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Infinite order of transcendental meromorphic solutions of some nonhomogeneous linear differential equations

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Abstract. In this paper, we investigate the order of growth of transcendental meromorphic solutions of the linear differential equation

$$f^{(k)}+\sum_{j=0}^{k-1}h_j(z)e^{P_j(z)}f^{(j)}=F,$$

where $k\geq 2$ is an integer, $P_j(z)$ $(j=0,...,k-1)$ are nonconstant polynomials, $h_j(z)$ $(j=0,...,k-1)$ and $F(\not\equiv 0)$ are meromorphic functions. Under some conditions, we prove that every transcendental meromorphic solution of the above equation is of infinite order.

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1. Introduction and main results

In this paper, we use the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see [8], [12]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of a meromorphic function f and $\sigma_2(f)$ to denote the hyper-order of f which is defined by (see [12])

$$\sigma_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r}. \tag{1}$$

We define the linear measure of a set $E\subset[0,+\infty)$ by $m(E)=\int_0^{+\infty}\chi_E(t)dt$, where χ_E is the characteristic function of E.

Many authors ([5], [7], [9]) have studied the second order linear differential equation

$$f''+h_1(z)e^{P(z)}f'+h_0(z)e^{Q(z)}f=0, \tag{2}$$

where $P(z)$ and $Q(z)$ are nonconstant polynomials, $h_1(z)$ and $h_0(z)(\not\equiv 0)$ are entire functions satisfying $\sigma(h_1)<\deg P$ and $\sigma(h_0)<\deg Q$. Gundersen showed in [7, *p.* 419] that if $\deg P\neq\deg Q$, then every nonconstant solution of the linear differential equation (2) is of infinite order. If $\deg P=\deg Q$, then equation (2) may have nonconstant solutions of finite order. Indeed, $f(z)=z$ satisfies $f''-z^3e^zf'+z^2e^zf=0$.

K. H. Kwon considered the case where $\deg P=\deg Q$ and proved the following result:

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Theorem 1.1. ([9]) *Let $P(z)$ and $Q(z)$ be nonconstant polynomials such that*

$$P(z) = a_n z^n + ... + a_1 z + a_0, \tag{3}$$

$$Q(z) = b_n z^n + ... + b_1 z + b_0, \tag{4}$$

where a_i, b_i $(i = 0, 1, ..., n)$ are complex numbers, $a_n \neq 0$ and $b_n \neq 0$. Let $h_j(z)$ $(j = 0, 1)$ be entire functions with $\sigma(h_j) < n$. Suppose that $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0 < c < 1)$. Then every nonconstant solution f of equation (2) is of infinite order and satisfies $\sigma_2(f) \geq n$.

In [4], Belaïdi and Abbas have studied some higher order linear differential equations with entire coefficients and have proved the following result:

Theorem 1.2. ([4]) *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, ..., k-1)$ be nonconstant polynomials with degree $n \geq 1$, where $a_{0,j}, ..., a_{n,j}$ $(j = 0, 1, ..., k-1)$ are complex numbers such that $a_{n,j} a_{n,s} \neq 0$ $(j \neq s)$ $(1 \leq s \leq k-1)$. Let $h_j(z) (\not\equiv 0)$ $(j = 0, 1, ..., k-1)$ be entire functions with $\sigma(h_j) < n$. Suppose that $\arg a_{n,j} \neq \arg a_{n,s}$ or $a_{n,j} = c_j a_{n,s}$ $(0 < c_j < 1)$ $(j \neq s)$. Then every transcendental solution f of equation*

$$f^{(k)} + \sum_{j=0}^{k-1} h_j(z) e^{P_j(z)} f^{(j)} = 0 \tag{5}$$

is of infinite order and satisfies $\sigma_2(f) = n$. Furthermore, if $\max\{c_1, ..., c_{s-1}\} < c_0$, then every solution $f (\not\equiv 0)$ of equation (5) is of infinite order and satisfies $\sigma_2(f) = n$.

In 2008, J. Tu and C. F. Yi have also considered equation (5) and obtained the following result:

Theorem 1.3. ([10]) *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, ..., k-1)$ be polynomials with degree $n \geq 1$, where $a_{0,j}, ..., a_{n,j}$ $(j = 0, 1, ..., k-1)$ are complex numbers. Let $h_j(z)$ $(j = 0, 1, ..., k-1)$ be entire functions with $\sigma(h_j) < n$. Suppose that there exist nonzero complex numbers $a_{n,s}$ and $a_{n,l}$ such that $0 \leq s < l \leq k-1$, $a_{n,s} = |a_{n,s}| e^{i\theta_s}$, $a_{n,l} = |a_{n,l}| e^{i\theta_l}$, $\theta_s, \theta_l \in [0, 2\pi)$, $\theta_s \neq \theta_l$, $h_s h_l \not\equiv 0$ and for $j \neq s, l$, $a_{n,j}$ satisfies either $a_{n,j} = d_j a_{n,s}$ $(0 < d_j < 1)$ or $a_{n,j} = d_j a_{n,l}$ $(0 < d_j < 1)$. Then every transcendental solution f of equation (5) satisfies $\sigma(f) = +\infty$. Furthermore, if f is a polynomial solution of equation (5), then $\deg f \leq s - 1$; if $s = 1$, then every nonconstant solution f of equation (5) satisfies $\sigma(f) = +\infty$.*

In this paper, we continue the research in this type of problems. The main purpose of this paper is to extend and improve the above results to some nonhomogeneous higher order linear differential equations with meromorphic coefficients. We will prove the following two results:

Theorem 1.4. *Let $k \geq 2$ be an integer, $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, \dots, k-1)$ be polynomials with degree $n \geq 1$, where $a_{0,j}, ..., a_{n,j}$ $(j = 0, ..., k-1)$ are complex numbers. Let $h_j(z)$ $(j = 0, ..., k-1)$ and F $(\not\equiv 0)$ be meromorphic functions having only finitely many poles with $\max\{\sigma(F), \sigma(h_j) : j = 0, ..., k-1\} < n$. Suppose that there exists an integer $s \in \{1, 2, ..., k-1\}$ such that $h_0 h_s \not\equiv 0$ and $a_{n,j} = c_j a_{n,s}$ $(0 < c_j < 1)$ $(j \neq s)$. Then every transcendental meromorphic solution of equation*

$$f^{(k)} + \sum_{j=0}^{k-1} h_j(z) e^{P_j(z)} f^{(j)} = F \tag{6}$$

is of infinite order. Furthermore, if $\max\{c_1, ..., c_{s-1}\} < c_0$*, then every meromorphic solution* $f(\not\equiv 0)$ *of equation* (6) *is of infinite order.*

Theorem 1.5. *Let* $k \geq 2$ *be an integer,* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, \ldots, k-1)$ *be polynomials with degree* $n \geq 1$*, where* $a_{0,j}, ..., a_{n,j}$ $(j = 0, ..., k-1)$ *are complex numbers. Let* $h_j(z)$ $(j = 0, ..., k-1)$ *and* F $(\not\equiv 0)$ *be meromorphic functions having only finitely many poles with* $\max\{\sigma(F), \sigma(h_j) : j = 0, ..., k-1\} < n$*. Suppose that there exist two integers* s, d *such that* $1 \leq s < d \leq k-1$*,* $h_s h_d \not\equiv 0$ *and* $a_{n,s} \neq a_{n,d}$*. Let* I *and* J *be two sets satisfying* $I \neq \emptyset$*,* $J \neq \emptyset$*,* $I \cap J = \emptyset$ *and* $I \cup J = \{0, \ldots, k-1\}/\{s, d\}$ *such that for* $j \in I$*,* $a_{n,j} = \alpha_j a_{n,s}$ $(0 < \alpha_j < 1)$ *and for* $j \in J$*,* $a_{n,j} = \beta_j a_{n,d}$ $(0 < \beta_j < 1)$*. Set* $a_{n,l} = |a_{n,l}| e^{i\theta_l}$*,* $\theta_l \in [0, 2\pi)$ $(l = s, d)$ *and* $\alpha = \max\{\alpha_j : j \in I\}$*.*
If $(\theta_s \neq \theta_d)$ *or* $(\theta_s = \theta_d$ *and* $|a_{n,d}| < (1-\alpha)|a_{n,s}|)$*, then every transcendental meromorphic solution of equation* (6) *is of infinite order. Furthermore, if* f *is a polynomial solution of* (6)*, then* $\deg f \leq s - 1$*.*

2. Preliminary lemmas

Lemma 2.1. ([1]) *Let* $P_j(z)$ $(j = 0, 1, ..., k)$ *be polynomials with* $\deg P_0(z) = n$ $(n \geq 1)$ *and* $\deg P_j(z) \leq n$ $(j = 1, 2, ..., k)$*. Let* $A_j(z)$ $(j = 0, 1, ..., k)$ *be meromorphic functions with finite order and* $\max\{\sigma(A_j) : j = 0, 1, ..., k\} < n$ *such that* $A_0(z) \not\equiv 0$*. We denote*

$$F(z) = A_k(z) e^{P_k(z)} + A_{k-1}(z) e^{P_{k-1}(z)} + ... + A_1(z) e^{P_1(z)} + A_0(z) e^{P_0(z)}. \quad (7)$$

If $\deg(P_0(z) - P_j(z)) = n$ *for all* $j = 1, ..., k$*, then* F *is a nontrivial meromorphic function with finite order and satisfies* $\sigma(F) = n$*.*

Lemma 2.2. ([6]) *Let* $f(z)$ *be a transcendental meromorphic function of finite order* σ*. Let* $\Gamma = \{(k_1, j_1), (k_2, j_2), ..., (k_m, j_m)\}$ *denotes a set of distinct pairs of integers satisfying* $k_i > j_i \geq 0$ $(i = 1, 2, ..., m)$ *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E_1 \subset [0, 2\pi)$ *that has linear measure zero such that if* $\theta \in [0, 2\pi) \setminus E_1$*, then there is a constant* $R_1 = R_1(\theta) > 1$ *such that for all* z *satisfying* $\arg z = \theta$ *and* $|z| \geq R_1$ *and for all* $(k, j) \in \Gamma$*, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}. \quad (8)$$

Lemma 2.3. ([3]) *Let* $P(z) = (\alpha + i\beta) z^n + ...$ $(\alpha, \beta$ *are real numbers,* $|\alpha| + |\beta| \neq 0)$ *be a polynomial with degree* $n \geq 1$ *and* $A(z)$ *be a meromorphic function with* $\sigma(A) < n$*. Set* $f(z) = A(z) e^{P(z)}$*,* $z = re^{i\theta}$*,* $\delta(P, \theta) = \alpha \cos(n\theta) - \beta \sin(n\theta)$*. Then for any given* $\varepsilon > 0$*, there exists a set* $E_2 \subset [0, 2\pi)$ *that has linear measure zero such that for any* $\theta \in [0, 2\pi) \setminus E_2 \cup H$*, where* $H = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ *is a finite set, there is a constant* $R_2 > 1$ *such that for* $|z| = r \geqslant R_2$*, we have*
(i) *if* $\delta(P, \theta) > 0$*, then*

$$\exp\{(1-\varepsilon)\delta(P, \theta) r^n\} \leq \left|f\left(re^{i\theta}\right)\right| \leq \exp\{(1+\varepsilon)\delta(P, \theta) r^n\}, \quad (9)$$

(ii) *if* $\delta(P, \theta) < 0$*, then*

$$\exp\{(1+\varepsilon)\delta(P, \theta) r^n\} \leq \left|f\left(re^{i\theta}\right)\right| \leq \exp\{(1-\varepsilon)\delta(P, \theta) r^n\}. \quad (10)$$

Lemma 2.4. ([2]) *Let $p \geq 1$ be an integer, $f(z)$ be a meromorphic function having only finitely many poles and suppose that*

$$G(z) = \frac{\log^{+}\left|f^{(p)}(z)\right|}{|z|^{\rho}}$$

is unbounded on some ray $\arg z = \theta$ with constant $\rho > 0$. Then there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, ...)$, where $r_m \to +\infty$ such that $G(z_m) \to \infty$ and

$$\left|\frac{f^{(j)}(z_m)}{f^{(p)}(z_m)}\right| \leqslant \frac{1}{(p-j)!}(1 + o(1))\,|z_m|^{p-j} \quad (j = 0, ..., p-1) \ \text{as } m \to +\infty. \tag{11}$$

Lemma 2.5. ([11]) *Let $f(z)$ be an entire function of finite order. Suppose that there exists a set $E_3 \subset [0, 2\pi)$ that has linear measure zero such that for any ray $\arg z = \theta \in [0, 2\pi) \setminus E_3$,*

$$\log^{+}\left|f\left(re^{i\theta}\right)\right| \leq Mr^{\sigma}, \tag{12}$$

where M (> 0) is a constant depending on θ and σ (> 0) is a constant independent of θ. Then $\sigma(f) \leq \sigma$.

3. Proof of Theorem 1.4

Proof. First we prove that every transcendental meromorphic solution f of equation (6) is of order $\sigma(f) \geq n$. Assume that f is a transcendental meromorphic solution f of equation (6) of order $\sigma(f) < n$. We can write equation (6) as

$$\sum_{j=0}^{k-1} h_j(z) f^{(j)} e^{P_j(z)} = B(z), \tag{13}$$

where $B = -f^{(k)} + F$ and $h_j f^{(j)}$ $(j = 0, 1, ..., k-1)$ are meromorphic functions of finite order with $\sigma\left(h_j f^{(j)}\right) < n$ $(j = 0, 1, ..., k-1)$ and $\sigma(B) < n$. We have $h_s f^{(s)} \not\equiv 0$. Indeed, if $h_s f^{(s)} \equiv 0$, it follows that $f^{(s)} \equiv 0$. Then f has to be a polynomial of degree less than s. This is a contradiction. Since $a_{n,j} = c_j a_{n,s}$ $(0 < c_j < 1)$ $(j \neq s)$, we get that $\deg(P_s(z) - P_j(z)) = n$ $(j \neq s)$. Thus by (13) and Lemma 2.1, we have $\sigma(B) = n$ and this contradicts the fact that $\sigma(B) < n$. Hence every transcendental meromorphic solution f of equation (6) is of order $\sigma(f) \geq n$.

Now Assume that f is a transcendental meromorphic solution of equation (6) with $\sigma(f) = \sigma < +\infty$. By Lemma 2.2, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $\theta \in [0, 2\pi) \setminus E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_1$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq |z|^{k\sigma} \quad (0 \leq i < j \leq k). \tag{14}$$

By Lemma 2.3, for any given $\varepsilon > 0$, there exists a set $E_2 \subset [0, 2\pi)$ that has linear measure zero such that if $z = re^{i\theta}$, $\theta \in [0, 2\pi) \setminus E_2 \cup H_1$ and r is sufficiently large, then $h_j e^{P_j(z)}$ $(j = 0, 1, ..., k-1)$ satisfy (9) or (10), where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0\}$.

Set $\rho = \max\{\sigma(F), \sigma(h_j) : j = 0, 1, ..., k-1\}$ and $c = \max\{c_j : j \neq s\}$. Since F

is a meromorphic function with only finitely many poles, then by Hadamard factorization theorem, we can write $F(z) = \frac{H(z)}{Q(z)}$, where $Q(z)$ is a polynomial with $\deg Q(z) = p \geq 1$ and $H(z)$ is an entire function with $\sigma(H) = \sigma(F)$.

For any given $\theta \in [0, 2\pi) \setminus E_1 \cup E_2 \cup H_1$, we have

$$\delta(P_s, \theta) > 0 \text{ or } \delta(P_s, \theta) < 0.$$

Case 1. $\delta(P_s, \theta) > 0$. For any given ε $\left(0 < 3\varepsilon < \min\left\{\frac{1-c}{1+c}, n - \rho\right\}\right)$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have

$$\left|h_s(z)e^{P_s(z)}\right| \geq \exp\{(1-\varepsilon)\delta(P_s, \theta) r^n\} \tag{15}$$

and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)c\delta(P_s, \theta)r^n\} \quad (j \neq s). \tag{16}$$

Now we prove that

$$G(z) = \frac{\log^+ \left|f^{(s)}(z)\right|}{|z|^{\rho+\varepsilon}}$$

is bounded on some ray $\arg z = \theta$. If $G(z)$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, ...)$, where $r_m \to +\infty$ such that $G(z_m) \to \infty$ and

$$\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right| \leqslant \frac{1}{(s-j)!}(1 + o(1))|z_m|^{s-j} \quad (j = 0, ..., s-1) \text{ as } m \to +\infty. \tag{17}$$

Since $G(z_m) \to \infty$, for sufficiently large number $A > 0$, we have

$$\left|f^{(s)}(z_m)\right| > \exp\left\{A|z_m|^{\rho+\varepsilon}\right\} \text{ as } m \to +\infty. \tag{18}$$

From (18), we have for m sufficiently large

$$\left|\frac{F(z_m)}{f^{(s)}(z_m)}\right| = \left|\frac{H(z_m)}{Q(z_m) f^{(s)}(z_m)}\right| \leq \frac{|H(z_m)|}{\lambda_1 r_m^p \exp\left\{A|z_m|^{\rho+\varepsilon}\right\}} \leq \frac{|H(z_m)|}{\exp\left\{A|z_m|^{\rho+\varepsilon}\right\}}, \tag{19}$$

where $\lambda_1 (> 0)$ is a constant. Since $\sigma(H) \leq \rho$, we obtain

$$\left|\frac{F(z_m)}{f^{(s)}(z_m)}\right| \leq \frac{|H(z_m)|}{\exp\left\{A|z_m|^{\rho+\varepsilon}\right\}} \to 0 \text{ as } m \to +\infty. \tag{20}$$

By (6), we get

$$\begin{aligned}
\left|h_s(z)e^{P_s(z)}\right| &\leq \left|\frac{f^{(k)}(z)}{f^{(s)}(z)}\right| + \left|h_{k-1}(z)e^{P_{k-1}(z)}\right|\left|\frac{f^{(k-1)}(z)}{f^{(s)}(z)}\right| \\
&+ ... + \left|h_{s+1}(z)e^{P_{s+1}(z)}\right|\left|\frac{f^{(s+1)}(z)}{f^{(s)}(z)}\right| + \left|h_{s-1}(z)e^{P_{s-1}(z)}\right|\left|\frac{f^{(s-1)}(z)}{f^{(s)}(z)}\right| \\
&+ \cdots + \left|h_1(z)e^{P_1(z)}\right|\left|\frac{f'(z)}{f^{(s)}(z)}\right| + \left|h_0(z)e^{P_0(z)}\right|\left|\frac{f(z)}{f^{(s)}(z)}\right| + \left|\frac{F(z)}{f^{(s)}(z)}\right|.
\end{aligned} \tag{21}$$

Substituting (14) – (17) and (20) into (21), we have for the above z_m

$$\begin{aligned}
&\exp\{(1-\varepsilon)\delta(P_s, \theta) r_m^n\} \\
&\leq M_1 r_m^{d_1} \exp\{(1+\varepsilon)c\delta(P_s, \theta) r_m^n\},
\end{aligned} \tag{22}$$

where M_1 (>0), d_1 (>0) are constants. From (22) and $0<\varepsilon<\dfrac{1-c}{3(1+c)}$, we get

$$\exp\{\frac{(1-c)}{3}\delta\left(P_s,\theta\right)r_m^n\}\leq M_1 r_m^{d_1}. \tag{23}$$

This is a contradiction. Therefore $G\left(z\right)$ is bounded on $\arg z=\theta$. Hence

$$\left|f^{(s)}\left(z\right)\right|\leq\exp\left\{M\left|z\right|^{\rho+\varepsilon}\right\} \tag{24}$$

on $\arg z=\theta$, where M (>0) is a constant. By (24) and (s)-fold iterated integration, we conclude that

$$\left|f\left(z\right)\right|\leq\exp\left\{M\left|z\right|^{\rho+2\varepsilon}\right\} \tag{25}$$

on $\arg z=\theta$.

Case 2. $\delta\left(P_s,\theta\right)<0$. For any given ε $(0<3\varepsilon<\min\{1,n-\rho\})$ and all z satisfying $\arg z=\theta$ and $|z|=r$ is sufficiently large , we have

$$\left|h_j(z)e^{P_j(z)}\right|\leq\exp\{(1-\varepsilon)\delta(P_j,\theta)r^n\}\ \ (j=0,...,k-1)\,. \tag{26}$$

By (6), we get

$$-1=h_{k-1}(z)e^{P_{k-1}(z)}\frac{f^{(k-1)}(z)}{f^{(k)}(z)}$$
$$+...+h_s(z)e^{P_s(z)}\frac{f^{(s)}(z)}{f^{(k)}(z)}+...+h_0(z)e^{P_0(z)}\frac{f(z)}{f^{(k)}(z)}+\frac{F\left(z\right)}{f^{(k)}(z)}. \tag{27}$$

Now we prove that

$$D\left(z\right)=\frac{\log^+\left|f^{(k)}\left(z\right)\right|}{\left|z\right|^{\rho+\varepsilon}}$$

is bounded on some ray $\arg z=\theta$. If $D\left(z\right)$ is unbounded on the ray $\arg z=\theta$, then by Lemma 2.4, there exists an infinite sequence of points $z_m=r_m e^{i\theta}$ $(m=1,2,...)$, where $r_m\to+\infty$ such that $D\left(z_m\right)\to\infty$ and

$$\left|\frac{f^{(j)}\left(z_m\right)}{f^{(k)}\left(z_m\right)}\right|\leqslant\frac{1}{(k-j)!}\left(1+o\left(1\right)\right)\left|z_m\right|^{k-j}\ \ (j=0,...,k-1)\ \text{ as } m\to+\infty. \tag{28}$$

From $D\left(z_m\right)\to\infty$, for sufficiently large number $B>0$, we have

$$\left|f^{(k)}\left(z_m\right)\right|>\exp\left\{B\left|z_m\right|^{\rho+\varepsilon}\right\}\ \text{ as } m\to+\infty. \tag{29}$$

By using the same reasoning as above, from (29), we have for m sufficiently large

$$\left|\frac{F\left(z_m\right)}{f^{(k)}\left(z_m\right)}\right|=\left|\frac{H\left(z_m\right)}{Q\left(z_m\right)f^{(k)}\left(z_m\right)}\right|\leq\frac{\left|H\left(z_m\right)\right|}{\exp\left\{B\left|z_m\right|^{\rho+\varepsilon}\right\}}. \tag{30}$$

Since $\sigma\left(H\right)\leq\rho$, we obtain

$$\left|\frac{F\left(z_m\right)}{f^{(k)}\left(z_m\right)}\right|\leq\frac{\left|H\left(z_m\right)\right|}{\exp\left\{B\left|z_m\right|^{\rho+\varepsilon}\right\}}\to 0\text{ as } m\to+\infty. \tag{31}$$

Substituting (14), (26), (28) and (31) into (27), we obtain as $r_m\to+\infty$

$$1\leq 0.$$

This is a contradiction. Therefore $D(z)$ is bounded on $\arg z = \theta$. Hence

$$\left|f^{(k)}(z)\right| \leq \exp\left\{M|z|^{\rho+\varepsilon}\right\} \tag{32}$$

on $\arg z = \theta$, where M (> 0) is a constant. By (32) and (k)-fold iterated integration, we obtain (25) on $\arg z = \theta$.

From equation (6), we know that the poles of f can only occur at the poles of F and $h_j(z)$ $(j = 0, 1, ..., k-1)$. Since F and $h_j(z)$ $(j = 0, 1, ..., k-1)$ are meromorphic functions having only finitely many poles, then f must have only finitely many poles. Therefore by Hadamard factorization theorem, we can write $f(z) = \frac{g(z)}{R(z)}$, where $R(z)$ is a polynomial and $g(z)$ is an entire function with $\sigma(g) = \sigma(f) \geq n$. From (25), we have

$$|g(z)| \leq \lambda_2 r^q \exp\left\{M|z|^{\rho+2\varepsilon}\right\} \tag{33}$$

on $\arg z = \theta$, where λ_2 (> 0) is a constant and $q = \deg R \geq 1$. Hence

$$|g(z)| \leq \exp\left\{M|z|^{\rho+3\varepsilon}\right\} \tag{34}$$

on $\arg z = \theta$. Therefore for any given $\theta \in [0, 2\pi) \setminus E_1 \cup E_2 \cup H_1$, where $E_1 \cup E_2 \cup H_1$ is a set of linear measure zero, we have (34) on $\arg z = \theta$. Then by Lemma 2.5, we have $\sigma(g) \leq \rho + 3\varepsilon < n$ and this contradicts the fact that $\sigma(g) \geq n$. Hence $\sigma(f) = +\infty$.

Suppose now that $\max\{c_1, ..., c_{s-1}\} < c_0$. if f is a rational solution of (6), then by $\max\{c_1, ..., c_{s-1}\} < c_0$ and

$$f = \frac{F}{h_0 e^{P_0(z)}} - \left(\frac{e^{-P_0(z)}}{h_0} f^{(k)} + \frac{h_{k-1}}{h_0} e^{P_{k-1}(z)-P_0(z)} f^{(k-1)} + ... + \frac{h_1}{h_0} e^{P_1(z)-P_0(z)} f'\right), \tag{35}$$

we obtain a contradiction since the left side of equation (35) is a rational function but the right side is a transcendental meromorphic function.

Now we prove that equation (6) cannot have a nonzero polynomial solution. Set $c' = \max\{c_1, ..., c_{s-1}\} < c_0$ and let $f(z)$ be a nonzero polynomial solution of equation (6) with $\deg f(z) = d$. We take a ray $\arg z = \theta \in [0, 2\pi) \setminus H_1$, where H_1 is defined as above such that $\delta(P_s, \theta) > 0$. By Lemma 2.3, for any given ε $(0 < 3\varepsilon < \min\{\frac{1-c}{1+c}, \frac{c_0-c'}{c_0+c'}, n-\rho\})$, there exists a set E_2 having linear measure zero such that for all z with $\arg z = \theta \in [0, 2\pi) \setminus E_2 \cup H_1$ and $|z| = r$ sufficiently large, we have (15) and (16).

If $d \geq s$, by (6), (15) and (16), we obtain for all z with $\arg z = \theta \in [0, 2\pi) \setminus E_2 \cup H_1$ and $|z| = r$ sufficiently large

$$B_1 r^{d-s} \exp\{(1-\varepsilon)\delta(P_s, \theta) r^n\} \leq |h_s(z) e^{P_s(z)} f^{(s)}(z)| \leq \sum_{j\neq s} |h_j(z) e^{P_j(z)} f^{(j)}(z)| + |F(z)|$$

$$\leq B_2 r^d \exp\{(1+\varepsilon) c\delta(P_s, \theta) r^n\} + \frac{\exp\{r^{\rho+\varepsilon}\}}{\lambda_1 r^p}, \tag{36}$$

where B_1 (> 0), B_2 (> 0) are constants. By (36), we get

$$\exp\left\{\frac{(1-c)}{3}\delta(P_s, \theta) r^n\right\} \leq B_3 r^{d_2} \exp\{r^{\rho+\varepsilon}\}, \tag{37}$$

where $B_3(>0)$ and d_2 are constants. Hence (37) is a contradiction.

If $d < s$, by (6), (15) and (16), we obtain for all z with $\arg z = \theta \in [0, 2\pi) \setminus E_2 \cup H_1$, and $|z| = r$ sufficiently large

$$B_4 r^{s-1} \exp\{(1-\varepsilon)c_0\delta(P_s,\theta)r^n\} \le |h_0(z)e^{P_0(z)}||f(z)| \le \sum_{j=1}^{s-1} |h_j(z)e^{P_j} f^{(j)}(z)| + |F(z)|$$

$$\le B_5 r^{s-2} \exp\{(1+\varepsilon)c'\delta(P_s,\theta)r^n\} + \frac{\exp\{r^{\rho+\varepsilon}\}}{\lambda_1 r^p}, \tag{38}$$

where B_4 (>0), B_5 (>0) are constants. By (38), we get

$$\exp\{\frac{(c_0-c')}{2}\delta(P_s,\theta)r^n\} \le \frac{B_6}{r}\exp\{r^{\rho+\varepsilon}\}, \tag{39}$$

where $B_6(>0)$ is a constant. This is a contradiction. Therefore, if $\max\{c_1, ..., c_{s-1}\} < c_0$, then every meromorphic solution of equation (6) is of infinite order. □

4. Proof of Theorem 1.5

Proof. First we prove that every transcendental meromorphic solution f of equation (6) is of order $\sigma(f) \ge n$. Assume that f is a transcendental meromorphic solution f of equation (6) of order $\sigma(f) < n$. We can write equation (6) in the form (13), where $B = -f^{(k)} + F$ and $h_j(z)f^{(j)}$ $(j = 0, 1, ..., k-1)$ are meromorphic functions of finite order with $h_s f^{(s)} \not\equiv 0$, $h_d f^{(d)} \not\equiv 0$, $\sigma(h_j f^{(j)}) < n$ $(j = 0, 1, ..., k-1)$ and $\sigma(B) < n$.

We have $\deg(P_s(z) - P_j(z)) = n$ $(j \ne s)$. Thus by (13) and Lemma 2.1, we obtain $\sigma(B) = n$ and this contradicts the fact that $\sigma(B) < n$. Hence every transcendental meromorphic solution f of equation (6) is of order $\sigma(f) \ge n$.

Assume f is a transcendental solution of equation (6) with $\sigma(f) = \sigma < +\infty$. By Lemma 2.2, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $\theta \in [0, 2\pi) \setminus E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \ge R_1$, we have (14). By Lemma 2.3, for any given $\varepsilon > 0$, there exists a set $E_2 \subset [0, 2\pi)$ that has linear measure zero such that if $z = re^{i\theta}$, $\theta \in [0, 2\pi) \setminus E_2 \cup H_2$ and r is sufficiently large, then $h_j e^{P_j(z)}$ $(j = 0, 1, ..., k-1)$ satisfy (9) or (10), where $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0 \text{ or } \delta(P_d, \theta) = 0\}$. Set $\rho = \max\{\sigma(F), \sigma(h_j) : j = 0, 1, ..., k-1\}$. Since F is a meromorphic function with only finitely many poles, then by Hadamard factorization theorem, we can write $F(z) = \frac{H(z)}{Q(z)}$, where $Q(z)$ is a polynomial with $\deg Q = p \ge 1$ and $H(z)$ is an entire function with $\sigma(H) = \sigma(F)$.

Case 1. Suppose that $\theta_s \ne \theta_d$. Set $H_3 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}$. Since $\theta_s \ne \theta_d$, then H_3 has linear measure zero. For any given $\theta \in [0, 2\pi) \setminus E_1 \cup E_2 \cup H_2 \cup H_3$, we have

$$\delta(P_s, \theta) \ne 0, \ \delta(P_d, \theta) \ne 0 \text{ and } \delta(P_s, \theta) > \delta(P_d, \theta) \text{ or } \delta(P_s, \theta) < \delta(P_d, \theta).$$

Set $\delta_1 = \delta(P_s, \theta)$ and $\delta_2 = \delta(P_d, \theta)$.

Subcase 1.1. $\delta_1 > \delta_2$. Here we also divide our proof into three subcases:

(a) $\delta_1 > \delta_2 > 0$. Set $\delta_3 = \max\{\delta(P_j, \theta) : j \neq s\}$. Then $0 < \delta_3 < \delta_1$. Thus for any given ε $\left(0 < 3\varepsilon < \min\left\{\frac{\delta_1 - \delta_3}{\delta_1 + \delta_3}, n - \rho\right\}\right)$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have

$$\left|h_s(z)e^{P_s(z)}\right| \geq \exp\{(1-\varepsilon)\delta_1 r^n\} \tag{40}$$

and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)\delta_3 r^n\} \ \ (j \neq s) \tag{41}$$

Now we prove that

$$G(z) = \frac{\log^+ \left|f^{(s)}(z)\right|}{|z|^{\rho+\varepsilon}}$$

is bounded on some ray $\arg z = \theta$. If $G(z)$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, ...)$, where $r_m \to +\infty$ such that $G(z_m) \to \infty$ and (17) holds. From (18), we have (19) for m sufficiently large. Since $\sigma(H) \leq \rho$, we obtain (20). Substituting (14), (17), (20), (40) and (41) into (21), for the above z_m, we obtain

$$\exp\{(1-\varepsilon)\delta_1 r_m^n\} \leq M_2 r_m^{d_3} \exp\{(1+\varepsilon)\delta_3 r_m^n\}, \tag{42}$$

where M_2 (> 0), d_3 (> 0) are constants. From (42) and $0 < \varepsilon < \frac{\delta_1 - \delta_3}{3(\delta_1 + \delta_3)}$, we get

$$\exp\{\frac{(\delta_1 - \delta_3)}{3} r_m^n\} \leq M_2 r_m^{d_3}. \tag{43}$$

This is a contradiction. Therefore $G(z)$ is bounded on $\arg z = \theta$. Hence (25) holds on $\arg z = \theta$.

(b) $\delta_1 > 0 > \delta_2$. Thus for any given ε $(0 < 3\varepsilon < \min\left\{\frac{1-\alpha}{1+\alpha}, n - \rho\right\})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (40),

$$\left|h_d(z)e^{P_d(z)}\right| \leq \exp\{(1-\varepsilon)\delta_2 r^n\} < 1, \tag{44}$$

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)\alpha\delta_1 r^n\} \ \ (j \in I) \tag{45}$$

and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1-\varepsilon)\delta(P_j, \theta) r^n\} < 1 \ \ (j \in J). \tag{46}$$

Now we prove that

$$G(z) = \frac{\log^+ \left|f^{(s)}(z)\right|}{|z|^{\rho+\varepsilon}}$$

is bounded on some ray $\arg z = \theta$. If $G(z)$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, ...)$, where $r_m \to +\infty$ such that $G(z_m) \to \infty$ and (17) holds. Substituting (14), (17), (20), (40), (44) – (46) into (21), for the above z_m, we obtain

$$\exp\{(1-\varepsilon)\delta_1 r_m^n\} \leq M_3 r_m^{d_4} \exp\{(1+\varepsilon)\alpha\delta_1 r_m^n\}, \tag{47}$$

where M_3 (> 0) and d_4 (> 0) are constants. From (47) and $0 < \varepsilon < \dfrac{1-\alpha}{3(1+\alpha)}$, we get

$$\exp\{\frac{(1-\alpha)}{3}\delta_1 r_m^n\} \leq M_3 r_m^{d_4}. \tag{48}$$

This is a contradiction. Therefore $G(z)$ is bounded on $\arg z = \theta$. Hence (25) holds on $\arg z = \theta$.

(c) $0 > \delta_1 > \delta_2$. For any given ε $(0 < 3\varepsilon < \min\{1, n-\rho\})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (26). Using similar reasoning as in case 2 in the proof of Theorem 1.4, (25) holds on $\arg z = \theta$.

Subcase 1.2. $\delta_1 < \delta_2$. Using the same reasoning as in subcase 1.1, we can also obtain (25) on $\arg z = \theta$.

Case 2. Suppose that $\theta_s = \theta_d$ and $|a_{n,d}| < (1-\alpha)|a_{n,s}|$. For any given $\theta \in [0, 2\pi) \setminus E_1 \cup E_2 \cup H_2$, where E_1, E_2 and H_2 are defined above, we have

$$\delta(P_s, \theta) > 0 \text{ or } \delta(P_s, \theta) < 0.$$

Subcase 2.1. $\delta(P_s, \theta) > 0$. For any given ε $(0 < 3\varepsilon < \min\left\{\frac{(1-\alpha)|a_{n,s}|-|a_{n,d}|}{(1+\alpha)|a_{n,s}|+|a_{n,d}|}, n-\rho\right\})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (15),

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)\alpha\delta(P_s,\theta)r^n\} \ (j \in I), \tag{49}$$

and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)\beta\delta(P_d,\theta)r^n\} \qquad (j \in J \cup \{d\}) \tag{50}$$

Now we prove that

$$G(z) = \frac{\log^+ \left|f^{(s)}(z)\right|}{|z|^{\rho+\varepsilon}}$$

is bounded on some ray $\arg z = \theta$. If $G(z)$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, ...)$, where $r_m \to +\infty$ such that $G(z_m) \to \infty$ and (17) holds. Substituting (14), (15), (17), (20), (49) and (50) into (21), for the above z_m, we obtain

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta(P_s,\theta)r_n\} \\ &\leq M_4 r_m^{d_5} \exp\{(1+\varepsilon)\alpha\delta(P_s,\theta)r_m^n\}\exp\{(1+\varepsilon)\delta(P_d,\theta)r_m^n\}, \end{aligned} \tag{51}$$

where M_4, d_5 (>0) are constants. By (51), we have

$$\exp\{\gamma r_m^n\} \leq M_4 r_m^{d_5}, \tag{52}$$

where

$$\gamma = (1-\varepsilon)\delta(P_s,\theta) - (1+\varepsilon)\delta(P_d,\theta) - (1+\varepsilon)\alpha\delta(P_s,\theta). \tag{53}$$

Since $0 < \varepsilon < \dfrac{(1-\alpha)|a_{n,s}| - |a_{n,d}|}{3[(1+\alpha)|a_{n,s}| + |a_{n,d}|]}$, $\theta_s = \theta_d$ and $\cos(\theta_s + n\theta) > 0$, we obtain

$$\begin{aligned} \gamma &= \{(1-\alpha)|a_{n,s}| - |a_{n,d}| - \varepsilon[(1+\alpha)|a_{n,s}| + |a_{n,d}|]\}\cos(\theta_s + n\theta) \\ &> \frac{((1-\alpha)|a_{n,s}| - |a_{n,d}|)}{3}\cos(\theta_s + n\theta) > 0. \end{aligned}$$

Since $\gamma > 0$, then (52) is a contradiction. Therefore $G(z)$ is bounded on $\arg z = \theta$. Hence (25) holds on $\arg z = \theta$.

Subcase 2.2 $\delta(P_s, \theta) < 0$. Using the same reasoning as in case 2 in the proof of Theorem 1.4, (25) holds on $\arg z = \theta$.

From equation (6), we know that the poles of f can only occur at the poles of F and $h_j(z)$ $(j = 0, 1, ..., k-1)$. Since F and $h_j(z)$ $(j = 0, 1, ..., k-1)$ are meromorphic functions having finitely many poles, then f must have only finitely many poles. Therefore by Hadamard factorization theorem, using similar arguments as in the proof of Theorem 1.4, we can write $f(z) = \frac{g(z)}{R(z)}$, where $R(z)$ is a polynomial and $g(z)$ is an entire function with $\sigma(g) = \sigma(f) \geq n$. From (25), we have (34) on $\arg z = \theta$. Then by Lemma 2.5, we have $\sigma(g) \leq \rho + 3\varepsilon < n$ and this contradicts the fact that $\sigma(g) \geq n$. Hence $\sigma(f) = +\infty$.

In the following, we show that if $f(z)$ is a polynomial solution of (6), then $\deg f \leq s-1$. Assume that f is a polynomial solution of (6) with $\deg f = b \geq s$.

(a) Assume that $\theta_s \neq \theta_d$.

(i) If $\theta_s \neq \theta_d + \pi$ or $\theta_d \neq \theta_s + \pi$, set $H_4 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) > \delta(P_d, \theta) > 0\}$. Then $m(H_4) > 0$. We can choose a curve $\Gamma = \{z : \arg z = \theta \in H_4\}$.
By (6), (40) and (41), for all $z \in \Gamma$ and $|z| = r$ sufficiently large, we obtain

$$\begin{aligned} B_7 r^{b-s} \exp\{(1-\varepsilon)\delta_1 r^n\} &\leq |h_s(z) e^{P_s(z)} f^{(s)}(z)| \\ &\leq \sum_{j\neq s} |h_j(z) e^{P_j(z)} f^{(j)}(z)| + \left|\frac{H(z)}{Q(z)}\right| \\ &\leq B_8 r^{d_6} \exp\{(1+\varepsilon)\delta_3 r^n\} \frac{\exp\{r^{\rho+\varepsilon}\}}{\lambda_1 r^p}, \end{aligned} \tag{54}$$

where $B_7(>0)$, $B_8(>0)$,d_6 and λ_1 are constants.
By (54), we obtain

$$\exp\left\{\frac{(\delta_1-\delta_3)}{3} r^n\right\} \leq B_9 r^{d_6} \exp\{r^{\rho+\varepsilon}\}, \tag{55}$$

where $B_9(>0)$ and d_6 are constants. This is a contradiction.

(ii) If $\theta_s = \theta_d + \pi$ or $\theta_d = \theta_s + \pi$, set $H_5 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) > 0 > \delta(P_d, \theta)\}$. Then $m(H_5) > 0$. We can choose a curve $G = \{z : \arg z = \theta \in H_5\}$.
By (6), (40) and (44) – (46), for all $z \in G$ and $|z| = r$ sufficiently large, we obtain

$$B_{10} r^{b-s} \exp\{(1-\varepsilon)\delta_1 r^n\} \leq B_{11} r^{d_7} \exp\{(1+\varepsilon)\alpha\delta_1 r^n\} \exp\{r^{\rho+\varepsilon}\}, \tag{56}$$

where $B_{10}(>0)$, $B_{11}(>0)$ and d_7 are constants.
By (56), we obtain

$$\exp\left\{\frac{(1-\alpha)}{3}\delta_1 r^n\right\} \leq B_{12} r^{d_8} \exp\{r^{\rho+\varepsilon}\}, \tag{57}$$

where $B_{12}(>0)$ and d_8 are constants. This is a contradiction.

(b) Assume that $\theta_s = \theta_d$ and $|a_{n,d}| < (1-\alpha)|a_{n,s}|$. We can take a ray $\arg z = \theta$ such that $\delta(P_s, \theta) > 0$. Thus $\delta(P_s, \theta) > \delta(P_d, \theta) > 0$.
By (6), (40) and (41), for all z with $\arg z = \theta$ and $|z| = r$ sufficiently large, we obtain (55) which is a contradiction.

Hence every polynomial solution f of (6) satisfies $\deg f \leq s-1$. □

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