Definition 2.2, MP]
(4) $r \vee q \preceq r \vee r$, [(2), (I20), Definition 2.2, MP]
(5) $r \vee r \equiv r$, [(I1)]
(6) $p \vee q \preceq r$. [(3), (4), (5), (I1), Definition 2.2, Theorem 2.3]

□

On the other hand, we verify:

Theorem 3.3. *The Lindenbaum-Tarski algebra of the Łukasiewicz $I-$prealgebra $\langle A, \rightarrow, D\rangle$ satisfy the properties:*

(W1) $1 \rightarrow x = x$,
(W2) $x \rightarrow (y \rightarrow x) = 1$,
(W3) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
(W4) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(W5) $((x \rightarrow y) \rightarrow (y \rightarrow x)) \rightarrow (y \rightarrow x) = 1$.

This is, the Łukasiewicz $I-$prealgebras of the Lindenbaum-Tarski algebras are the algebras that satisfy the identities W1, ..., W5.

From the third of Example 2.1 and the Theorem 3.3 we get a proof that the Lindenbaum-Tarski algebra of the $\omega-$IŁ is a Łukasiewicz residuation algebra [1].

4. Łukasiewicz $I_{n+1}-$prealgebras

In this section, we will analize a particular class of Łukasiewicz I–prealgebras.

Definition 4.1. A Łukasiewicz $I-$prealgebra $\langle A, \rightarrow, D\rangle$ is a Łukasiewicz $I_{n+1}-$ prealgebra if for every p, $q \in A$ the following property is verified:

(R5) $(p \rightarrow_n q) \vee p \in D$.

Onwards, to sum up, we write

(Ab3) $p \mapsto q = p \rightarrow_n q$,

The operation $\mapsto$, which we will call weak implication, defined in (Ab3), has the following properties.

Theorem 4.1. *In very Łukasiewicz $I_{n+1}-$prealgebra $\langle A, \rightarrow, D\rangle$ we verify:*

(DR1) $p \mapsto (q \mapsto p) \in D$,
(DR2) $(p \mapsto (q \mapsto r)) \mapsto ((p \mapsto q) \mapsto (p \mapsto r)) \in D$,
(DR3) $((p \mapsto q) \mapsto p) \mapsto p \in D$.

The proof of the Theorem 4.1 is a consequence of the following properties:

Lemma 4.2. *For every Łukasiewicz $I_{n+1}-$prealgebra $\langle A, \rightarrow, D\rangle$ the following properties are verified:*

(a) $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \rightarrow r)$
(b) $p \rightarrow (q \mapsto p) \in D$.
(c) $p \mapsto (q \mapsto p) \in D$.
(d) $(p \mapsto q) \rightarrow p \equiv p$.
(e) $(p \mapsto q) \mapsto p \equiv p$.

(f) $((p \mapsto q) \mapsto p) \mapsto p \in D$.
(g) $p \to (p \mapsto q) \equiv p \mapsto q$.
(h) $p \mapsto (p \mapsto q) \equiv p \mapsto q$.
(i) $p \mapsto (q \to r) \equiv q \to (p \mapsto r)$.
(j) $p \mapsto (q \mapsto r) \equiv q \mapsto (p \mapsto r)$.
(k) $p \to q \preceq p \mapsto q$.
(l) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \mapsto r)$.
(ll) $p \mapsto (q \to r) \preceq (p \mapsto q) \mapsto (p \mapsto r)$.
(m) $p \mapsto (q \mapsto r) \preceq (p \mapsto q) \mapsto (p \mapsto r)$.

Proof. (a) The proof is adjacent to (I9) and the Definition 2.3.
(b)
(1) $p \to (q \to p) \in D$, [R1]
(2) $q \to (p \to (q \to p)) \in D$, [(1), (I2)]
(3) $p \to (q \to (q \to p)) \in D$, [(2), (I9), MP]
(4) $p \to (q \to_2 p) \in D$. [(3), Ab1]
If $n = 2$, the proof is done. On the contrary:
(5) $q \to (p \to (q \to_2 p)) \in D$, [(4), (I2)]
(6) $p \to (q \to (q \to_2 p)) \in D$, [(5), (I9), MP]
(7) $p \to (q \to_3 p) \in D$. [(6), Ab1]
Repeating the process, we obtain:
$p \to (q \mapsto p) \in D$.
(c) It is a consequence of (b) and of Ab3.
(d)
(1) $((p \mapsto q) \to p) \to p \in D$, [R5, Ab3, Ab2]
(2) $p \to ((p \mapsto q) \to p) \in D$, [R1]
(3) $(p \mapsto q) \to p \equiv p$. [(1), (2), Definition 2.2]
(e)
(1) $(p \mapsto q) \to p \equiv p$, [(d)]
(2) $(p \mapsto q) \to ((p \mapsto q) \to p) \equiv (p \mapsto q) \to p$, [(1), Theorem 2.4]
$\equiv p$ [(1)]
(3) $(p \mapsto q) \to_2 p \equiv p$. [(2), Ab3]
If $n = 2$ the proof is finished, if $n \geq 3$ repeating the process we get to:
(j) $(p \mapsto q) \to_n p \equiv p$,
(j+1) $(p \mapsto q) \mapsto p \equiv p$. [(j), Ab3]
(f)
(1) $(p \mapsto q) \mapsto p \equiv p$, [(e)]
(2) $((p \mapsto q) \mapsto p) \to p \equiv p \to p$, [(1), Theorem 2.4]
(3) $((p \mapsto q) \mapsto p) \to p \in D$, [(2), (I12), Definition 2.2]
(4) $((p \mapsto q) \mapsto p) \to (((p \mapsto q) \mapsto p) \to p) \in D$, [(3), (I2)]
(5) $((p \mapsto q) \mapsto p) \to_2 p \in D$. [(4), Ab3]
If $n = 2$ the proof is finished. If not, repeating the process we get to:
(l) $((p \mapsto q) \mapsto p) \to_n p \in D$,
(l+1) $((p \mapsto q) \mapsto p) \mapsto p \in D$. [(l), Ab3]
(g)
(1) $((p \mapsto q) \to p) \to p \in D$, [R5, Ab2]
(2) $(p \to (p \mapsto q)) \to (p \mapsto q) \in D$, [(1), R3, MP]
(3) $p \to (p \mapsto q) \preceq p \mapsto q$, [(2), Definition 2.2]

(4) $p \mapsto q \preceq p \to (p \mapsto q)$, [R1, Definition 2.2]
(5) $p \to (p \mapsto q) \equiv p \mapsto q$. [(3), (4), Definition 2.3]

(h)
(1) $p \to (p \mapsto q) \equiv p \mapsto q$, [(g)]
(2) $p \to (p \to (p \mapsto q)) \equiv p \to (p \mapsto q)$ [(1), Theorem 2.4]
$\equiv p \mapsto q$, [(g)]
(3) $p \to_2 (p \mapsto q) \equiv p \mapsto q$. [(2), Ab1]

If $n = 2$ the proof is finished. If not, repeating the process, we get to:
(j) $p \to_n (p \mapsto q) \equiv p \mapsto q$,
(j+1) $p \mapsto (p \mapsto q) \equiv p \mapsto q$. [(j), Ab1]

(i)
(1) $p \to (q \to r) \equiv q \to (p \to r)$, [(a)]
(2) $p \to (p \to (q \to r)) \equiv p \to (q \to (p \to r))$ [(1), Theorem 2.4]
$\equiv q \to (p \to (p \to r))$, [(a)]
(3) $p \to_2 (q \to r) \equiv q \to (p \to_2 r)$. [(2), Ab1]

If $n = 2$, from (3) and Ab3 we obtain
(4) $p \mapsto (q \to r) \equiv q \to (p \mapsto r)$.

If $n > 2$, repeating the process we obtain:
(l) $p \to_n (q \to r) \equiv q \to (p \to_n r)$,
(l+1) $p \mapsto (q \to r) \equiv q \to (p \mapsto r)$. [(k), Ab3]

(j)
(1) $q \to (p \mapsto r) \equiv p \mapsto (q \to r)$, [(j)]
(2) $q \to (q \to (p \mapsto r)) \equiv q \to (p \mapsto (q \to r))$ [(1), Theorem 2.4]
$\equiv p \mapsto (q \to (q \mapsto r))$, [(j)]
(3) $q \to (q \mapsto r) \equiv q \mapsto r$, [(g)]
(4) $q \to_2 (p \mapsto r) \equiv p \mapsto (q \mapsto r)$. [(2), (3), Theorem 2.4, Ab1]

If $n = 2$, we obtain:
(5) $q \mapsto (p \mapsto r) \equiv p \mapsto (q \mapsto r)$. [(4), Ab3]

If $n > 2$, repeating the process we obtain:
(l) $q \to_n (p \mapsto r) \equiv p \mapsto (q \mapsto r)$,
(l+1) $q \mapsto (p \mapsto r) \equiv p \mapsto (q \mapsto r)$. [(l), Ab3]

(k)
(1) $p \to q \preceq p \to (p \to q)$, [R1, Definition 2.2]
(2) $p \to q \preceq p \to_2 q$. [(l), Ab1]

If $n = 2$, the proof is over. On the contrary, repeating the process we get:
(i) $p \to q \preceq p \to_n q$,
(i+1) $p \to q \preceq p \mapsto q$. [(i), Ab3]

(l)
(1) $q \preceq (q \to r) \to r$, [(I5), Ab2]
(2) $p \mapsto q \preceq p \mapsto ((q \to r) \to r)$, [(1), (I14), Ab3]
(3) $p \mapsto q \preceq (q \to r) \to (p \mapsto r)$, [(2), (i)]
(4) $(p \mapsto q) \to ((q \to r) \to (p \mapsto r)) \in D$, [(3), Definition 2.2]
(5) $(q \to r) \to ((p \mapsto q) \to (p \mapsto r)) \in D$, [(4), (I9), Definition 2.2]
(6) $q \to r \preceq (p \mapsto q) \to (p \mapsto r)$, [(5), Definition 2.2]
(7) $p \mapsto (q \to r) \preceq p \mapsto ((p \mapsto q) \to (p \mapsto r))$, [(6), (I14), Ab3]
(8) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \to (p \mapsto r))$, [(7), (i)]
(9) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \mapsto r)$, [(8), (g)]

(ll)
(1) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \mapsto r)$, [(i)]
(2) $p \to (p \mapsto (q \to r)) \preceq p \to ((p \mapsto q) \to (p \mapsto r))$, [(1), (I14)]
(3) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \to (p \mapsto r))$, [(2), (I9), (g)]
(4) $p \mapsto (q \to r) \preceq (p \mapsto q) \to (p \mapsto r))$. [(3), (g)]
(m)
(1) $p \mapsto (q \mapsto r) = p \mapsto (q \to (q \to_{n-1} r))$. [Ab1, Ab3]
For $n = 2$, we verify:
(2) $p \mapsto (q \mapsto r) \preceq (p \mapsto q) \to (p \mapsto (q \to_{n-1} r))$, [(1), (ll)]
(3) $p \mapsto (q \to_{n-1} r) = p \mapsto (q \to r)$ [$n = 2$]
$\preceq (p \mapsto q) \to (p \mapsto r)$, [(g)]
(4) $(p \mapsto q) \to (p \mapsto (q \to_{n-1} r)) \preceq (p \mapsto q) \to ((p \mapsto q) \to (p \mapsto r))$, [(3), Theorem 2.4 (ii)]
(5) $(p \mapsto q) \to (p \mapsto (q \to_{n-1} r)) \preceq (p \mapsto q) \mapsto (p \mapsto r)$, [(4), $n = 2$, Ab1, Ab3, (g)]
(6) $p \mapsto (q \mapsto r) \preceq (p \mapsto q) \mapsto (p \mapsto r)$. [(2), (5), (I1)]
For $n = 3$, we have:
(7) $p \mapsto (q \to (q \to_{n-1} r)) \preceq (p \mapsto q) \to (p \mapsto (q \to_{n-1} r))$, [(j)]
(8) $p \mapsto (q \to_{n-1} r) = p \mapsto (q \to (q \to_{n-2} r))$ [Ab1]
$\preceq (p \mapsto q) \to (p \mapsto (q \to_{n-2} r))$, [(j)]
(9) $p \mapsto (q \to_{n-2} r) = p \mapsto (q \to r)$, [$n = 3$]
(10) $p \mapsto (q \to_{n-2} r) \preceq (p \mapsto q) \to (p \mapsto r)$, [(4), (j)]
(11) $(p \mapsto q) \to (p \mapsto (q \to_{n-2} r)) \preceq (p \mapsto q) \to ((p \mapsto q) \to (p \mapsto r))$, [(5), Theorem 2.4(ii)]
(12) $p \mapsto (q \to_{n-1} r) \preceq (p \mapsto q) \to_2 (p \mapsto r)$, [(4), (11), (I1), Ab1]
(13) $(p \mapsto q) \to (p \mapsto (q \to_{n-1} r)) \preceq (p \mapsto q) \to_3 (p \mapsto r)$, [(12), Theorem 2.4(ii), Ab1]
(14) $p \mapsto (q \mapsto r) \preceq (p \mapsto q) \mapsto (p \mapsto r)$. [(1), (7), (13), (I1), Ab3, $n = 3$]
For $n \geq 4$ we proceed in an analogous way. □

An interesting result to remark is the following.

Lemma 4.3. *If $\mathcal{A} = \langle A, \to, D \rangle$ is a prealgebra which verifies* R1 *to* R5*, then the following conditions are equivalent:*
(i) *$\mathcal{A}$ verifies the modus ponens rule,*
(ii) *$\mathcal{A}$ verifies the weak modus ponens rule* (MPD) $\dfrac{p \in D, p \mapsto q \in D}{q}$,

(MP) ⇒ (MPD):
(1) $p \in D$, [hip.]
(2) $p \mapsto q \in D$, [hip.]
(3) $p \to (p \to_{n-1} q) \in D$, [(2), Ab1]
(4) $p \to_{n-1} q \in D$, [(3), (1), (MP)]
if $n = 2$ the proof is done. On the contrary, repeating the process, we get to:
(j) $p \to q \in D$,
(j+1) $q \in D$. [(1), (j), (MP)]
(MPD) ⇒ (MP):
(1) $p \in D$, [hip.]
(2) $p \to q \in D$, [hip.]

(3) $p \mapsto q \in D$, [(2), (I1)]
(4) $q \in D$. [(1), (3), (MPD)]

5. Lindenbaum-Tarski algebras of the Łukasiewicz $I_{n+1}-$prealgebras

Theorem 5.1. *If $\mathcal{A} = \langle A, \to, D\rangle$ is a Łukasiewicz $I_{n+1}-$prealgebra, then the algebra of Lindenbaum-Tarski $\langle A/\equiv, \to, \mathbf{1}\rangle$ of $\mathcal{A}$ is a Łukasiewicz residuation algebra that verifies the additional identity:*
(I6) $(x \to_n y) \vee x = 1$.

Proof. It is consequence of Theorem 3.3, (R5) and Theorem 2.5. □

That is to say, the Lindenbaum-Tarski algebra of the Łukasiewicz $I_{n+1}-$prealgebras are $(n+1)-$valued Łukasiewicz residuation algebras.

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