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Generalized co-annihilators in residuated lattices

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Abstract. The aim of this paper is to extend the notion of the generalized co-annihilator of residuated lattices. We introduce the concept of a generalized co-annihilator of a given subset of a residuated lattice $\mathfrak{A}$. We prove that generalized co-annihilators relative to a filter F of $\mathfrak{A}$ are again filters, and moreover pseudocomplements in the lattice $\mathbf{Fi}(\mathfrak{A})[F, A]$ of all filters of $\mathfrak{A}$ containing F. Also, for a given filter F of $\mathfrak{A}$ we prove that the set $Co - An^F(\mathfrak{A})$ of all generalized co-annihilators relative to F forms a complete Boolean lattice.

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1. Introduction

It is well known that certain information processing, especially inferences based on certain information, is based on the classical logic. Naturally, it is necessary to establish some rational logic systems as the logical foundation for uncertain information processing. For this reason, various kinds of non-classical logic systems have been extensively proposed and researched. In fact, non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. On the other hand, various logical algebras have been proposed as the semantical systems of non-classical logic systems, for example, residuated lattices, divisible residuated lattices, MTL-algebras, Girard monoids, BL-algebras, Gödel algebras, lattice implication algebras, etc. Among these logical algebras, residuated lattices are very basic and important algebraic structures because the other logical algebras are all particular cases of residuated lattices.

Commutative residuated lattices are the algebraic counterpart of logics without contraction rule. The concept of a commutative residuated lattice firstly introduced by by W. Krull in [15] who discussed decomposition into isolated component ideals. After him, they were investigated by M. Ward and R.P. Dilworth in [26], as the main tool in the abstract study of ideal lattices in ring theory. The properties of a residuated lattice were presented in [10, 19, 20, 21, 22, 23]. For a survey of residuated lattices we refer to [14].

Non-commutative residuated lattices, sometimes called pseudo-residuated lattices, biresiduated lattices or generalized residuated lattices, are the algebraic counterparts of substructural logics; i.e. logics which lack at least one of the three structural rules, namely contraction, weakening and exchange. Complete studies on non-commutative residuated lattices were developed in [2] and [14].

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In ring theory the annihilator of a set is a concept generalizing torsion and orthogonality. Also, Baer rings and Rickart rings are various attempts to give an algebraic analogue of von Neumann algebras, using axioms about annihilators of various sets. A. Filipoiu in [8] used the notion of annihilator for Baer extensions of MV-algebras. In [1, 13, 16] the notion of annihiltors studied for BCK-algebras. L. Leuştean [17] used the notion of co-annihilator for Baer extensions of BL-algebras. The aim of this paper is to extend results proved by L. Leuştean [17], for the case of Bl-algebras to the non-commutative residuated lattices.

In Section 2 of the article we recall some definitions and facts about residuated lattices that we use in the sequel.

Let $\mathfrak{A}$ be a residuated lattice $\mathfrak{A}$ and a F be a filter of $\mathfrak{A}$. For any subset X of A we define the generalized co-annihilator $(F : X)$ and we show that $(F : X)$ is a filter of $\mathfrak{A}$ containing F. These and some other properties of generalized co-annihilators are contained in Section 3 of this paper.

2. A brief excursion into residuated lattices

Various logical algebra shave been proposed as the semantical systems of non-classical logic systems, for example, residuated lattices, MV-algebras, BL-algebras, Gödel algebras, lattice implicational algebras, MTL-algebras, NM-algebras and R_0-algebras, etc. Among these logical algebras, residuated lattices are very basic and important algebraic structures because the other logical algebras are all particular cases of residuated lattices.

In the following, we recall some basic definitions and properties of residuated lattices and give some examples in this concept.

Definition 2.1. [14] A *residuated lattice* is an algebra $\mathfrak{A} = (A; \vee, \wedge, \odot, \rightarrow_l, \rightarrow_r, 1)$ of type $(2, 2, 2, 2, 2, 0)$ satisfying the following conditions:

RL_1 $(A; \vee, \wedge, 0, 1)$ is a bounded lattice;

RL_2 $(A, \odot, 1)$ is a monoid;

RL_3 $x \odot y \leq z$ iff $x \leq y \rightarrow_l z$ iff $y \leq x \rightarrow_r z$ for $x, y, z \in A$.

The operations $\rightarrow_l$ and $\rightarrow_r$ are referred to as the left and right residual of $\odot$, respectively. Note that, in general, 1 is not the top element of the lattice reduct $\mathfrak{A}$, $\ell(\mathfrak{A})$. A residuated lattice with a constant 0 (which can denote any element) is called a pointed residuated lattice or a full Lambek algebra (FL-algebra). If 1 is a top a element of $\ell(\mathfrak{A})$, then $\mathfrak{A}$ is called an integral residuated lattice. A FL-algebra $\mathfrak{A}$ in which $(A, \vee, \wedge, 0, 1)$ is a bounded lattice is called a FL_w-algebra. A FL_w-algebra is sometimes called a bounded integral residuated lattice. A residuated lattice $\mathfrak{A}$ is called commutative if $\rightarrow_l = \rightarrow_r$. It is obvious that $\mathfrak{A}$ is a commutative residuated lattice if and only if $\odot$ is a commutative binary operation. A residuated lattice $\mathfrak{A}$ in which $x \odot y = x \wedge y$ for all $x, y \in A$ is called a Heyting algebra or pseudo-Boolean algebra [25]. Clearly, a Heyting algebra is a commutative residuated lattice.

In this paper, a residuated lattice will be a FL_w-algebra. A residuated lattice $\mathfrak{A}$ is nontrivial if and only if $0 \neq 1$. In a residuated lattice $\mathfrak{A}$, for any $a \in A$, we put $\neg_l a := a \rightarrow_l 0$ and $\neg_r a := a \rightarrow_r 0$. Also, if X is a non-empty subset of A, we put $\neg_l X = \{\neg_l x | x \in X\}$ and $\neg_r X = \{\neg_r x | x \in X\}$. Also, $\neg_l \neg_l X$, $\neg_l \neg_r X$, $\neg_r \neg_l X$ and $\neg_r \neg_r X$ are denoted by $\neg_{ll} X$, $\neg_{lr} X$, $\neg_{rl} X$ and $\neg_{rr} X$, respectively.

The following proposition provides some rules of calculus in a residuated lattice (see [2, 3, 6]).

Proposition 2.1. *Let $\mathfrak{A}$ be a residuated lattice. Then the following rules of calculus hold for any $a, b, c, d \in A$.*
(1) $a \to_l (b \to_l c) = (a \odot b) \to_l c$;
(2) $a \to_r (b \to_r c) = (b \odot a) \to_r c$;
(3) $a \leq b$ *if and only if* $a \to_l b = 1$ *if and only if* $a \to_r b = 1$.
(4) $a \odot b \leq a \wedge b$. *In particular,* $a \odot b = 1$ *implies* $a = b = 1$.
(5) $a \leq b$ *implies* $a \odot c \leq b \odot c$ *and* $c \odot a \leq c \odot b$.
(6) $a \to_l b \leq \neg_l b \to_r \neg_l a$, $a \to_r b \leq \neg_r b \to_l \neg_r a$.
(7) $\neg_{lrl} a = \neg_l a$, $\neg_{rlr} a = \neg_r a$.
(8) $(b \to_l c) \odot (a \to_l b) \leq a \to_l c$, $(a \to_r b) \odot (b \to_r c) \leq a \to_r c$.
(9) $a \to_l \neg_r b = b \to_r \neg_l a$.
(10) $1 \to_l a = 1 \to_r a = a$.
(11) $b \leq a \to_l b$ *and* $a \leq b \to_r b$.
(12) *If* $a \leq b$ *then* $c \to_l a \leq c \to_l b$ *and* $c \to_r a \leq c \to_r b$.
(13) *If* $a \leq b$ *then* $b \to_l c \leq a \to_l c$ *and* $b \to_r c \leq a \to_r c$.
(14) $(a \vee c) \odot (b \vee c) \leq (a \odot b) \vee c$.
(15) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$.

In the following, we give some examples of residuated lattice.

Example 2.1. Assume that R is a ring with unit and let $I(R)$ be the collection of all ideals of R. This set, ordered by inclusion, is a lattice. The meet of two ideals is their intersection and their join is the ideal generated by the union. We define multiplication of two ideals I, J in the usual way

$$I \odot J := \{ \sum_{x \in X, y \in Y} xy : \ X, Y \text{ are finite subsets of } I, J, \text{ respectivly.} \}.$$

Also, we put $I \to_l J := \{k : Ik \subseteq J\}$ and $I \to_r J := \{k : kI \subseteq J\}$. Then $(I(R); \vee, \wedge, \odot, \to_l, \to_r, 0, R)$ forms a residuated lattice.

Example 2.2. [4] Let $A_5 = \{0, a, b, c, 1\}$ be a lattice whose Hasse diagram is below (see Figure 1). Define $\odot$, $\to_l$ and $\to_r$ on A_5 as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	a	b
c	0	0	0	c	c
1	0	a	b	c	1

$\to_l$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\to_r$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

Routine calculation shows that $\mathfrak{A}_5 = (A_5; \vee, \wedge, \odot, \to_l, \to_r, 0, 1)$ is a residuated lattice.

Example 2.3. [18] Let $A_7 = \{0, a, b, c, d, e, 1\}$ be a lattice whose Hasse diagram is below (see Figure 2). Define $\odot$, $\to_l$ and $\to_r$ on A_7 as follows:

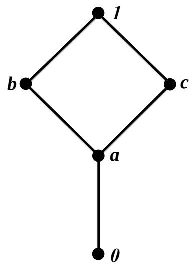

FIGURE 1. The Hasse diagram of $\mathfrak{A}_5$.

$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a
b	0	a	a	a	a	a	b
c	0	a	a	c	c	c	c
d	0	a	a	c	c	c	d
e	0	a	b	c	d	e	e
1	0	a	b	c	d	e	1

$\rightarrow_l$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	d	1	d	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	d	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

$\rightarrow_r$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	e	1	e	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	e	1	1	1
e	0	a	b	c	d	1	1
1	0	a	b	c	d	e	1

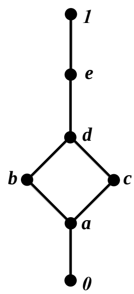

FIGURE 2. The Hasse diagram of $\mathfrak{A}_7$.

Routine calculation shows that $\mathfrak{A}_7 = (A_7; \vee, \wedge, \odot, \rightarrow_l, \rightarrow_r, 0, 1)$ is a residuated lattice.

Example 2.4. [18] Let $A_{10} = \{0, a, b, c, d, e, f, g, h, 1\}$ be a lattice whose Hasse diagram is below (see Figure 3). Define $\odot$, $\rightarrow_l$ and $\rightarrow_r$ on A_{10} as follows:

$\odot$	0	a	b	c	d	e	f	g	h	1
0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b	b	b
c	0	0	0	0	0	0	c	0	c	c
d	0	0	0	0	0	0	c	b	d	d
e	0	0	0	0	0	e	e	e	e	e
f	0	0	b	0	b	e	f	e	f	f
g	0	0	0	c	c	e	e	g	g	g
h	0	0	b	c	d	e	f	g	h	h
1	0	a	b	c	d	e	f	g	h	1

$\rightarrow_l$	0	a	b	c	d	e	f	g	h	1
0	1	1	1	1	1	1	1	1	1	1
a	h	1	1	1	1	1	1	1	1	1
b	g	g	1	g	1	1	1	1	1	1
c	f	f	f	1	1	1	1	1	1	1
d	e	e	f	g	1	1	1	1	1	1
e	d	d	d	d	d	1	1	1	1	1
f	b	b	b	d	d	g	1	g	1	1
g	c	c	d	c	d	f	f	1	1	1
h	a	a	b	c	d	e	f	g	1	1
1	0	a	b	c	d	e	f	g	h	1

$\rightarrow_r$	0	a	b	c	d	e	f	g	h	1
0	1	1	1	1	1	1	1	1	1	1
a	h	1	1	1	1	1	1	1	1	1
b	f	f	1	f	1	1	1	1	1	1
c	g	g	g	1	1	1	1	1	1	1
d	e	e	g	f	1	1	1	1	1	1
e	d	d	d	d	d	1	1	1	1	1
f	c	c	d	c	d	g	1	g	1	1
g	b	b	b	d	d	f	f	1	1	1
h	a	a	b	c	d	e	f	g	1	1
1	0	a	b	c	d	e	f	g	h	1

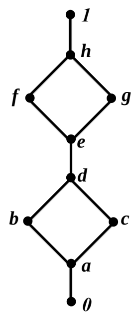

FIGURE 3. The Hasse diagram of $\mathfrak{A}_{10}$.

Routine calculation shows that $\mathfrak{A}_{10} = (A_{10}; \vee, \wedge, \odot, \rightarrow_l, \rightarrow_r, 0, 1)$ is an involutive residuated lattice.

Let $\mathfrak{A}$ be a residuated lattice. A non-empty subset F of A is called a filter of $\mathfrak{A}$ if it satisfies the following conditions for all $x, y \in A$:

$(fi1)$ $x, y \in F$ implies $x \odot y \in F$;

$(fi2)$ $x \leq y$ and $x \in F$ imply $y \in F$.

Proposition 2.2. [11] *Let $\mathfrak{A}$ be a residuated lattice. For a subset F of A the following assertions are equivalent:*

(1) *F is a filter;*

(2) *$1 \in F$ and $x, x \rightarrow_l y \in F$, then $y \in F$;*

(3) *$1 \in F$ and $x, x \rightarrow_r y \in F$, then $y \in F$.*

Proposition 2.3. *Let $\mathfrak{A}$ be a residuated lattice and F be a non-empty subset of A. Then, F is a filter of $\mathfrak{A}$ if and only if F satisfies the following assertions:*
(1) $x, y \in F$ *implies* $x \odot y \in F$;
(2) $x \in F$ *and* $y \in F$ *imply* $x \vee y \in F$.

Proof. Let F be a filter of $\mathfrak{A}$. Obviously, for any $x, y \in F$, we have $x \odot y \in F$. Now, consider $x \in F$ and $y \in A$. Thus we have $x \leq x \vee y$ and this shows that $x \vee y \in F$. Conversely, let $x \leq y$ and $x \in F$. So we have $y = x \vee y \in F$. □

Trivial examples of filters are $\mathbf{1} = \{1\}$ and A. A filter F of $\mathfrak{A}$ is called *proper* if $F \neq A$. Clearly, F is a proper filter if and only if $0 \notin F$. A proper filter P of $\mathfrak{A}$ is called *prime* provided that it is prime as a filter of $\ell(\mathfrak{A})$, that is

$$x \vee y \in P \text{ implies that } x \in P \text{ or } y \in P.$$

In the sequel, we shall denote the set of filters of $\mathfrak{A}$ by $F(\mathfrak{A})$, and the set of prime filters of $\mathfrak{A}$ by $Spec(\mathfrak{A})$.

Example 2.5. Consider the residuated lattice $\mathfrak{A}_5$ in Example 2.2. Then $F(\mathfrak{A}_5) = \{F_1 = \mathbf{1}, F_2 = \{b, 1\}, F_3 = \{c, 1\}, F_4 = A_5\}$.

Example 2.6. Consider the residuated lattice $\mathfrak{A}_7$ in Example 2.3. Then $F(\mathfrak{A}_7) = \{F_1 = \mathbf{1}, F_2 = \{e, 1\}, F_3 = \{c, d, e, 1\}, F_4 = \{a, b, c, d, e, 1\}, F_5 = A_7\}$.

Example 2.7. Consider the residuated lattice $\mathfrak{A}_{10}$ in Example 2.4. Then $F(\mathfrak{A}_{10}) = \{F_1 = \mathbf{1}, F_2 = \{h, 1\}, F_3 = \{f, h, 1\}, F_4 = \{g, h, 1\}, F_5 = \{e, f, g, h, 1\}, F_6 = A_{10}\}$.

Let $\mathfrak{A}$ be a residuated lattice. It is obvious that $(A; F(\mathfrak{A}))$ is an algebraic closed set system. The closure operator associated with this system is denoted by $Fi^{\mathfrak{A}} : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$. Thus for any subset X of A, $Fi^{\mathfrak{A}}(X) = \cap\{F \in F(\mathfrak{A}) | X \subseteq F\}$ is the smallest filter of $\mathfrak{A}$ contains X. $Fi^{\mathfrak{A}}(X)$ is called *the filter generated by* X. For each $x \in A$, the filter generated by $\{x\}$ is denoted by $Fi^{\mathfrak{A}}(x)$ and it is called the *principal* filter of $\mathfrak{A}$. When there is no ambiguity we will drop the superscript $\mathfrak{A}$.

If $\{F_i\}_{i\in I}$ is a family of all filters of $\mathfrak{A}$, we define $\wedge_{i\in I} F_i = \cap_{i\in I} F_i$ and $\veebar_{i\in I} F_i = Fi(\cup_{i\in I} F_i)$. According to [4], $(F(\mathfrak{A}), \wedge, \veebar)$ is a complete Browerian algebraic lattice which its compact elements are exactly the principal filter of $\mathfrak{A}$.

Example 2.8. Consider Example 2.7 and let $X = \{f\}$. Then $Fi(X) = \{f, h, 1\}$.

Proposition 2.4. [4] *Let $\mathfrak{A}$ be a residuated lattice and X be a subset of A. Then we have*

$$Fi(X) = \{a \in A | x_1 \odot \cdots \odot x_n \leq a, for\ some\ integer\ n,\ x_1, \cdots, x_n \in X\}.$$

Proposition 2.5. [4] *Let $\mathfrak{A}$ be a residuated lattice and $x, y \in A$. The following conditions hold.*
(1) $Fi(F, x) := F \veebar Fi(x) = \{a \in A | (f \odot x^n)^m \leq a,\ f \in F,\ n, m \in \mathbb{N}\}$;
(2) $Fi(x \vee y) = Fi(x) \cap Fi(y)$;
(3) $x \leq y$ *implies* $Fi(y) \subseteq Fi(x)$;
(4) $Fi(x) \veebar Fi(y) = Fi(x \wedge y) = Fi(x \odot y) = Fi(y \odot x)$;
(5) $Fi(x \rightarrow_l y) \veebar Fi(x) = Fi(x \rightarrow_r y) \veebar Fi(x)$.

Let $\mathfrak{A}$ be a residuated lattice. We put $d^l(a, b) = (a \rightarrow_l b) \odot (b \rightarrow_l a)$ and $d^r(a, b) = (a \rightarrow_r b) \odot (b \rightarrow_r a)$, for any $a, b \in A$. With any filter of a residuated lattice $\mathfrak{A}$ we associate two binary relations $\equiv^l_F$ and $\equiv^r_F$ on A by defining

$$(a,b) \in \equiv_F^l \text{ if and only if } d^l(a,b) \in F,$$
$$(a,b) \in \equiv_F^r \text{ if and only if } d^r(a,b) \in F,$$

It is easy to check that the binary relations $\equiv_F^l$ and $\equiv_F^r$ are equivalence relations on A. $\equiv_F^l$ and $\equiv_F^r$ are called the left equivalence relation and the right equivalence relation induced by F, respectively. In the following, for any $a \in A$ the equivalence classes $a/\equiv_F^l$ and $a/\equiv_F^r$ are denoted by $[a]_F^l$ and $[a]_F^l$, respectively.

Definition 2.2. [7] Let $\mathfrak{A}$ be a residuated lattice. A filter F of $\mathfrak{A}$ is called normal if $x \to_l y \in F$ if and only if $x \to_r y \in F$, for any $x, y \in A$. We shall denote by $F_n(\mathfrak{A})$ the set of normal filters of $\mathfrak{A}$.

Example 2.9. Consider the residuated lattice $\mathfrak{A}_{10}$ in Example 2.4. Then we have $F_n(\mathfrak{A}_{10}) = \{F_1 = \mathbf{1}, F_2 = \{h,1\}, F_5 = \{e,f,g,h,1\}, F_6 = A_{10}\}$

Let $\mathfrak{A}$ be a residuated lattice. The set of all complemented elements in the lattice reduct $\mathfrak{A}$ is denoted by $B(\mathfrak{A})$ and it is called the Boolean center of $\mathfrak{A}$. Complements are generally not unique unless the lattice is distributive. In residuated lattices however, although the underlying lattices need not be distributive, according to [5], the complements are unique.

Proposition 2.6. [5] *Let $\mathfrak{A}$ be a residuated lattice, $e \in B(\mathfrak{A})$ and $a \in A$. Then we have*
(1) $e^c = \neg_l e = \neg_r e$;
(2) $e^n = e$, *for each integer* n;
(3) $e \odot a = a \odot e = e \wedge a$;
(4) $\neg_{rl} e = \neg_{lr} e = e$.

Proposition 2.7. [5] *Let $\mathfrak{A}$ be a residuated lattice and $e \in B(\mathfrak{A})$. Then $Fi(e)$, $Fi(\neg_l e)$ and $Fi(\neg_r e)$ are normal filters of $\mathfrak{A}$ and we have*

$$Fi(e) = \{a \in A | e \leq a\}.$$

Proposition 2.8. *Let $\mathfrak{A}$ be a residuated lattice, F be a filter of A and $e \in B(\mathfrak{A})$. Then we have*

$$F \veebar Fi(e) = \{a \in A | f \odot e \leq a,\ f \in F,\ n \in \mathbb{N}\}.$$

Proof. It is straightforward by Proposition 2.5((1)) and Proposition 2.6(2 and 3). □

Proposition 2.9. *Let $\mathfrak{A}$ be a residuated lattice, $e \in B(\mathfrak{A})$ and $a \in A$. Then the following are equivalent.*
(1) $a \vee \neg_l e = 1$;
(2) $e \leq a$;
(3) $a \in Fi(e)$;
(4) $a \vee \neg_r e = 1$.

Proof.
$1 \Rightarrow 2)$ If $a \vee \neg_l e = 1$, then

$$\begin{aligned} e = e \odot 1 &= e \odot (a \vee \neg_l e) && \\ &= (e \odot a) \vee (e \odot \neg_l e) && \text{Proposition 2.1 (15)} \\ &= (e \odot a) \vee 0 && \text{Proposition 2.6 (1)} \\ &= e \odot a \leq e \wedge a \leq a. && \end{aligned}$$

$2 \Leftrightarrow 3)$ It is obvious by Proposition 2.7.

$2 \Rightarrow 1)$ If $e \leq a$, then $e \vee a = a$, hence $a \vee \neg_l e = (a \vee e) \vee \neg_l e = a \vee (e \vee \neg_l e) = a \vee 1 = 1$. Similarly, we can show that 2, 3 and 4 are equivalent.

□

Proposition 2.10. [4] *Let $\mathfrak{A}$ be a residuated lattice and $\{F_i\}_{i \in I}$ be a non empty family of normal filters of $\mathfrak{A}$. Then $\wedge_{i \in I} F_i$ and $\veebar_{i \in I} F_i$ are normal filters of $\mathfrak{A}$.*

As a consequence of Proposition 2.10 we conclude $(F_n(\mathfrak{A}), \wedge, \veebar)$ is a complete sublattice of $(F(\mathfrak{A}), \wedge, \veebar)$.

It is obvious that if F is a normal filter of the residuated lattice $\mathfrak{A}$ then the right and the left equivalence relations induced by F are equal and both of them are denoted by $\equiv_F$. So $(x, y) \in \equiv_F$ if and only if $d^l(x, y) \in F$ if and only if $d^r(x, y) \in F$. According to [11], if F is a normal filter of a residuated lattice $\mathfrak{A}$ then $\equiv_F$ is a congruence relation on $\mathfrak{A}$. In this case, For any $a \in A$, let a/F be the equivalence class $a/\equiv_F$ and $A/F = \{a/F | a \in A\}$. A/F becomes a residuated lattice with the natural operations induced from those of $\mathfrak{A}$ and it is denoted by $\mathfrak{A}/F$. If $a, b \in A$, then $a/F \leq b/F$ if and only if $a \rightarrow_l b \in F$ if and only if $a \rightarrow_r b \in F$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be residuated lattices. A mapping $h : A \longrightarrow B$ is called a homomorphism, in symbols $h : \mathfrak{A} \longrightarrow \mathfrak{B}$, if it preserves the fundamental operations. If $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a homomorphism we put $coker(h) = h^{\leftarrow}(1)$. It is easy to check that $coker(h)$ is a normal filter of $\mathfrak{A}$. Also, it is obvious that h is a monomorphism if and only if $coker(h) = \{1\}$.

Proposition 2.11. *Let $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism.*
(1) *If h is onto and $F \in F(\mathfrak{A})(F \in F_n(\mathfrak{A}))$ such that $coker(h) \subseteq F$ then $h(F) \in F(\mathfrak{B})(h(F) \in F_n(\mathfrak{B}))$.*
(2) *If $F \in F(\mathfrak{B})(F \in F_n(\mathfrak{B}))$ then $h^{\leftarrow}(F) \in F(\mathfrak{A})(h^{\leftarrow}(F) \in F_n(\mathfrak{A}))$ and $coker(h) \subseteq h^{\leftarrow}(F)$.*

Proof. It is straightforward. □

3. Generalized co-annihilators

In this section, we introduce and investigate the notion of generalized co-annihilator of residuated lattices. Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$ and X be a subset of A. The *generalized co-annihilator of X relative to F* is denoted by $(F : X)$ and defined as follow:

$$(F : X) = \{a \in A | x \vee a \in F, \forall x \in X\}.$$

If $X = \{x\}$, then $(F : \{x\})$ is denoted by $(F : x)$. Also, $(\mathbf{1}, X)$ is called the *co-annihilator of X* and it is denoted by $X^{\perp}$. It is easy to see that $A^{\perp} = \{1\}$ and $\emptyset^{\perp} = \{1\}^{\perp} = A$. If $X = \{x\}$, then $X^{\perp}$ is denoted by $x^{\perp}$. In the following proposition, we collect the properties of generalized co-annihilators.

Proposition 3.1. *Let $\mathfrak{A}$ be a residuated lattice, F, G be filters of $\mathfrak{A}$ and X, Y be subsets of A. Then the following assertions hold:*
(1) *$(F : X)$ is a filter of $\mathfrak{A}$;*
(2) *$F \subseteq (F : X)$;*
(3) *$X \subseteq Y$ implies $(F : Y) \subseteq (F : X)$;*
(4) *$(F : X) = (F : Fi(X))$. In particular, $(F : 0) = (F : A) = F$;*

(5) *$F \subseteq G$ implies $(F : X) \subseteq (G : X)$;*
(6) *$(F : X) = A$ if and only if $X \subseteq F$;*
(7) *$X \subseteq (F : (F : X))$;*
(8) *$(F : X) = (F : (F : (F : X)))$;*
(9) *$\bigcap_{i\in I}(F : X_i) = (F : \bigcup_{i\in I} X_i)$;*
(10) *$(F : X) = (F : X - F)$;*
(11) *$(F : X) = \bigcap_{x\in X}(F : x)$. In particular, $X^\perp = \bigcap_{x\in X} x^\perp$;*
(12) *$\bigcap_{i\in I}(F_i : X) = (\bigcap_{i\in I} F_i : X)$;*
(13) *if X contains F then $X \cap (F : X) = F$;*
(14) *$((F : X) : Y) = ((F : Y) : X) = (F : X\vee Y)$, where $X\vee Y = \{x\vee y | x \in X, y \in Y\}$.*

Proof. (1): It is obvious that $1 \in (F : X)$. Let $a, b \in (F : X)$. Thus for any $x \in X$ we have $a \vee x, b \vee x \in F$ and it implies that $(a \vee x) \odot (b \vee x) \in F$. By Proposition 2.1(14), we have $(a\vee x)\odot(b\vee x) \leq (a\odot b)\vee x$. Therefore, we have $(a\odot b)\vee x \in F$, for any $x \in X$. It shows that $a \odot b \in (F : X)$. Also, it is easy to check that $a \leq b$ and $a \in (F : X)$ implies $b \in (F : X)$. Hence, $(F : X)$ is a filter of $\mathfrak{A}$.

(2): Let $a \in F$. Then $a \vee x \geq a \in F$ and it implies $a \in (F : X)$.

(3): Let $a \in (F : Y)$. Then for any $x \in X$ we have $x \in Y$ and it implies $a \vee x \in F$. Hence, $a \in (F : X)$.

(4): Since $Fi(X)$ is the smallest filter of $\mathfrak{A}$ containing X so by part (3) we have $(F : Fi(X)) \subseteq (F : X)$. Now, assume that $a \in (F : X)$ and $b \in Fi(X)$. By Proposition 2.4, there are $x_1, \cdots, x_n \in X$ such that $x_1 \odot \cdots \odot x_n \leq b$. For each $1 \leq i \leq n$ we have $a \vee x_i \in F$. Hence, $a \in (F : Fi(X))$ and this shows that $(F : X) \subseteq (F : Fi(X))$.

(5): Let $F \subseteq G$ and $a \in (F : X)$. Then for any $x \in X$ we have $a \vee x \in F \subseteq G$ and it implies that $a \in (G : X)$.

(6): Let $(F : X) = A$ and $x \in X$. So we have $x = x \vee x \in F$ and it shows that $X \subseteq F$. Now, assume that $X \subseteq F$ and $a \in A$. For each $x \in X$ we have $x \leq a \vee x$ and it means that $a \in (F : X)$. Hence, $(F : X) = A$.

(7): Let $x \in X$ and $a \in (F : X)$. So we have $a \vee x \in F$ and it implies that $x \in (F : (F : X))$.

(8): By (3) and (7) we have $(F : (F : (F : X))) \subseteq (F : X)$. On the other hand by (7) we have $(F : X) \subseteq (F : (F : (F : X)))$ and it shows that the equality holds.

(9): By (3) we have $(F : \bigcup_{i\in I} X_i) \subseteq \bigcap_{i\in I}(F : X_i)$. Now, assume that $a \in \bigcap_{i\in I}(F : X_i)$ and $x \in \bigcup_{i\in I} X_i$. Thus for some $i \in I$ we have $x \in X_i$ and it states that $a \vee x \in F$. Hence, $a \in (F : \bigcup_{i\in I} X_i)$ and it shows that $\bigcap_{i\in I}(F : X_i) \subseteq (F : \bigcup_{i\in I} X_i)$.

(10): Since we have $X = (X - F) \cup (X \cap F)$ so by (6) and (9) we obtain that $(F : X) = (F : (X - F) \cup (X \cap F)) = (F : X - F) \cap (F : X \cap F) = (F : X - F) \cap A = (F : X - F)$.

(11): It is straightforward by (9).

(12): By (5) we have $(\bigcap_{i\in I} F_i : X) \subseteq \bigcap_{i\in I}(F_i : X)$. Let $a \in \bigcap_{i\in I}(F_i : X)$ and $x \in X$. Therefore, $a \vee x \in F_i$, for each $i \in I$ and it implies that $a \vee x \in \bigcap_{i\in I} F_i$. Thus $a \in (\bigcap_{i\in I} F_i : X)$ and it results $\bigcap_{i\in I}(F_i : X) \subseteq (\bigcap_{i\in I} F_i : X)$.

(13): By (2) and hypothesis we have $F \subseteq X \cap (F : X)$. Now, assume that $x \in X \cap (F : X)$. Thus for each $y \in X$ we have $x \vee y \in F$. Let $y = x$ and so $x \in F$. Therefore, $X \cap (F : X) \subseteq F$.

(14): Let $a \in ((F : X) : Y)$ and $z \in X \vee Y$. So there are $x \in X$ and $y \in Y$ such that $z = x \vee y$. Now, we have $a \vee y \in (F : X)$ and it implies that $(a \vee x) \vee y \in F$. Hence, $a \vee z \in F$ and it means that $a \in (F : X \vee Y)$. Conversely, let $a \in (F : X \vee Y)$ and $y \in Y$. Thus for any $x \in X$ we have $a \vee (x \vee y) \in F$ and so $a \vee y \in (F : X)$ and it shows that $a \in ((F : X) : Y)$. Analogously, we can show that $((F : Y) : X) = (F : X \vee Y)$.

□

Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. An F-divisor in $\mathfrak{A}$ is an element a for which there exists $x \in A - F$ such that $a \vee x \in F$. The set of all F-divisor in $\mathfrak{A}$ is denoted by $D_F(\mathfrak{A})$. Obviously, we have $F \subseteq D_F(\mathfrak{A})$. Also, one can see that $D_F(\mathfrak{A}) = F$ if $F \in Spec(\mathfrak{A})$.

Proposition 3.2. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then*

$$D_F(\mathfrak{A}) = \cup_{x \in A-F}(F : x).$$

Proof. Let $a \in D_F(\mathfrak{A})$. So there is $x \in A - F$ such that $a \vee x \in F$ and it shows that $a \in (F : x) \subseteq \cup_{x \in A-F}(F : x)$. Now, assume that $a \in \cup_{x \in A-F}(F : x)$. Hence there is $x \in A - F$ such that $a \in (F : x)$ and it means $a \vee x \in F$. Thus $a \in D_F(\mathfrak{A})$. □

Proposition 3.3. *Let $\mathfrak{A}$ be a residuated lattice, F be a filter of $\mathfrak{A}$ and X a be filter of $\ell(\mathfrak{A})$. If X is linearly ordered and $X \nsubseteq F$, then $(F : X) \in Spec(\mathfrak{A})$. In particular, if $X \neq \mathbf{1}$ is linearly ordered, then $X^{\perp} \in Spec(\mathfrak{A})$.*

Proof. Since $X \nsubseteq F$ so by Proposition 3.1(6), $(F : X)$ is a proper filter. Now, suppose that $a \vee b \in (F : X)$, $a \notin (F : X)$ and $b \notin (F : X)$. It implies that there are $x_a, x_b \in X$ such that $a \vee x_a \notin F$ and $b \vee x_b \notin F$. Let $x = x_a \wedge x_b$. Since X is a filter of the lattice $\ell(\mathfrak{A})$, so $x \in X$ and consequently $a \vee x, b \vee x \in X$. Now, by linearity of X let $a \vee x \leq b \vee x$. Thus we have $b \vee x_b \geq b \vee x = b \vee (b \vee x) \geq b \vee (a \vee x) = (a \vee b) \vee x \in F$. IT implies $b \vee x_b \in F$ and this contradiction proves the proposition. Also, if we let $F = \mathbf{1}$ and $X \neq \mathbf{1}$ is linearly ordered , we can conclude that $X^{\perp} \in Spec(\mathfrak{A})$. □

Lemma 3.4. *Let $\mathfrak{A}$ be a residuated lattice, F be a filter of $\mathfrak{A}$ and X be a filter of $\ell(\mathfrak{A})$ such that for each $x_1, x_2 \in X$ we have $x_1 \vee x_2 \in F$. If $(F : X) \in Spec(\mathfrak{A})$, then $X \nsubseteq F$ and for any $x_1, x_2 \in G$ we have $x_1 \to_l x_2 \in F$, $x_2 \to_l x_1 \in F$, $x_1 \to_r x_2 \in F$ or $x_2 \to_r x_1 \in F$.*

Proof. Since $(F : X)$ is a proper filter so by Proposition 3.1(6) we have $X \nsubseteq F$. Assume that $x_1, x_2 \in X$. By Proposition 2.1(11) we get that $x_1 \to_l x_2, x_1 \to_r x_2, x_2 \to_l x_1, x_2 \to_r x_1 \in X$. Also, by hypothesis and Proposition 3.1(2) we conclude that $(x_1 \to_l x_2) \vee (x_1 \to_r x_2) \vee (x_2 \to_l x_1) \vee (x_2 \to_r x_1) \in F \subseteq (F : X)$. Since, $(F : X)$ is a prime filter of $\mathfrak{A}$ then we can assume that $x_1 \to_l x_2 \in (F : X)$. By Proposition 3.1(11), we can conclude that $x_1 \to_l x_2 \in (F : x_1 \to_l x_2)$ and it shows that $x_1 \to_l x_2 = (x_1 \to_l x_2) \vee (x_1 \to_l x_2) \in F$. □

A residuated lattice $\mathfrak{A}$ is called a pseudo-MTL algebra if it satisfies the pseudo-prelinearity condition $(a \to_l b) \vee (b \to_l a) = (a \to_r b) \vee (b \to_r a) = 1$.

Corollary 3.5. *Let $\mathfrak{A}$ be a pseudo-MTL algebra and X be a filter of $\ell(\mathfrak{A})$. Then $X^{\perp} \in Spec(\mathfrak{A})$ if and only if X is linearly ordered and $X \neq \mathbf{1}$.*

Proof. Let $X^{\perp} \in Spec(\mathfrak{A})$ and consider $x_1, x_2 \in X$. Similar to the proof of Lemma 3.4, we can show that $x_1 \rightarrow_l x_2, x_2 \rightarrow_l x_1, x_1 \rightarrow_r x_2, x_2 \rightarrow_r x_1 \in X$. Also, since $\mathfrak{A}$ satisfies the pseudo-prelinearity condition so $(x_1 \rightarrow_l x_2) \vee (x_2 \rightarrow_l x_1) = (x_1 \rightarrow_r x_2) \vee (x_2 \rightarrow_r x_1) = 1 \in X^{\perp}$. Similar to the proof of Lemma 3.4, it results that either $x_1 \rightarrow_l x_2 = 1$, $x_2 \rightarrow_l x_1 = 1$, $x_1 \rightarrow_r x_2 = 1$ or $x_2 \rightarrow_r x_1 = 1$ and it shows that X is linearly ordered. Also, since $X^{\perp}$ is a proper filter so by Proposition 3.1(6) we obtain that $X \neq 1$. Conversely, it is obvious by Proposition 3.3. □

Proposition 3.6. *Let $\mathfrak{A}$ be a residuated lattice and F be a linearly ordered filter of $\mathfrak{A}$ such that it contains an element $x \neq 1$ and either $x \vee \neg_l x = 1$ or $x \vee \neg_r x = 1$. Then x is the least element of F.*

Proof. Let $x \vee \neg_l x = 1$ and $a \in F$. Then $a = a \vee 0 = a \vee (x \odot \neg_l x) \geq (a \vee x) \odot (a \vee \neg_l x)$ where the last inequality follows by Proposition 2.1(14). By Corollary 3.5, $F^{\perp}$ is a prime filter. Since $x \vee \neg_l x = 1$, either $x \in F^{\perp}$ or $\neg_l x \in F^{\perp}$ and as $x \vee x = x \neq 1$, we necessarily have $\neg_l x \in F^{\perp}$. Now $a \in F$, hence $a \vee \neg_l x = 1$, whence $a \vee x \leq a$ and it implies $a \vee x = a$. Thus $x \leq a$ and the proof is complete. □

Corollary 3.7. *Let $\mathfrak{A}$ be a residuated lattice and F be a linearly ordered filter of $\mathfrak{A}$. Then $card(F \cap B(\mathfrak{A})) \leq 2$.*

Proof. Let $a_1, a_2 \in F \cap B(\mathfrak{A}) - \{1\}$. So by Proposition 3.6, we have $a_1 \leq a_2$ and $a_2 \leq a_1$ and it shows that $a_1 = a_2$. □

Proposition 3.8. *Let $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ be an epimorphism.*
(1) *If F is a filter of $\mathfrak{A}$ containing $coker(h)$ and $X \subseteq A$, then $h(F : X) = (h(F) : h(X))$;*
(2) *if F is a filter of $\mathfrak{B}$ and $Y \subseteq B$, then $h^{\leftarrow}(F : Y) = (h^{\leftarrow}(F) : h^{\leftarrow}(Y))$.*

Proof. (1): By Proposition 2.11(1), $h(F)$ is a filter of $\mathfrak{B}$. If $X = \emptyset$ then by Proposition 3.1(6) we have $(F : X) = A$ and $(h(F) : h(X)) = B$. Since h is onto so the equality holds. Let X be a nonempty subset of A. Assume that $b \in (h(F) : h(X))$. So for each $y \in h(X)$ we have $b \vee y \in h(F)$. Hence, there are $x \in X$, $a \in A$ and $f \in F$ such that $h(x) = y$, $h(a) = b$ and $b \vee y = h(f)$. It means that $h(a) \vee h(x) = h(f)$. By Proposition 2.1(3), we have $h(f) \rightarrow_l h(a \vee x) = 1$ and it results $h(f \rightarrow_l (a \vee x)) = 1$. Thus $f \rightarrow_l (a \vee x) \in coker(h) \subseteq F$ and by Proposition 2.2(2) we can conclude that $a \vee x \in F$. So $a \in (F : X)$ and it implies that $b \in h(F : X)$. Now, let $b \in h(F : X)$ and $y \in h(X)$. So there are $a \in (F : X)$ and $x \in X$ such that $h(a) = b$ and $h(x) = y$ and it results $a \vee x \in F$. Therefore, $h(a) \vee y \in h(F)$ and it implies that $b \in (h(F) : h(X))$.

(2): By Proposition 2.11(2), $h(F)$ is a filter of $\mathfrak{A}$. If $Y = \emptyset$ then we have $(F : Y) = B$ and $(h^{\leftarrow}(F) : h^{\leftarrow}(Y)) = A$. Since h is onto so the equality holds. Suppose that $a \in (h^{\leftarrow}(F) : h^{\leftarrow}(Y))$. We want to show that $h(a) \in (F : Y)$. Consider $y \in Y$. So there is $x \in A$ such that $h(x) = y$. Thus we have $h(a) \vee y = h(a) \vee h(x) = h(a \vee x)$. On the other hand, we have $x \in h^{\leftarrow}(Y)$ and it implies that $a \vee x \in h^{\leftarrow}(F)$. Therefore, $h(a \vee x) \in F$ and it means $h(a) \in (F : Y)$. Now, assume that $a \in h^{\leftarrow}(F : Y)$ and $x \in h^{\leftarrow}(Y)$. Hence, $h(a) \in (F : Y)$ and $h(x) \in Y$ and it implies that $h(a \vee x) = h(a) \vee h(x) \in F$. Therefore, $a \vee x \in h^{\leftarrow}(F)$ and it concludes that $a \in (h^{\leftarrow}(F) : h^{\leftarrow}(Y))$.

□

Let $\mathfrak{A}$ be a residuated lattice and F be a normal filter of $\mathfrak{A}$. The mapping $\pi_F^{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathfrak{A}/F$ defined by $\pi_F^{\mathfrak{A}}(a) = a/F$ is called *the natural homomorphism.* It is obvious that the natural homomorphism $\pi_F^{\mathfrak{A}}$ is onto and $coker(\pi_F^{\mathfrak{A}}) = F$. Therefore, by Proposition 2.11 we have

$$F(\mathfrak{A}/F) = \{H/F | F \subseteq H \in F(\mathfrak{A})\}.$$

Lemma 3.9. *Let $\mathfrak{A}$ be a residuated lattice and F be a normal filter of $\mathfrak{A}$. Then for any filter G of $\mathfrak{A}$ which contains F and for any subset X of A $(G : X)/F$ is a filter of $\mathfrak{A}/F$.*

Proof. Let G be a filter of $\mathfrak{A}$ contains F and X be a subset of A. By Proposition 3.1(2) we have $F \subseteq (G : X)$ and it shows that $(G : X)/F$ is a filter of $\mathfrak{A}/F$. □

Corollary 3.10. *Let $\mathfrak{A}$ be a residuated lattice, F be a normal filter of $\mathfrak{A}$, G be a filter of $\mathfrak{A}$ which contains F and X be a subset of A which contains F. Then we have*

$$(G/F : X/F) = (G : X)/F.$$

Proof. Consider the natural epimorphism π_F in Proposition 3.8. Then we have $\pi_F^{\leftarrow}(G/F : X/F) = (G : \pi_F^{\leftarrow}(\pi_F(X)))$. By Proposition 3.1(3), we have

$$(G : \pi_F^{\leftarrow}(\pi_F(X))) \subseteq (G : X) \quad \text{so} \quad (G/F : X/F) \subseteq (G : X)/F.$$

Now, assume that $a/F \in (G : X)/F$. By Lemma 3.9, $(G : X)/F$ is a filter of $\mathfrak{A}/F$ and it implies that $a \in (G : X)$. Consider $y/F \in X/F$. So there is $x \in X$ such that $y/F = x/F$ and it implies that $a/F \vee y/F = a/F \vee x/F = a \vee x/F \in G/F$. Hence, $a/F \in (G/F : X/F)$ and it shows that $(G : X)/F \subseteq (G/F : X/F)$. □

Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. We define

$$Co - An^F(\mathfrak{A}) = \{(F : G) | G \in Fi(\mathfrak{A})[F, A]\}.$$

In the sequel, the set $Co - An^{\mathbf{1}}(\mathfrak{A})$ will be denoted by $Co - An(\mathfrak{A})$.

Lemma 3.11. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then for each $H, K \in Co - An^F(\mathfrak{A})$ and for any family $\{H_i\}_{i\in I} \subseteq Co - An^F(\mathfrak{A})$ the following conditions hold:*

(1) *Let L be a filter of $\mathfrak{A}$. Then $L \in Co - An^F(\mathfrak{A})$ if and only if $L = (F : (F : L))$;*
(2) *$Co - An^F(\mathfrak{A}) = \{(F : X) | X \subseteq A\}$;*
(3) *$\cap_{i\in I} H_i \in Co - An^F(\mathfrak{A})$;*
(4) *$F, A \in Co - An^F(\mathfrak{A})$;*
(5) *we define $\sqcup_{i\in I} H_i = (F : (F : \cup_{i\in I} H_i))$. Then $\sqcup_{i\in I} H_i$ is the lowest upper bound of $\cup_{i\in I} H_i$ in the poset $Co - An^F(\mathfrak{A})$;*
(6) *$H \sqcup K = (F : (F : H) \cap (F : K))$;*
(7) *$H \veebar K \subseteq H \sqcup K$.*

Proof. (1): If $L \in Co - An^F(\mathfrak{A})$ then there is $G \in Fi(\mathfrak{A})[F, A]$ such that $L = (F : G)$. So by Proposition 3.1(8) the equality holds. Also, if $L = (F : (F : L))$ let $G = (F : L)$. It is obvious that $G \in Fi(\mathfrak{A})[F, A]$ and it implies that $L \in Co - An^F(\mathfrak{A})$.

(2): It is obvious that $Co - An^F(\mathfrak{A}) \subseteq \{(F : X) | X \subseteq A\}$. Now, let $L \in \{(F : X) | X \subseteq A\}$. Therefore, exists $X \subseteq A$ such that $L = (F : X)$ and it implies that $L = (F : (F : L))$. Hence, by (1) we conclude that $L \in Co - An^F(\mathfrak{A})$ and it shows that the result holds.

(3): For each $i \in I$, let G_i be a filter of $\mathfrak{A}$ contains F such that $H_i = (F : G_i)$. By Proposition 3.1(9) and (2) we have

$$\cap_{i\in I} H_i = \cap_{i\in I}(F : G_i) = (F : \cup_{i\in I} G_i) \in Co - An^F(\mathfrak{A}).$$

(4): By Proposition 3.1((4) and (6)), we have $(F : A) = F$ and $(F : F) = A$ and it shows that $F, A \in Co - An^F(\mathfrak{A})$.

(5): By Proposition 3.1(2), we have $F \subseteq (F : \cup_{i\in I} H_i)$. So by (2), we have $\sqcup_{i\in I} H_i \in Co - An^F(\mathfrak{A})$. For each $i \in I$, we have $H_i \subseteq \cup_{i\in I} H_i$. Hence, by Proposition 3.1(3), we get that $(F : \cup_{i\in I} H_i) \subseteq (F : H_i)$, for each $i \in I$. Again by Proposition 3.1(3) and (1), we can conclude that $H_i \subseteq (F : (F : \cup_{i\in I} H_i))$ and it shows that $(F : (F : \cup_{i\in I} H_i))$ is an upper bound of the set $\cup_{i\in I} H_i$ in the poset $Co-An^F(\mathfrak{A})$. Now, let K be an upper bound of the set $\cup_{i\in I} H_i$ in the poset $Co - An^F(\mathfrak{A})$. Thus $(F : K) \subseteq (F : \cup_{i\in I} H_i)$ and it concludes that $(F : (F : \cup_{i\in I} H_i)) \subseteq K$. Hence, $\sqcup_{i\in I} H_i$ is the lowest upper bound of $\cup_{i\in I} H_i$ in the poset $Co - An^F(\mathfrak{A})$.

(6): By (5) we have $H \sqcup K = (F : (F : H \cup K))$. Also, by 3.1(9) we have $(F : H) \cap (F : K) = (F : H \cup K)$ and this shows that the equality holds.

(7): Since $H \cup K \subseteq H \sqcup K$, it is clear.

□

Proposition 3.12. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the interval $\boldsymbol{Fi}(\mathfrak{A})[F, A]$ is a pseudocomplemented lattice. In particular, the lattice $\boldsymbol{Fi}(\mathfrak{A})$ is a pseudocomplemented lattice.*

Proof. Let G be a filter of $\mathfrak{A}$ containing F. By Proposition 3.1((1) and (2)), we can obtain that $(F : G) \in Fi(\mathfrak{A})[F, A]$. Thus, by Proposition 3.1(13), we have $G \wedge_{\mathbf{Fi}(\mathfrak{A})[F,A]} (F : G) = G \cap (F : G) = F$. Now, we show that $(F : G)$ is the greatest element of the interval $Fi(\mathfrak{A})[F, A]$ such that its meet with G is F. Let K be a filter of $\mathfrak{A}$ contains F such that $G \cap K = F$. Assume that $k \in K$ and $g \in G$. Therefore, $g, k \leq g \vee k$ and it shows that $g \vee k \in G \cap K = F$. Hence, for each $g \in G$ we have $g \vee k \in F$ and it shows that $k \in (F : G)$. One can see that if we let $F = \mathbf{1}$ then the lattice $\mathbf{Fi}(\mathfrak{A})$ is a pseudocomplemented lattice such that for any filter F of $\mathfrak{A}$, its pseudocomplement is $F^{\perp}$. □

Proposition 3.13. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then $\boldsymbol{Co\text{-}An}^F(\mathfrak{A}) = (Co - An^F(\mathfrak{A}), \cap, \sqcup, (F : -), F, A)$ is a complete Boolean lattice.*

Proof. By Lemma 3.11((3),(4), and (5)), $\mathbf{Co\text{-}An}^F(\mathfrak{A})$ is a bounded complete lattice. Now, we need prove that $\mathbf{Co\text{-}An}^F(\mathfrak{A})$ is distributive. Consider $H, K, L \in Co - An^F(\mathfrak{A})$. It is obvious that

$$H \sqcup (K \cap L) \subseteq (H \sqcup K) \cap (H \sqcup L). \tag{1}$$

Also, $H \cap L \subseteq H \sqcup (K \cap L)$ and $K \cap L \subseteq H \sqcup (K \cap L)$. Therefore, by Proposition 3.1(13) we get that $(H \cap L) \cap (F : H \sqcup (K \cap L)) = F$ and $(K \cap L) \cap (F : H \sqcup (K \cap L)) = F$. By Proposition 3.12, we conclude that $L \cap (F : H \sqcup (K \cap L)) \subseteq (F : H)$ and $L \cap (F : H \sqcup (K \cap L)) \subseteq (F : K)$. Consequently, $L \cap (F : H \sqcup (K \cap L)) \subseteq (F : H) \cap (F : K)$ and by Proposition 3.1(13) we obtain that $(L \cap (F : H \sqcup (K \cap L))) \cap (F : (F : H) \cap (F : K)) = F$. Hence, $((F : (F : H) \cap (F : K)) \cap L) \cap (F : H \sqcup (K \cap L)) = F$ and by Proposition 3.12, it implies that $(F : (F : H) \cap (F : K)) \cap L \subseteq (F : (F : H \sqcup (K \cap L)))$. Now, the left-hand side is $(H \sqcup K) \cap L$ by Lemma 3.11(6), and the right-hand side is $H \sqcup (K \cap L)$ by Lemma 3.11(1). Thus we obtain

$$(H \sqcup K) \cap L \subseteq H \sqcup (K \cap L). \tag{2}$$

Now, by 2 we get

$$\begin{array}{ll}(H \sqcup K) \cap (H \sqcup L) & \subseteq H \sqcup (K \cap (H \sqcup L)) \\ & = H \sqcup ((H \sqcup L) \cap K) \\ & \subseteq H \sqcup (H \sqcup (K \cap L)) \\ & = (H \sqcup H) \sqcup (K \cap L) \\ & H \sqcup (K \cap L).\end{array} \tag{3}$$

Now, we conclude distributivity by (1) and (3).

Also, for any $H \in Co - An^F(\mathfrak{A})$ we have $H \cap (F : H) = F$ and

$$\begin{array}{ll}H \sqcup (F : H) & = (F : (F : H) \cap (F : (F : H))) \\ & = (F : (F : H) \cap H) \\ & = (F : F) \\ & = A.\end{array}$$

It shows that $\mathbf{Co\text{-}An}^F(\mathfrak{A})$ is a complete Boolean algebra. □

Corollary 3.14. *Let $\mathfrak{A}$ be a residuated lattice. Then*

$$\boldsymbol{Co\text{-}An}(\mathfrak{A}) = (Co - An(\mathfrak{A}), \cap, \sqcup, ^{\perp}, \boldsymbol{1}, A)$$

is a complete Boolean lattice.

Proof. By taking $F = \mathbf{1}$, it follows by Proposition 3.13. □

Proposition 3.15. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions hold for any $x, y \in A$ and $e \in B(\mathfrak{A})$:*

(1) *$x \leq y$ implies $(F : x) \subseteq (F : y)$;*
(2) *$(F : x) = A$ if and only if $x \in F$;*
(3) *$(F : x) \cap (F : y) = (F : x \odot y)$;*
(4) *$(F : (F : x)) \cap (F : (F : y)) = (F : (F : x \vee y))$;*
(5) *$(F : x) \veebar (F : y) \subseteq (F : x) \sqcup (F : y) = (F : x \vee y)$;*
(6) *if F is a normal filter of $\mathfrak{A}$, then $(F : x/F) = (F : x)$;*
(7) *$(F : e) = F \veebar Fi(\neg_l e) = F \veebar Fi(\neg_r e)$. In particular, $e^{\perp} = Fi(\neg_l e) = Fi(\neg_r e)$.*

Proof. (1): It follows by Proposition 2.5(3) and Proposition 3.1((3) and (4)).

(2): It is evident by Proposition 3.1(6).

(3): It follows by Proposition 2.5(4) and Proposition 3.1((4) and (9)).

(4): Since $x, y \leq x \vee y$ so by Proposition 3.1(3) and (1) follows that $(F : (F : x \vee y)) \subseteq (F : (F : x)) \cap (F : (F : y))$. Let $a \in (F : (F : x)) \cap (F : (F : y))$ and $b \in (F : x \vee y)$. It states that $b \vee x \in (F : y)$ and it means that $a \vee (b \vee x) \in F$. Hence $a \vee b \in (F : x)$ and it implies that $a \vee b = a \vee (a \vee b) \in F$. Therefore, $a \in (F : (F : x \vee y))$.

(5): By Proposition 3.1(8), Lemma 3.11(6) and (3) we have the following sequence of formulas:

$$\begin{array}{ll}(F : x) \sqcup (F : y) & = (F : ((F : (F : x)) \cap (F : (F : x)))) \\ & = (F : (F : (F : x \vee y))) = (F : x \vee y).\end{array}$$

Also, by (1) we have $(F:x),(F:y)\subseteq(F:x\vee y)$ and it implies $(F:x)\veebar(F:y)\subseteq(F:x\vee y)$ since $(F:x\vee y)$ is a filter.

(6): By Proposition 3.1(3), it is obvious that $(F:x/F)\subseteq(F:x)$. Now, let $a\in(F:X)$ and $y\in x/F$. Therefore, $d^l(x,y)\in F$ and this means $d^l(a\vee x,a\vee y)\in F$. On the other hand, we have $a\vee x\in F$ and this implies that $a\vee y\in F$. Thus $a\in(F:x/F)$ and this shows the equality.

(7): Let $a\in(F:e)$. So $a\vee e\in F$ and by Proposition 2.1(15), we conclude that $(a\vee e)\odot\neg_l e=(a\odot\neg_l e)\vee(e\odot\neg_l e)=a\odot\neg_l e\leq a$. Now, by Proposition 2.8 we get that $a\in F\veebar Fi(\neg_l e)$. Conversely, let $a\in F\veebar Fi(\neg_l e)$. By Proposition 2.6 and 2.8, there exist $f\in F$ and an integer n such that $f\odot\neg_l e\leq a$. So $(f\odot\neg_l e)\vee e\leq a\vee e$ and by Proposition 2.1(14) we obtain $(f\vee e)\odot(\neg_l e\vee e)\leq(f\odot\neg_l e)\vee e$. Since, $f\odot e\in F$ and $\neg_l e\vee e=1$ we have $a\vee e\in F$ and it means $a\in(F:e)$. Analogously, we can show that $(F:e)=F\veebar Fi(\neg_r e)$. Also, if we let $F=\mathbf{1}$, then we have $e^\perp=Fi(\neg_l e)=Fi(\neg_r e)$.

□

Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. We define

$$Co-Anu^F(\mathfrak{A})=\{(F:x)|x\in A\},$$

and instead of $Co-Anu^{\mathbf{1}}(\mathfrak{A})$ we write $Co-Anu(\mathfrak{A})$. The elements of $Co-Anu(\mathfrak{A})$ will be called *coannulets* of $\mathfrak{A}$.

Theorem 3.16. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then*

$$\boldsymbol{Co\text{-}Anu}^F(\mathfrak{A})=(Co-Anu^F(\mathfrak{A}),\cap,\sqcup,F=(F:0),A=(F:1))$$

is a sublattice of $\boldsymbol{Co\text{-}An}^F(\mathfrak{A})$.

Proof. It is a direct consequence of Proposition 3.15. □

Let $\mathfrak{A}$ be a residuated lattice and $x,y\in A$. We shall denote by $(x:y)$ the co-annihilator $(Fi(x):y)$.

Proposition 3.17. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions hold for any $x,y\in A$:*

(1) $(x:x)=A$;

(2) $(x:y)=(x:x\wedge y)=(x:x\odot y)=(x:y\odot x)=(x:\{x,y\})$;

(3) $(x:y)=(x\vee y:y)$.

Proof. (1): Apply Proposition 3.15(2).

(2): By Proposition 3.15(3) and (1), it follows that $(x:y)=(x:y)\cap A=(x:y)\cap(x:x)=(x:x\wedge y)=(x:x\odot y)=(x:y\odot x)=(x:\{x,y\})$.

(3): By Proposition 2.5(2), Proposition 3.1(12) and (1), it follows that $(x\vee y:y)=(Fi(x\vee y):y)=(Fi(x)\cap Fi(y):y)=(Fi(x):y)\cap(Fi(y):y)=(x:y)\cap(y:y)=(x:y)\cap A=(x:y)$.

□

Proposition 3.18. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions hold for any $x,y\in A$:*

(1) $x\in(F:(F:x))$;

(2) $x\leq y$ *implies* $(F:(F:y))\subseteq(F:(F:x))$;

(3) $y\in(F:x)$ *implies* $(F:(F:x))\subseteq(F:y)$.

Proof. (1): It is an immediate consequence of Proposition 3.1(7).
(2): By Proposition 3.15(1), we have $(F : x) \subseteq (F : y)$ and by Proposition 3.1(3) we get the result.
(3): Apply Proposition 3.1(3).

□

Proposition 3.19. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The maximal element in the set of $\{(F : X)|X \subseteq A, F \cap X = \emptyset\}$ is a prime filter.*

Proof. Let $\mathfrak{U} = \{(F : X)|X \subseteq A, F \cap X = \emptyset\}$ and $(F : X)$ be the maximal element of $\mathfrak{U}$. Applying Proposition 3.1(6), it follows that $(F : X)$ is a proper filter. Assume that $a \vee b \in (F : X)$ and $b \notin (F : X)$. Therefore, $(a \vee b) \vee x \in F$ and $b \vee x \notin F$, for some $x \in X$. It states that $a \in (F : b \vee x)$ and by Proposition 3.15(2) and 3.18(2) we can conclude $(F : X) \subseteq (F : x) \subseteq (F : b \vee x) \neq A$. It implies that $a \in (F : b \vee x) = (F : X)$ and it shows that $(F : X)$ is a prime filter of $\mathfrak{A}$. □

There is a natural question whether the equation $(F : x) \veebar (F : y) \subseteq (F : x) \sqcup (F : y)$ in Proposition 3.15(5) holds or not. It follows from two examples below that this equation does not hold in general.

Example 3.1. Let $A_6 = \{0, a, b, c, d, 1\}$ be a lattice whose Hasse diagram is below (see Figure 4). Define $\odot = \wedge$ and $\rightarrow$ on A_5 as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	c	1	c	1	1
c	0	b	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

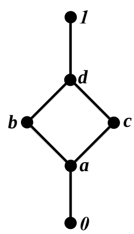

FIGURE 4. The Hasse diagram of $\mathfrak{A}_6$.

Routine calculation shows that $\mathfrak{A}_6 = (A_6; \odot, \rightarrow, 1)$ is a Heyting algrbra. One can see that $F = \{d, 1\}$ is a filter of $\mathfrak{A}_6$. Now, we have $(F : b) = \{c, d, 1\}$ and $(F : c) = \{b, d, 1\}$. Therefore, $(F : b) \veebar (F : c) = \{a, b, c, d, 1\}$ and $(F : b \vee c) = A_6$ and it shows that $(F : b \vee c)$ is not a subset of $(F : b) \veebar (F : c)$.

Example 3.2. Let $B_5 = \{0, a, b, c, 1\}$ be a lattice whose Hasse diagram is below (see Figure 5). Define $\odot = \wedge$ and $\rightarrow$ on B_5 as follows:

$\to_l$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

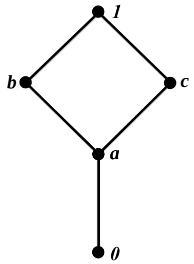

FIGURE 5. The Hasse diagram of $\mathfrak{B}_5$.

Routine calculation shows that $\mathfrak{B}_5 = (B_6; \odot, \to, 1)$ is a Heyting algrbra. Now, we have $b^\perp = \{c, 1\}$ and $c^\perp = \{b, 1\}$. Therefore, $b^\perp \veebar c^\perp = \{a, b, c, 1\}$ and $(b \vee c)^\perp = B_5$ and it shows that $(b \vee c)^\perp$ is not a subset of $b^\perp \veebar c^\perp$.

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