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Generalized ring-groupoids

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Abstract. In this work, we are going to present the concept of generalized ring-groupoid. Also, we are going to investigate some characterizations about the generalized ring-groupoids. We are going to introduce the concept of generalized subring-groupoid. So we construct the category of generalized ring-groupoids. Furthermore, we are going to discuss a new class of the generalized ring-groupoids, which we will say it "M-ring-groupoid". In the end of the paper, we are going to give the product of generalized ring-groupoids.

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1. Introduction

The concept of generalized ring was first defined by Molaei [13] in 2003. Later, some algebraic properties of the generalized ring which is a new concept in literature have been studied in [7]. There is the concept of generalized group in the structure of generalized ring. The concept was again defined by Molaei [12] is an interesting generalization of groups. While there is only one identity element in a group, each element in a generalized group has a unique identity element. With this property, every group is a generalized group.

Another algebraic notion covered in the present study is groupoid which was defined by Brandt [1] in 1926. But, in the category theoretical approach, a groupoid is a small category whose every morphism is an isomorphism. After introducing of topological and differentiable groupoids by Ehresmann [4] in 1950s, it has been studied by many mathematicians with different approaches [3, 9]. One of these different approaches is structured groupoid which is obtained with adding another algebraic structure such that the composition of groupoid is compatible with the operation of the added algebraic structure [2, 5, 10, 14]. The best knowns of the structured groupoids are the concepts of group-groupoid and ring-groupoid. The group-groupoid which is a group object in the category of groupoids was defined by Brown and Spencer [2]. The concept of ring-groupoid defined by [15] has been studied by many mathematicians [10, 11].

In this study, we extend the concept of ring-groupoid to the concept of generalized ring-groupoid by adding the structure of generalized ring to a groupoid such that the composition of the groupoid and the operations of the generalized ring are compatible. In other words, a generalized ring-groupoid is a generalized ring object in the category of groupoids. Thus, we construct the category of the generalized ring-groupoids. Also,

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we present two concept related to the generalized ring-groupoids: generalized subring-groupoid and M-ring-groupoid.

2. Preliminaries

This section of the paper is devoted to give basic definitions and concepts related to the generalized rings and groupoids. We will consider these concepts under two headings: generalized rings and groupoids.

2.1. Generalized Rings. In this subsection, it is given some basic recalls of the concept of generalized ring which was first defined by Molaei. Let us start with the definition of a generalized group that the existing in the structure of a generalized ring.

Definition 2.1. [12] A generalized group G is a non-empty set admitting an operation called multiplication subject to the set of rules given below:
i) $(ab)c = a(bc)$, for all $a, b, c \in G$
ii) For each $a \in G$, there exists a unique $e(a) \in G$ such that $ae(a) = e(a)a = a$
iii) For each $a \in G$, there exists $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e(a)$.

Let us list some properties of generalized groups via following lemma.

Lemma 2.1. [12] *Let G be a generalized group. Then,*
i) For each $a \in G$, there is a unique element $a^{-1} \in G$.
ii) For each $a \in G$, we have $e(a) = e(a^{-1})$ and $e(e(a)) = e(a)$.
iii) For each $a \in G$, we have $(a^{-1})^{-1} = a$.

It is easily from Definition 2.1 that every group is a generalized group. But it is not true in general that every generalized group is a group.

Let us state the relation between group and generalized group by the following lemma.

Lemma 2.2. [12] *Let G be a generalized group and $ab = ba$ for all $a, b \in G$. Then, G is a group.*

In other words, every abelian generalized group is a group.

Example 2.1. [12] Let $G = IR \times (IR \setminus \{0\})$. Then G with the multiplication $(a, b) \cdot (c, d) = (bc, bd)$ is a generalized group in which for all $(a, b) \in G$, $e(a, b) = (a/b, 1)$ and $(a, b)^{-1} = (a/b^2, 1/b)$.

Example 2.2. [5] Let G with the multiplication m be a generalized group. Then, $G \times G$ with the multiplication

$$m_1((a, b), (c, d)) = (m(a, c), m(b, d))$$

is a generalized group. For any element $(a, b) \in G \times G$, the identity element is $e_1(a, b) = (e(a), e(b))$ and the inverse element is $(a, b)^{-1} = (a^{-1}, b^{-1})$.

Definition 2.2. [12] If $e(ab) = e(a)e(b)$ for all $a, b \in G$, then G is called normal generalized group.

Definition 2.3. [12] A non-empty subset H of a generalized group G is a generalized subgroup of G if and only if for all $a, b \in H$, $ab^{-1} \in H$.

Definition 2.4. [12] A generalized subgroup N of the generalized group G is said to be normal if there exist a generalized group H and a homomorphism $f : G \to H$ such that for each $a \in G$, $N_a = kerf_a$ provided that $N_a \neq \emptyset$, where $N_a = N \cap G_a$.

Example 2.3. [12] Let G be a generalized group of Example 2.1. Then $N = \{(a, b) : a = b \text{ or } a = 3b\}$ is a generalized normal subgroup of G.

Definition 2.5. [12] Let G and H be two generalized groups. A generalized group homomorphism from G to H is a map $f : G \to H$ such that $f(ab) = f(a)f(b)$ for all $a, b \in G$.

Theorem 2.3. [12] *Let $f : G \to H$ be a homomorphism of the distinct generalized groups G and H. Then,*
i) $f(e(a)) = e(f(a))$ is an identity element in H for all $a \in G$.
ii) $f(a^{-1}) = (f(a))^{-1}$
iii) If K is a generalized subgroup of G, then $f(K)$ is a generalized subgroup of H.

Now we can give definition of a generalized ring.

Definition 2.6. [13] A generalized ring R is a non-empty set R with two different operations $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ with the following axioms:
i) $(x + y) + z = x + (y + z)$, where $x, y, z \in R$
ii) For all $x \in R$, there exists a unique $e(x) \in R$ such that $x + e(x) = e(x) + x = x$
iii) For all $x \in R$, there exists $-x \in R$ such that $x + (-x) = (-x) + x = e(x)$.
iv) $(xy)z = x(yz)$, where $x, y, z \in R$
v) For all $x, y, z \in R$, $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$.

The properties (i), (ii) and (iii) mean that $(R, +)$ is a generalized group.

Remark 2.1. Using (iii) and the associavity of $+$, one easily verifies $e(x)+e(x) = e(x)$ for every $x \in R$. Hence $e(e(x)) = e(x)$ follows by definitions and so $e^2 = e$ for the corresponding function $e : R \to R$.

A generalized ring with its operations is a ring iff e is a constant function.

Example 2.4. [7] The two dimensional Euclidean space IR^2 with the operations $(a_1, b_1) + (a_2, b_2) = (a_1, b_2)$ and $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ is a generalized ring.

A generalized ring R is called an M-ring if $e(xy) = e(x)e(y)$ and $e(x + y) = e(x) + e(y)$, for all $x, y \in R$.

R is an M-ring if $e(x + y) = e(x) + e(y)$, for all $x, y \in R$. In other words, the identity function e is a generalized ring homomorphism if $e(x + y) = e(x) + e(y)$, for all $x, y \in R$.

If there is $1 \in R$ such that $x.1 = 1.x = x$, for all $x \in R$, then R is called a generalized ring with an identity.

One can easily prove that the identity of a generalized ring is unique.

Theorem 2.4. [7] *If R is a generalized ring, then $e(ab) = e(a)e(b)$, for all $a, b \in R$.*

Proof. Let $a, b \in R$ be given $ab+ae(b) = a(b+e(b)) = ab$, $ae(b)+ab = a(e(b)+b) = ab$. So $e(ab) = ae(b)$, $e(a)e(b) + ae(b) = (e(a) + a)e(b) = ae(b)$, $ae(b) + e(a)e(b) = (a + e(a))e(b) = ae(b)$. So $e(ae(b)) = e(a)e(b)$. Hence $e(e(ab)) = e(a)e(b)$. Thus $e(ab) = e(a)e(b)$, because $e^2 = e$. □

Corollary 2.5. *If R is a generalized ring, then $e(a)e(b) = ae(b) = e(a)b = e(ab)$, for all $a, b \in R$.*

Previous theorem implies that a generalized ring R is an M-ring if and only if $(R, +)$ is a normal generalized group.

Theorem 2.6. [7] *If R is a generalized ring, and if there is $x \in R$ such that $Rx = \{e(y) \mid y \in R\}$, then R is an M-ring.*

Proof. If $a, b \in R$, then there are $a_x \in R$ and $b_x \in R$ such that $e(a) = a_x x$ and $e(b) = b_x x$. So $e(a) + e(b) = (a_x + b_x)x$. Thus $e(a) + e(b) = e(z)$ for some $z \in R$. Hence $e(e(a)+e(b)) = e(e(z)) = e(z) = e(a)+e(b)$. In Remark 2.3 of [6] it was proved that $e(e(a) + e(b)) = e(a + b)$. So $e(a + b) = e(a) + e(b)$. Thus R is an M-ring. □

A subset I of an M-ring R is called a g-ideal (see [13]) if there exist a generalized ring D and a generalized ring homomorphism $f : R \to D$ such that $kerf = I$, where $kerf = \{r \in R \mid f(r) = f(e(a))$ for some $a \in R\}$. The set $R/I = \{x + kerf_r \mid x \in R_r$ and $f_r = f|_{R_r}\}$ with the operations $(x + kerf_r) + (y + kerf_k) = (x + y) + kerf_{r+k}$ and $(x + kerf_r)(y + kerf_k) = (xy) + kerf_{rk}$ is an M-ring (for the proof see Theorem 2.3 of [13]).

Definition 2.7. [7] If R and K are generalized rings, then a mapping $f : R \to K$ is called an embedding if f is a monomorphism.

2.2. Groupoids. In this section, we introduce the elementary concepts of the groupoid theory. Then, it is given some recalls about the concept of ring-groupoid which is a ring object in the category of groupoids.

Definition 2.8. [3, 9] A groupoid consists of two sets G and G_0, called respectively the groupoid and the base, together with two maps α and β from G to G_0, called respectively the source and the target maps, a map $\epsilon : G_0 \to G$, $x \mapsto \epsilon(x) = \widetilde{x} = 1_x$, called the object inclusion map, a map $i : G \to G$, $x \mapsto i(x) = x^{-1}$, called the inversion, and a partial multiplication $(x, y) \mapsto m(x, y) = xy$ in G defined on the set $G_2 = G * G = \{(x, y) \mid \beta(x) = \alpha(y)\}$. These maps verify the following conditions:

G1) (associativity): $x(yz) = (xy)z$ for all $x, y, z \in G$ such that $\alpha(x) = \beta(y)$ and $\alpha(y) = \beta(x)$.
G2) (units): For each $x \in G$, we have $(\epsilon(\alpha(x)), x) \in G_2$, $(x, \epsilon(\beta(x))) \in G_2$and $\epsilon(\alpha(x))x = x\epsilon(\beta(x)) = x$.
G3) (inverses): For each $x \in G$, we have $(x, i(x)) \in G_2$, $(i(x), x) \in G_2$and $xi(x) = \epsilon(\alpha(x))$, $i(x)x = \epsilon(\beta(x))$.

The maps $\alpha, \beta, m, \epsilon, i$ are called structure maps of groupoid. For a groupoid G on G_0 and $x, y \in G_0$, we will write $St_G x$ for $\alpha^{-1}(x)$, $CoSt_G y$ for $\beta^{-1}(y)$ and $G(x, y)$ for $St_G x \cap CoSt_G y$. The set $St_G x$ is the star of G at x and $CoSt_G y$ is the co-star of G at y. The set $G(x, x)$, obviously a group under the restriction of the partial multiplication in G, is called the vertex group at x.

The following examples of groupoids are well-known.

Example 2.5. [3, 9] A group can be regarded as a groupoid with only one object.

Example 2.6. [3, 9] Any set G can be regarded as a groupoid on itself with $\alpha = \beta = id_G$ and every element a unity.

Example 2.7. [3] For a set X, the cartesian product $X \times X$ is a groupoid over X, called the Banal groupoid. The maps α and β are the natural projections onto the second and first factors, respectively. The object inclusion map is $x \mapsto (x, x)$ and the partial multiplication is given by $(x, y)(y, z) = (x, z)$. The inverse of (x, y) is simply (y, x).

Definition 2.9. [3, 9] Let G and $G^{'}$ be groupoids on B and $B^{'}$, respectively. A homomorphism $G \to G^{'}$ is a pair of (f, f_0) of maps $f : G \to G^{'}$, $f_0 : B \to B^{'}$ such that $\alpha^{'} \circ f = f_0 \circ \alpha$, $\beta^{'} \circ f = f_0 \circ \beta$ and $f(ab) = f(a)f(b)\ \forall (a, b) \in G_2$.

We denote the groupoid homomorphism (f, f_0) by f for brevity.

Thus, we can construct the category *Gpd* of the groupoids and their homomorphisms.

Now let us recall the concept of ring-groupoid which is a ring object in the category of groupoids.

Definition 2.10. [15] A ring-groupoid R is a groupoid endowed with a structure of ring such that following ring structure maps are groupoid homomorphisms.

i) $m : R \times R \to R$, $(a, b) \mapsto a + b$, group operation

ii) $n : R \times R \to R$, $(a, b) \mapsto ab$, ring operation

iii) $u : R \to R$, $a \mapsto -a$, inverse in group

iv) $e : * \to R$.

Also, there exist following interchange laws in a ring-groupoid R.

(1) $(c \circ a) + (d \circ b) = (c + d) \circ (a + b)$,

(2) $(c \circ a)(d \circ b) = (cd) \circ (ab)$.

A ring groupoid homomorphism is a groupoid homomorphism preserving ring structure.

Example 2.8. Given a ring R, we can construct a ring-groupoid $R \times R$ over R. In this ring-groupoid we define the ring operation by $(a, b)(c, d) = (ac, bd)$ for all $a, b, c, d \in R$ (for more details, see [15]).

Definition 2.11. [15] Let R and S be two ring-groupoids. A homomorphism $f : R \to S$ of ring-groupoids is a homomorphism of underlying groupoids preserving ring structure.

Thus, the ring-groupoids and their homomorphisms form a category which is denoted by *RGd*.

3. Generalized Ring-Groupoids

In this section we present the concept of generalized ring-groupoid which is a generalized ring object in the category of groupoids. In addition, we construct the category of generalized ring-groupoids. From [8] with this aim, let us recall the concept of generalized group-groupoid which is lie in the structure of a generalized ring-groupoid.

Definition 3.1. A generalized group-groupoid is a groupoid (G, G_0) such that the following conditions are hold:

i) (G, w, v, σ) and $(G_0, w_0, v_0, \sigma_0)$ are generalized groups.

ii) The maps $(w, w_0) : (G \times G, G_0 \times G_0) \to (G, G_0)$, $v : \{\lambda\} \to G$ and $(\sigma, \sigma_0) : (G, G_0) \to (G, G_0)$ are groupoid homomorphisms.

Also, there exists an interchange law between the groupoid composition and the generalized group operation:

$$w\left(m\left(b,a\right), m\left(d,c\right)\right) = m\left(w\left(b,d\right), w\left(a,c\right)\right).$$

We shall denote a generalized group-groupoid by $(G, G_0, \circ, +)$.

We use the following equality for interchange law:

$$(b \circ a) + (d \circ c) = (b + d) \circ (a + c).$$

In other words, a generalized group-groupoid is a groupoid endowed with a structure of generalized group such that the structure maps of groupoid are generalized group homomorphisms.

Example 3.1. [8] Let G be a generalized group. Then we constitute a generalized group-groupoid $G \times G$ with object set G. For each object $(x, y) \in G \times G$, the identity arrow is $(e(x), e(y))$, and the inverse is $(-x, -y)$.

A generalized group homomorphism $f : G \to H$ between the generalized group-groupoids G and H is a groupoid homomorphism preserving the structure of generalized group [8].

Therefore, the generalized group-groupoids and their homomorphisms form a category denoted by $GG - Gd$.

Now let us give definition of a generalized ring-groupoid.

Definition 3.2. A generalized ring-groupoid R is a groupoid R endowed with a structure of generalized ring such that the following maps are groupoid homomorphisms:
1) $m : R \times R \to R$, $(a, b) \mapsto a + b$, generalized group operation,
2) $u : R \to R$, $a \mapsto -a$,
3) $e : * \to R$, where $*$ is a singleton,
4) $n : R \times R \to R$, $(a, b) \mapsto ab$, generalized ring operation.

Also, there exist two interchange laws between the groupoid composition and the operations of the generalized ring:

$$(c \circ a) + (d \circ b) = (c + d) \circ (a + b)$$
$$(c \circ a) \cdot (d \circ b) = (c \cdot d) \circ (a \cdot b).$$

We shall denote a generalized ring-groupoid by $(R, R_0, \circ, +, \cdot)$.

In a generalized ring-groupoid, if e is the identity of R_0, then 1_e is that of R.

We can rewrite the definition of a generalized ring-groupoid in terms of the generalized group-groupoid as follows:

Definition 3.3. A generalized ring-groupoid R is a generalized group-groupoid R endowed with a structure of generalized ring such that the map $n : R \times R \to R$, defined by $(a, b) \mapsto ab$, is a homomorphism of groupoids. Also, in a generalized ring-groupoid, we have the following interchange law:

$$(c \circ a)(d \circ b) = (cd) \circ (ab).$$

Proposition 3.1. *Let R be a generalized ring-groupoid. Then, the maps of source, target and object are generalized ring homomorphisms.*

Proof. Since $R = ((R, R_0, \circ, +)$ is a generalized group-groupoid, the maps of source, target and object are generalized group homomorphisms. Let $a, b \in R$ and $x, y \in R_0$. Since n is a groupoid homomorphism, the equalities $\alpha n(a,b) = f_0(\alpha \times \alpha)(a,b)$, $\beta n(a,b) = f_0(\beta \times \beta)(a,b)$ and $\epsilon f_0(x,y) = n(\epsilon \times \epsilon)(x,y)$ imply to be $\alpha(a,b) = \alpha(a)\alpha(b)$, $\beta(a,b) = \beta(a)\beta(b)$ and $\epsilon(x,y) = \epsilon(x)\epsilon(y)$, respectively.

Thus, the maps of source, target and object are generalized ring homomorphisms. □

Example 3.2. Let R be a generalized ring. Then $R \times R$ is a generalized ring-groupoid with the object set R. We know from [8] that $R \times R$ with the operation $(x,y)+(z,t) = (x+z, y+t)$ is a generalized group-groupoid over R. So it is enough to show that $R \times R$ is a generalized ring, and then the generalized ring map $n : (R \times R) \times (R \times R) \to R \times R$ is a groupoid homomorphism. We also must verify the second interchange law.

If we show that the conditions (iv) and (v) in Definition 3.2 are hold, we conclude that $R \times R$ is a generalized ring. Now let us control these conditions.

We define the generalized ring operation of $R \times R$ as follows:

$$(x,y)(z,t) = (xz, yt).$$

iv) We have

$$\begin{aligned} ((x,y)(z,t))(p,s) &= (xz,yt)(p,s) \\ &= ((xz)p,(yt)s) \\ &= (x(zp),y(ts)) \\ &= (x,y)(zp,ts) \\ &= (x,y)((z,t)(p,s)). \end{aligned}$$

So, fourth condition is hold.

v)

$$\begin{aligned} (x,y)[(z,t)+(p,s)] &= (x,y)(z+p,t+s) \\ &= (x(z+p),y(t+s)) \\ &= (xz+xp,yt+ys) \\ &= (xz,yt)+(xp,ys) \\ &= (x,y)(z,t)+(x,y)(p,s)) \end{aligned}$$

and

$$\begin{aligned} [(x,y)+(z,t)](p,s) &= (x+z,y+t)(p,s) \\ &= ((x+z)p),(y+t)s) \\ &= (xp+zp,ys+ts) \\ &= (xp,ys)+(zp,ts) \\ &= (x,y)(p,s)+(z,t)(p,s)). \end{aligned}$$

Hence, the condition (v) also is hold. Therefore, $R \times R$ is a generalized ring.

Now let us show that the second interchange law is satisfied.

$$\begin{aligned} [(z,y)\circ(y,x)][(z^{'},y^{'})\circ(y^{'},x^{'})] &= (z,x)(z^{'},x^{'}) \\ &= (zz^{'},xx^{'}) \end{aligned}$$

and

$$\begin{aligned}[(z,y)(z',y')] \circ [(y,x)(y',x')] &= (zz',yy') \circ (yy',xx') \\ &= (zz',xx').\end{aligned}$$

Hence, we have the equality

$$[(z,y) \circ (y,x)][(z',y') \circ (y',x')] = [(z,y)(z',y')] \circ [(y,x)(y',x')].$$

Consequently, $R \times R$ is a generalized ring-groupoid.

Definition 3.4. Let R and S be two generalized ring-groupoids. A generalized ring-groupoid homomorphism $f : R \to S$ is a groupoid homomorphism satisfying the generalized ring structure.

Therefore, the generalized ring-groupoids and their homomorphisms form a category denoted by $GR - Gd$.

Proposition 3.2. *There is a functor from the category GR of the generalized rings to the category $GR - Gd$ of the generalized ring-groupoids.*

Proof. Let R be a generalized ring. Then, from Example 3.2, the cartesian product $R \times R$ is a generalized ring-groupoid. If $f : R \to S$ is a homomorphism of the generalized rings, then

$$\begin{gathered}\Gamma(f) : R \times R \longrightarrow S \times S \\ (a,b) \longmapsto (f(a), f(b))\end{gathered}$$

is a homomorphism of the generalized ring-groupoids. Thus, Γ is a functor from the category GR to the category $GR - Gd$. □

Now let us define the concept of generalized subring-groupoid.

Definition 3.5. Let R be a generalized ring-groupoid and be $S \subset R$. S is called a generalized subring-groupoid if $(S, S_0, \circ, +, .)$ has a structure of generalized ring-groupoid.

Furthermore, S is wide, if $S_0 = R_0$, and S is full, if $S(x,y) = R(x,y)$ for all $x, y \in S_0$.

Proposition 3.3. *Let R be a generalized ring-groupoid. Then, the set of identities $\epsilon(R_0)$ is a wide generalized subring-groupoid.*

Proof. Denote by A the set of identities $\epsilon(R_0)$ for brevity. If $1_x, 1_y \in A$, then $1_x + \overline{1_y} \in A$. Hence (A, A_0) is a wide subgroupoid of R. It remains to prove that A is closed under the generalized ring operation.

Since the object map ϵ preserves the generalized ring structure, we have

$$1_x 1_y = (1_x \circ 1_x)(1_y \circ 1_y) = (1_x 1_y) \circ (1_x 1_y) = 1_{xy} \circ 1_{xy} = 1_{xy}.$$

This implies that $1_x 1_y \in A$.

On the other hand, for $1_z \in A$

$$1_x(1_y + 1_z) = 1_x 1_{y+z} = 1_{x(y+z)} = 1_{xy+xz} = 1_{xy} + 1_{xz} = 1_x 1_y + 1_x 1_z.$$

Therefore, $A = \epsilon(R_0)$ is a wide generalized subring-groupoid. □

We define a special class of the generalized ring-groupoids.

Definition 3.6. A generalized ring-groupoid R is called an M-ring-groupoid if R has a structure of M-ring.

It is obvious that the category of M-ring-groupoids is a subcategory of the category of generalized ring-groupoids. Also, every M-ring-groupoid is a generalized ring-groupoid.

Since the set of arrows and the set of objects in an M-ring-groupoid are M-rings, then we can define the concept of a g-ideal ring-groupoid as follows:

Definition 3.7. A generalized subgroup-groupoid S of an M-ring-groupoid R is a left g-ideal ring-groupoid if

$$\begin{aligned} l \quad : \quad & R \times S \to S \\ & (r,s) \mapsto rs, \forall r \in R, \forall s \in S \end{aligned}$$

is a groupoid homomorphism. Similarly, S is a right g-ideal ring-groupoid if

$$\begin{aligned} k \quad : \quad & S \times R \to S \\ & (s,r) \mapsto sr, \forall r \in R, \forall s \in S \end{aligned}$$

is a groupoid homomorphism. Furthermore, S is a g-ideal ring-groupoid if it is both left and right g-ideal ring-groupoid.

From Definition 3.7, the sets of arrows and objects of S are left g-ideal rings, because l is a groupoid homomorphism. Also, every left (right) g-ideal ring-groupoid is a generalized subring-groupoid.

Proposition 3.4. *Let S be a generalized subgroup-groupoid of an M-ring groupoid R. If the set of arrows of S is a left g-ideal ring, then S_0 is also a left g-ideal of R_0.*

Proof. Let $x \in S_0$ and $y \in R_0$. Then, we have $1_x \in S$ and $1_y \in R$. Since the set of arrows of S is a left g-ideal ring, then we have $1_y 1_x = 1_{yx} \in S$. Since S is a generalized subgroup-groupoid, then we have $yx \in S_0$. Thus, S_0 is a left g-ideal of R_0. □

The interchange law in a g-ideal ring-groupoid is hold as follows: Let R be an M-ring-groupoid and I be a left g-ideal ring-groupoid such that $a, c \in I$. For $b, d \in R$, if $a \circ c$ and $b \circ d$ are defined, then we have $(b \circ d)(a \circ c) = (ba) \circ (dc)$. Since the set of arrows of I is a left g-ideal, then $ba, dc \in I$. Also, since I is a generalized subgroup-groupoid, which means that ba and dc are defined in I, then we have $(ba) \circ (dc) \in I$.

A similar result to Proposition 3.4 can also be given for a right g-ideal ring-groupoid.

Finally, let us present the product of generalized ring-groupoids.

Proposition 3.5. *Let $\{R_i : i \in I\}$ be a family of generalized ring-groupoids. Then, $(R = \prod R_i, R_0 = \prod (R_i)_0, \circ, +, .)$ is a generalized ring-groupoid.*

Proof. The arrows of R are all tuples $(r_i)_{i \in I}$ for each $r_i \in R_i$ and its objects are all tuples $(x_i)_{i \in I}$ for each $x_i \in (R_i)_0$. It is easily proved that $(R, R_0, \circ, +)$ is a generalized group-groupoid. We define the generalized ring operation on R as follows:

$$\begin{aligned} (r_i)_{i \in I}(s_i)_{i \in I} &= (r_i s_i)_{i \in I}, for\ each\ (r_i, s_i) \in R_i \times R_i \\ (x_i)_{i \in I}(y_i)_{i \in I} &= (x_i y_i)_{i \in I}, for\ each\ (x_i, y_i) \in (R_i)_0 \times (R_i)_0 \end{aligned}$$

For the source map α, since

$$\begin{aligned}\alpha((r_i)_{i\in I}+(s_i)_{i\in I}) &= \alpha((r_i+s_i)_{i\in I})\\ &= (\alpha_i(r_i+s_i))_{i\in I}\\ &= (\alpha_i(r_i))_{i\in I}+(\alpha_i(s_i))_{i\in I}\\ &= \alpha((r_i)_{i\in I})+\alpha((s_i)_{i\in I})\end{aligned}$$

and

$$\begin{aligned}\alpha((r_i)_{i\in I}(s_i)_{i\in I}) &= \alpha((r_is_i)_{i\in I})\\ &= (\alpha_i(r_is_i)_{i\in I}\\ &= (\alpha_i(r_i))_{i\in I}(\alpha_i(s_i))_{i\in I}\\ &= \alpha((r_i)_{i\in I})\alpha((s_i)_{i\in I}),\end{aligned}$$

then α is a generalized ring homomorphism. Similarly, it can be easily shown that β and ϵ are also generalized ring homomorphisms.

Let us show that the interchange law is hold. Let us take any elements $(r)_{i\in I}$, $(s)_{i\in I}$, $(t)_{i\in I}$ and $(v)_{i\in I}\in R$ such that $\alpha((r)_{i\in I})=\beta((s)_{i\in I})$ and $\alpha((t)_{i\in I})=\beta((v)_{i\in I})$. Then,

$$\begin{aligned}[(r_i)_{i\in I}\circ(s_i)_{i\in I}][(t_i)_{i\in I}\circ(v_i)_{i\in I}] &= (r_i\circ s_i)_{i\in I}(t_i\circ v_i)_{i\in I}\\ &= ((r_i\circ s_i)(t_i\circ v_i))_{i\in I}\\ &= ((r_it_i)\circ(s_iv_i))_{i\in I}\\ &= (r_it_i)_{i\in I}\circ(s_iv_i)_{i\in I}\\ &= (r_i)_{i\in I}(t_i)_{i\in I}\circ(s_i)_{i\in I}(v_i)_{i\in I}.\end{aligned}$$

Thus, the interchange law between the groupoid composition and the generalized ring operation is satisfied. Moreover, we have

$$\begin{aligned}(r_i)_{i\in I}[(s_i)_{i\in I}+(t_i)_{i\in I}] &= (r_i)_{i\in I}[(s_i+t_i)_{i\in I}\\ &= (r_i(s_i+t_i)_{i\in I}\\ &= (r_is_i+r_it_i)_{i\in I}\\ &= (r_is_i)_{i\in I}+(r_it_i)_{i\in I}.\end{aligned}$$

Consequently, R is a generalized ring-groupoid.

□

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