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Hausdorff topological BE-algebras

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Abstract. In this paper, we introduce the notion of (semi) topological BE-algebras and derive here conditions that imply a BE-algebra to be a (semi) topological BE-algebra. We prove that for each cardinal number α there is at least a (semi) topological BE-algebra of order α. Also we study separation axioms on (semi) topological BE-algebras and show that for any infinite cardinal number α there is a Hausdorff (semi) topological BE-algebra of order α with nontrivial topology.

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1. Introduction

In 1966, Y. Imai and K. Iséki in [7] introduced a class of algebras of type $(2, 0)$ called BCK-algebras which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of impliction algebra. K. Iséki posed an interesting problem whether the class of BCK-algebras form a variety. In connection with this problem Y. Komori in [10] introduced a notion of BCC-algebras which is a generalization of notion BCK-algebras and proved that class of all BCC-algebras is not a variety. In [2], H. S. Kim introduced the concept of BE-algebra as a generalization of dual BCK-algebra. In [8], S. S. Ahn and K. S. So introduced ideals and upper sets in BE-algebras and discussed several properties of ideals. In [15] A. Walendziak introduced commutative BE-algebras and discussed some of its properties. H. S. Kim and K. J. Lee in [9] generalized the notions of upper sets and introduced the concept of extended upper sets and with the help of this concept they gave several discriptions of filters in BE-algebras.

Algebra and topology are the two fundamental domains of mathematics. Many of the most important objects of mathematics represent a blend of algebraic objects and topological structures. Topological groups, topological fields and topological lattices are objects of this kind. The rules that describe the relation between a topology and algebraic operation are almost always transparent and natural- the operation has to be continuous or separately continuous. In this paper, we will define (left, right, semi) topological BE-algebras and show that for each cardinal number α there is at least a topological BE-algebra of order α. In section 5, we will study connection between (semi) topological BE-algebras and T_i spaces, when $i = 0, 1, 2, 5/2$. We prove that for

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any infinte cardinal number α there is at least a Hausdorff topological BE-algebra of order α which its topology is nontrivial.

2. Preliminary

In this section we collect the relevant definitions and results from topology and BE-algebras theory to make this paper self-contained and easy to read. The material can be found in [6, 15, 14, 9, 8, 3, 7].

Topological Space

Recall that a set A with a family $\mathcal{U}$ of its subsets is called a *topological space*, denoted by $(A,\mathcal{U})$, if $A,\emptyset \in \mathcal{U}$, the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called *open sets* of A and the complement of $U \in \mathcal{U}$, that is $A \setminus U$, is said to be a *closed set.* If B is a subset of A, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$. A subfamily $\{U_\alpha : \alpha \in I\}$ of $\mathcal{U}$ is said to be a *base* of $\mathcal{U}$ if for each $x \in U \in \mathcal{U}$, there exists an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{U}$ is the union of members of $\{U_\alpha\}$. A subset P of A is said to be a *neighborhood* of $x \in A$, if there exists an open set U such that $x \in U \subseteq P$. A *directed set* I is a partially ordered set such that, for any i and j of I, there is a $k \in I$ with $k \geq i$ and $k \geq j$. If I is a directed set, then the subset $\{x_i : i \in I\}$ of A is called a *net.* A net $\{x_i ; i \in I\}$ *converges* to $x \in A$ if for each neighborhood U of x, there exists a $j \in I$ such that for all $i \geq j$, $x_i \in U$. If $B \subseteq A$ and $x \in \overline{B}$, then there is a net in B that converges to x.

Topological space $(A,\mathcal{U})$ is said to be a:

(i) T_0-space if for each $x \neq y \in A$, there is at least one in an open neighborhood excluding the other,

(ii) T_1-space if for each $x \neq y \in A$, each has an open neighborhood not containing the other,

(iii) *Hausdorff* space if for each $x \neq y \in A$, there two disjoint open neighborhoods U, V of x and y, respectively,

(iv) *Uryshon* space if for each $x \neq y \in A$, there are two open neighborhoods U, V of x and y, respectively, such that $\overline{U} \cap \overline{V} = \phi$.

BE- Algebras

A *BE-algebra* is a non empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms, for all $x, y, z \in X$:

(BE1) $x * x = 1$,

(BE2) $x * 1 = 1$,

(BE3) $1 * x = x$,

(BE4) $x * (y * z) = y * (x * z)$.

Definition 2.1. Let $(X, *, 1)$ be a BE-algebra, then X is said to be:

(i) *transitive* if for any $x, y, z \in X$, $(y * z) \leq (x * y) * (x * z)$,

(ii) *self distributive* if for any $x, y, z \in X$, $x * (y * z) = (x * y) * (x * z)$,

(iii) *commutative* if for any $x, y, z \in X$, $(x * y) * y = (y * x) * x$,

(iv) *bounded* with unit 0, if $0 \in X$ and $0 * x = 1$, for every $x \in X$.

In a bounded BE-algebra, $x * 0$ denoted by x' and $(x')'$ by x''. On any BE-algebra X one define:

$$x \leq y \Leftrightarrow x * y = 1.$$

If X is a commutative BE-algebra, then the relation $\leq$ is a partial order on X.

Definition 2.2. Let $(X, *, 1)$ be a BE-algebra and $I \subseteq X$. The set I is called *ideal* when :
(i) if $a \in I$, then for each $x \in X$, $x * a \in I$
(ii) if $a, b \in I$, and $y \in I, then$ $(a * (b * x)) * x \in I$.
If I is an ideal in X, then $x \in I$ and $x \leq y$ imply $y \in I$.

Definition 2.3. A subset F of X is called a *filter* when it satisfies the conditions:
(F1) $1 \in F$,
(F2) if $x, x * y \in F$ then $y \in F$.
If F is a filter in X, then $x \in F$ and $x \leq y$ imply $y \in F$.
A filter F of X is said to be *normal* if for each $x, y, z \in X$,

$$x * y \in F \Rightarrow [(z * x) * (z * y) \in F, \quad (y * z) * (x * z) \in F.$$

Define the binary operations $\vee, \wedge$, and $+$ on X as the following: for any $x, y \in X$,

$$x \vee y = (y * x) * x, \quad x \wedge y = (x' \vee y')', \quad x + y = (x * y) * y.$$

In BE-algebra X, for any $x, y, z \in X$, the following hold:
(B1) $x \leq y * x$,
(B2) $x \leq ((x * y) * y)$,
(B3) $(x * y) \leq (y \vee x) * y$,
if X is self distributive, then:
(B4) $x \leq y$ implies $z * x \leq z * y$ and $y * z \leq x * z$,
(B5) $(y \vee x) * y \leq (x * y)$,
(B6) $x * (x * y) = x * y$,
if X is a bounded BE-algebra with unit 0, then:
(B7) $1' = 0, 0' = 1$,
(B8) $x \leq x''$,
(B9) $x * y' = y * x'$,
(B10) $x \vee 0 = x$,
if X is a bounded and self distributive BE-algebra with unit 0, we have:
(B11) $x * y \leq y' * x'$,
(B12) $x \leq y$ implies $y' \leq x'$,
if X is a commutative BE-algebra, we have:
(B13) $x * (x + y) = 1$,
(B14) $x * y = y * z = 1 \Rightarrow x * z = 1$,
(B15) $x * y = 1 \Rightarrow (x + z) * (y + z) = 1$,
(B16) $x * z = y * z = 1 \Rightarrow (x + y) * z = 1$,
if X is a bounded and commutative BE-algebra, the following hold:
(B17) $x'' = x$,
(B18) $x' \wedge y' = (x \vee y)'$,
(B19) $x' \vee y' = (x \wedge y)'$,
(B20) $x' * y' = y * x$.
If X is a commutative or self distributive BE-algebra, then it is transitive.
If X is a transitive BE-algebra, then every filter of X is normal.
Let F be a filter in BE-algebra X, in the following way we define the binary relation $\equiv^F$ on X :

$$x \equiv^F y \Leftrightarrow x * y \in F, \ y * x \in F,$$

if F is a normal filter, then $\equiv^F$ is a congruence relation, i.e. $\equiv^F$ is an equeivalence relation and for each $a, b, x, y \in X$, if $x \equiv^F y$ and $a \equiv^F b$, then $a * x \equiv^F b * y$. In this case, if $F(x) = \{y \in X : x \equiv^F y\}$, then $X/F = \{F(x) : x \in X\}$ is a BE-algebra with the following operation:

$$F(x) * F(y) = F(x * y).$$

3. Topological BE-algebras

Definition 3.1. Let $\mathcal{T}$ be a topology on a BE-algebra $(X, *, 1)$. Then:
(i) $(X, *, \mathcal{T})$ is *(right) left topological* BE-algebra if $x * y \in U \in \mathcal{T}$, then there is a (V) $W \in \mathcal{T}$ such that $(x \in V)$ $y \in W$ and $(V * y \subseteq U)$ $x * W \subseteq U$. In this case, we also say that $*$ is *continuous in (first)second variable*,
(ii) $(X, *, \mathcal{T})$ is *semitopological* BE-algebra if it is left and right topological BE-algebra, i.e. if $x * y \in U \in \mathcal{T}$, then there are $V, W \in \mathcal{T}$ such that $x \in V, y \in W$ and $x * W \subseteq U$ and $V * y \subseteq U$. In this case we also say that $*$ is *continuous in each variable separately*,
(iii) $(X, *, \mathcal{T})$ is *topological* BE-algebra if $*$ is continuous , i.e. if $x * y \in U \in \mathcal{T}$, then there are two neighborhoods V, W of x, y, respectively, such that $V * W \subseteq U$.

Example 3.1. Let $X = \{1, a, b\}$ be a BE-algebra with the following table:

*	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then $\mathcal{T} = \{\{1\}, \{a, b\}, X, \phi\}$ and $\mathcal{U} = \{\{1, a\}, \{b\}, X, \phi\}$ are two topologies on X such that $(X, *, \mathcal{T})$ is a topological BE-algebra and $(X, *, \mathcal{U})$ is a left topological BE-algebra. Moreover, $(X, *, \mathcal{U})$ is not a right topological BE-algebra.

Let $(X, *, 1)$ be a BE-algebra. Then:
(i) a family Ω of subsets of X is *prefilter* if for each $U, V \in \Omega$, there exists a $W \in \Omega$ such that $W \subseteq U \cap V$,
(ii) for each $V \subseteq X$ and $x \in X$, we denote

$$V[x] = \{y \in X : y * x \in V\} \quad V(x) = \{y \in X : y * x, x * y \in V\}.$$

Theorem 3.1. *Let $\mathcal{F}$ be a prefilter of normal filters in a BE-algebra $(X, *, 1)$. Then there is a topology $\mathcal{T}$ on X such that $(X, *, \mathcal{T})$ is a topological BE-algebra.*

Proof. Define $\mathcal{T} = \{U \subseteq X : \forall x \in U\ \exists F \in \mathcal{F}\ s.t\ F(x) \subseteq U\}$. For each $x \in X$ and $F \in \mathcal{F}$, the set $F(x) \in \mathcal{T}$ because if y is an arbitrary element of $F(x)$, then $F(y) \subseteq F(x)$. It is easy to see that $\mathcal{T}$ is a topology on X. We prove that $*$ is continuous. For this, suppose $x*y \in U \in \mathcal{T}$, then for some $F \in \mathcal{F}$, $F(x*y) \subseteq U$. Now $F(x)$ and $F(y)$ are two open neighborhoods of x and y, respectively, such that $F(x)*F(y) \subseteq F(x*y) \subseteq U$. □

Corollary 3.2. *Let $\mathcal{F}$ be a prefilter of filters in BE-algebra X. If X is commutative or self distributive or transitive BE-algebra, then there a topology $\mathcal{T}$ on X such that $(X, *, \mathcal{T})$ is a topological BE-algebra.*

Proof. If X is commutative or self distributive or transitive, then $\mathcal{F}$ is a prefilter of normal filters in X. By Theorem 3.1, there exists a topology $\mathcal{T}$ on X such that $(X, *, \mathcal{T})$ is a topological BE-algebra. □

Theorem 3.3. *Let F be a filter in commutative or self distributive BE-algebra $(X, *, 1)$. Then there is a topology $\mathcal{T}$ on X such that $(X, *, 1, \mathcal{T})$ is a left topological BE-algebra.*

Proof. Clearly X is a transitive BE-algebra. Let $\mathcal{T} = \{U \subseteq X : \forall x \in U \ F[x] \subseteq U\}$. First we show that for any $x \in X$, $F[x] \in \mathcal{T}$. Suppose $x \in X$ and $y \in F[x]$, then $y * x \in F$. Take $z \in F[y]$. By transitivity , $(y * x) * ((z * y) * (z * x)) = 1 \in F$. Hence $(z * y) * (z * x) \in F$. Since F is filter and $(z * y) * (z * x)$ and $z * y$, both, are in F, $z * x$ is in F so. Hence $F[y] \subseteq F[x]$. This implies that $F[x] \in \mathcal{T}$. Now we prove that $*$ is continuous in second variable. Let $x * y \in U \in \mathcal{T}$, then $F[x * y] \subseteq U$. If $z \in F[y]$, then $z * y \in F$. By transitivity $z * y \leq (x * z) * (x * y)$, hence $x * z \in F[x * y]$. This proves that $x * F[y] \subseteq F[x * y] \subseteq U$. □

Theorem 3.4. *Let $(X, *, 1, \mathcal{T})$ be a topological BE-algebra and $0 \notin X$. Suppose $X_0 = X \cup \{0\}$ and $\mathcal{T}^* = \mathcal{T} \setminus \{\phi\}$. If $1 \in \cap\mathcal{T}^*$, then there are an operation $\otimes$ and a topology $\mathcal{T}_0$ on X_0 such that $(X_0, \otimes, 1, \mathcal{T}_0)$ is a topological bounded self distributive BE-algebra and $1 \in \cap\mathcal{T}_0^*$.*

Proof. Define the operation $\otimes$ on X_0 by

$$x \otimes y = \begin{cases} x * y & \text{if } x, y \in X \\ 0 & \text{if } x \in X, y = 0 \\ 1 & \text{if } x = 0, y \in X \\ 1 & \text{if } x = y = 0. \end{cases}$$

Assume that $\mathcal{T}_0 = \{U \cup \{0\} : U \in \mathcal{T}\} \cup \{\phi\}$. It is easy to verify that $(X_0, \otimes, 1)$ is a bounded self distributive BE-algebra and $\mathcal{T}_0$ is a topology on X_0. Let $x \otimes y \in U \cup \{0\}$. In the following cases we find two sets $V, W \in \mathcal{T}_0$ such that $x \in V$, $y \in W$ and $V \otimes W \subseteq U \cup \{0\}$.
Case 1. If $x, y \in X$, then $x * y = x \otimes y \in U$. Since $*$ is continuous, there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V * W \subseteq U$. If $z_1 \in V \cup \{0\}$ and $z_2 \in W \cup \{0\}$, then $z_1 \otimes z_2 \in \{z_1 * z_2, 0, 1\} \subseteq U \cup \{0\}$. Hence $V \cup \{0\} \otimes W \cup \{0\} \subseteq U \cup \{0\}$.
Case 2. If $x \in X$ and $y = 0$, then $x \in X_0 \in \mathcal{T}_0$, $y = 0 \in \{0\} \in \mathcal{T}_0$ and $X_0 \otimes \{0\} = \{1, 0\} \subseteq U \cup \{0\}$.
Case 3. If $x = 0$ and $y \in X$, then $x = 0 \in \{0\} \in \mathcal{T}_0$, $y \in X_0 \in \mathcal{T}_0$ and $\{0\} \otimes X_0 = \{1, 0\} \subseteq U \cup \{0\}$.
Case 4. If $x = y = 0$, then $x = y = 0 \in \{0\} \in \mathcal{T}_0$ and $\{0\} \otimes \{0\} = \{1\} \subseteq U \cup \{0\}$.
The Cases $1, 2, 3$ and 4 prove that $(X_0, \otimes, \mathcal{T}_0)$ is a topological BE-algebra. But it is obvious that $1 \in \cap\mathcal{T}_0^*$. □

Theorem 3.5. *For any integer $n \geq 3$ there exists a topological bounded self distributive commutative BE-algebra of order n.*

Proof. Let $X = \{1, a\}$ be the self distributive commutative BE-algebra with the following table:

*	1	a
1	1	a
a	1	1

Then $\mathcal{T} = \{X, \phi\}$, is a topology on X such that $(X, *, \mathcal{T})$ is a topological BE-algebra. Let $u_1 \notin X$. Since $1 \in \cap\mathcal{T}^*$, by Theorem 3.4, there is an operation $\otimes$ and a topology

$\mathcal{T}_1$ on $X_1 = X \cup \{u_1\}$ such that $(X_1, *, \mathcal{T}_1)$ is a topological bounded self distributive BE-algebra of order 3 with unit u_1 and $1 \in \cap \mathcal{T}^*{}_1$.

Take $(X_n, *, \mathcal{T}_n)$ a topological bounded self distributive BE-algebra of order n with unit u_n such that $1 \in \cap \mathcal{T}^*{}_n$. Let $X_{n+1} = X_n \cup \{u_{n+1}\}$, where $u_{n+1} \notin X_n$. By Theorem 3.4, there is a topology $\mathcal{T}_{n+1}$ on X_{n+1} such that $(X_{n+1}, *, \mathcal{T}_{n+1})$ is a topological bounded self distributive commutative BE-algebra of order $n+1$ with unit u_{n+1} and $1 \in \cap \mathcal{T}^*{}_{n+1}$.

□

Theorem 3.6. *Let α be an infinite cardinal number. Then there is a topological self distributive BE-algebra of order α.*

Proof. Let X be a set with cardinal number α. Consider $X^0 = \{x_0 = 1, x_1, x_2, ...\}$ as a countable subset of X and define the operation $*$ on X^0 by

$$x_i * x_j = \begin{cases} 1 & \text{if } i = j \\ x_j & \text{if } i \neq j. \end{cases}$$

Then $(X^0, *, 1)$ is a self distributive BE-algebra. The set $F_n = \{1, x_1, ..., x_n\}$, for any $n \geq 1$ is a normal filter of X^0. Since $B_0 = \{F_n : n \geq 1\}$ is a prefilter of normal filters in X^0, by Theorem 3.1, there is a nontrivial topology $\mathcal{T}^0$ on X^0 such that $(X^0, *, \mathcal{T}^0)$ is a topological BE-algebra. Now define the binary operation $\circ$ on X by

$$x \circ y = \begin{cases} x * y & \text{if } x, y \in X^0 \\ y & \text{if } x \in X^0, y \notin X^0 \\ 1 & \text{if } x \notin X^0, y \in X^0 \\ 1 & \text{if } x = y \notin X^0 \\ y & \text{if } x \neq y, x, y \notin X^0. \end{cases}$$

It is routine to check that $(X, \circ, 1)$ is a self distributive BE-algebra of order α and the set $B = \mathcal{T}^0 \cup \{\{x\} : x \notin X^0\}$ is a subbase for a topology $\mathcal{T}$ on X. Since $\{1\} \notin \mathcal{T}$, $\mathcal{T}$ is a nontrivial topology on X. In the following cases we will show that $(X, \circ, \mathcal{T})$ is a topological BE-algebra. For this, let $x \circ y \in U \in B$.

Case 1. If $x, y \in X^0$, then $x \circ y = x * y \in U \in \mathcal{T}^0$. Since $*$ is continuous in $(X^0, \mathcal{T}^0)$, there are $V, W \in \mathcal{T}^0$ containing x, y, respectively, such that $V * W \subseteq U$. Hence $V \circ W \subseteq U$ which implies that $\circ$ is continuous in $(X, \mathcal{T})$.

Case 2. If $x \in X^0$ and $y \notin X^0$, then X^0 and $\{y\}$ are two elements of $\mathcal{T}$ such that $x \in X^0$, $y \in \{y\}$ and $X^0 \circ \{y\} = \{y\} \subseteq U$.

Case 3. If $x \notin X^0$ and $y \in X^0$, then $x \circ y = 1 \in U$. Now $\{x\}$ and X^0, both, belong to $\mathcal{T}$ and $x \in \{x\}$, $y \in X^0$ and $\{x\} \circ X^0 = \{1\} \subseteq U$.

Case 4. If $x = y \notin X^0$, then $x \circ y = 1 \in U$. Then $\{x\}$ is an open set in $\mathcal{T}$ which contains x, y and $\{x\} \circ \{x\} = \{1\} \subseteq U$.

Case 5. If $x \notin X^0$ and $y \notin X^0$, then $x \in \{x\} \in \mathcal{T}$ and $y \in \{y\} \in \mathcal{T}$ and $\{x\} \circ \{y\} \subseteq U$.

The cases $1, 2, 3, 4, 5$ show that operation $\circ$ is continuous in $(X, \mathcal{T})$. □

Theorem 3.7. *Let $(X, *, 1, \mathcal{T})$ be a topological BE-algebra and α be a cardinal number. If $\alpha \geq |X|$, then there is a topological BE-algebra $(Y, \circ, 1, \mathcal{U})$ such that $\alpha \leq |Y|$ and X is a subalgebra of Y.*

Proof. Suppose

$$\Gamma = \{(H, \circledast, 1, \mathcal{U}) : (H, \circledast, 1, \mathcal{U}) \textit{ is a topological } BE-algebra, \ X \subseteq H \ \ \circledast|_X = *\}.$$

The following relation is a prtial order on Γ.

$$(H, \circledast, 1, \mathcal{U}) \leq (K, \odot, 1, \mathcal{V}) \Leftrightarrow H \subseteq K,\ \odot|_H = \circledast,\ \mathcal{U} \subseteq \mathcal{V}.$$

Let $\{(H_i, \circledast_i, 1, \mathcal{U}_i) : i \in I\}$ be a chain in Γ. Put $H = \cup H_i$ and $\mathcal{U} = \cup\, \mathcal{U}_i$. If x and y are two elements of H, then for some $i \in I$, $x, y \in H_i$. Define $x \circledast y = x \circledast_i y$. We prove that $\circledast$ is an operation on H. Suppose $x, y \in H_i \cap H_j$. Since $\{(H_i, \circledast_i, 1, \mathcal{U}_i) : i \in I\}$ is a chain, $H_i \subseteq H_j$ or $H_j \subseteq H_i$. Without the lost of generality, assume that $H_i \subseteq H_j$. Then $\circledast_j|_{H_i} = \circledast_i$. So $x \circledast_j y = x \circledast_i y$. This proves that $\circledast$ is an operation on H. Now it is easy to see that $(H, \circledast, 1)$ is a BE-algebra such that $\circledast|_X = *$. On the other hand, Since $\{(H_i, \circledast_i, 1, \mathcal{U}_i) : i \in I\}$ is a chain, $\mathcal{U}$ is a topology on H. We prove that $(H, \circledast, \mathcal{U})$ is a topological BE-algebra. Let $x \circledast y \in U \in \mathcal{U}$. Then there is an $i \in I$ such that $x \circledast y = x \circledast_i y \in U \in \mathcal{U}_i$. Since $\circledast_i$ is continuous in $(H_i, \mathcal{U}_i)$, there are $V, W \in \mathcal{U}_i$ such that $x \in V$, $y \in W$, and $V \circledast_i W \subseteq U$. This proves that $\circledast$ is continuous in $(H, \mathcal{U})$. Thus $(H, \circledast, 1, \mathcal{U})$ is an upper bound for $\{(H_i, \circledast_i, 1, \mathcal{U}_i) : i \in I\}$ in Γ. By Zorn's Lemma, Γ has a maximal element. Suppose $(Y, \circ, 1, \mathcal{U})$ is a maximal element of Γ. We prove that $|Y| \geq \alpha$. If $|Y| < \alpha$, then for some nonempty set C, $|Y \cup C| = \alpha$. Take $a \in Y \setminus C$ and put $H = Y \cup \{a\}$. Then it is easy to claim that H with the following operation is a BE-algebra.

$$x \circledast y = \begin{cases} x \circ y & \text{if } x, y \in Y \\ a & \text{if } x \in Y, y = a \\ 1 & \text{if } x = a, y \in Y \\ 1 & \text{if } x = y = a. \end{cases}$$

The set $B = \mathcal{U} \cup \{\{a\}\}$ is a subbase for a topology $\mathcal{V}$ on H. In the following cases we prove that $(H, \circledast, \mathcal{V})$ is a topological BE-algebra. Let $x, y \in H$ and $x \circledast y \in U \in B$.

Case 1. If $U \in \mathcal{U}$, then or x, y, both, are in Y or $x \in Y$ and $y = a$ or $x = y = a$. If $x, y \in Y$, then since $\circ$ is continuous in $(Y, \mathcal{U})$, there are $V, W \in \mathcal{U} \subseteq B$ such that $x \in V$, $y \in W$ and $V \circledast W = V \circ W \subseteq U$. If $x \in Y$ and $y = a$, then Y and $\{a\}$ are two open sets in $\mathcal{V}$ containing x and y, respectively, such that $x \circledast y \in Y \circledast \{a\} = \{a\} \subseteq U$. If $x = y = a$, then $\{a\}$ is an open neighborhood of x, y in $\mathcal{V}$ such that $\{a\} \circledast\ \{a\} = \{1\} \subseteq U$.

Case 2. If $U = \{a\}$, then $x = a \in \{a\} \in B$ and $y \in Y \in B$ and $x \circledast y \in \{a\} \circledast Y \subseteq U$. Thus by cases $1, 2$, $(H, \circledast, \mathcal{V})$ is a topological BE-algebra. But $(H, \circledast, \mathcal{V})$ is a member of Γ which $(Y, \circ, 1, \mathcal{U}) < (H, \circledast, 1, \mathcal{V})$, a contradiction. Therefore, $|Y| \geq \alpha$ and X is a subalgebra of Y. □

Theorem 3.8. *Let α be an infinite cordinal number. Then there is a left topological BE-algebra of order α which is not a topological BE-algebra.*

Proof. Let X be a set with cardinal number α. Suppose $X^0 = \{x_0 = 1, x_1, x_2, ...\}$ is a countabel subset of X. Define

$$x_i * x_j = \begin{cases} 1 & \text{if } i \leq j \\ x_j & \text{if } i > j. \end{cases}$$

It is easy to prove that $(X^0, *, 1)$ is a BE-algebra. If $U_i = \{x_i, x_{i+1}, x_{i+2}, ...\}$, then $B = \{U_i : i = 0, 2, 3, ...\}$ is a base for a topology $\mathcal{T}^0$ on X^0. We prove that $(X^0, *, \mathcal{T}^0)$ is a left topological BE-algebra. Let $x_i * x_j \in U \in \mathcal{T}^0$. If $i \leq j$, then $x_i * x_j = 1 \in U$. Since X^0 is only open neighborhood of 1, $U = X^0$. Cleraly, $x_j \in X^0$ and $x_i * X^0 \subseteq U$. If $i > j$, then $x_i * x_j = x_j$. Since B is a base for $\mathcal{T}^0$, $x_j \in U_j \subseteq U$. Since $i > j$, $x_i * U_j = U_j \subseteq U$. Therefore, $(X^0, *, \mathcal{T}^0)$ is a left topological BE-algebra. But this space is not topological BE-algebra because $x_1 \in U_1$, $x_2 \in U_2$ and $x_1 * x_2 = x_1 \in U_1$ but $U_1 * U_2 \nsubseteq U_1$. Consider X with the following operation

$$x \circ y = \begin{cases} x * y & \text{if } x, y \in X^0 \\ y & \text{if } x \in X^0, y \notin X^0 \\ 1 & \text{if } x \notin X^0, y \in X^0 \\ 1 & \text{if } x = y \notin X^0 \\ y & \text{if } x \neq y, x, y \notin X^0, \end{cases}$$

then $(X, \circ, 1)$ is a BE-algebra. As the proof of Theorem 3.6, we can claim that $B = \mathcal{T}^0 \cup \{\{x\} : x \notin X^0\}$ is a subbase for a topology $\mathcal{T}$ on X such that $(X, \circ, \mathcal{T})$ is a left topological BE-algebra and it's not right topological BE-algebra. But $\circ$ is not continuous in $(X, \mathcal{T})$ because $*$ is not continuous in $(X^0, \mathcal{T}^0)$. □

Definition 3.2. Let $(X, *, 1)$ be a BE-algera. A non empty subset V on X is *BE-filter* if for each $x, y \in X$, $x \leq y$ and $x \in V$ imply $y \in V$.
Clearly every filter and each ideal is a BE-filter, but it's converse is not correct.

Proposition 3.9. *Let $(X, *, 1)$ be a BE-algebra. Then:*
(i) arbiterary unions and intersections of BE-filters in X is a prefilter in X,
(ii) if V is a BE-filter, then $1 \in V$,
(iii) if V is a BE-filter, then $X * V \subseteq V$.

Proof. The proof is easy. □

Theorem 3.10. *Let Ω be a family of BE-filters in self distributive BE-algebra $(X, *, 1)$ such that is closed under intersection. If for each $x \in V \in \Omega$, there is a $U \in \Omega$ such that $U[x] \subseteq V$, then there is a topology $\mathcal{T}$ on X such that $(X, *, 1, \mathcal{T})$ is a left topological BE-algebra.*

Proof. It is not difficult to prove that $\mathcal{T} = \{U \subseteq X : \forall x \in U,\ \exists V \in \Omega \ s.t\ V[x] \subseteq U\}$ is a topology on X. Let $U \in \Omega$ and $x \in X$. We show that $U[x] \in \mathcal{T}$. For this, suppose $y \in U[x]$, then $y * x \in U$. Consider $V \in \Omega$ such that $V[x * y] \subseteq U$. Let $z \in V[y]$. Since $z * y \leq (x * z) * (x * y)$ and $z * y \in V$, we get that $(x * z) * (x * y) \in V$. Hence $x*z \in V[x*y] \subseteq U$. This shows that $y \in V[y] \subseteq U[x]$. Therefore, $U[x]$ is an open set, for each $U \in \Omega$ and $x \in X$. Also, obviously, the set $B = \{U[x] : U \in \Omega, x \in X\}$ is a base for $\mathcal{T}$. Now we prove that $*$ is continuous in second variable. Let $y*x \in U[y*x] \in B$. If $z \in U[x]$, since $z*x \leq (y*z)*(y*x)$ and $z*x \in U$, we conclude that $(y*z)*(y*x) \in U$. So $y*z \in U[y*x]$. Thus, $y*U[x] \subseteq U[y*x]$. Therefore, $(X, *, 1, \mathcal{T})$ is a left topological BE-algebra. □

Theorem 3.11. *Let Ω be a family of BE-filters in self distributive BE-algebra $(X, *, 1)$ such that is closed under intersections. Let for each $x \in V \in \Omega$, there is a $U \in \Omega$ such that $U(x) \subseteq V$. Then there is a topology $\mathcal{T}$ on X such that $(X, *, 1, \mathcal{T})$ is a semitopological BE-algebra.*

Proof. Define $\mathcal{T} = \{U \subseteq X : \forall x \in U,\ \exists V \in \Omega \ s.t\ V(x) \subseteq U\}$. Easily, one can prove that $\mathcal{T}$ is a topology on X. We show that $B = \{U(x) : U \in \Omega, x \in X\}$ is a base for $\mathcal{T}$. Let $x \in U(a) \in B$. Then there exists a $V \in \Omega$ such that $V(x * a)$ and $V(a * x)$, both, are the subsets of U. We show that $x \in V(x) \subseteq U(a)$. Let $y \in V(x)$. Then $y * x$ and $x * y$ belong to V. Since X is transitive,

$$y * x \leq (a * y) * (a * x), \quad x * y \leq (a * x) * (a * y),$$

hence $(a*y)*(a*x)$ and $(a*x)*(a*y)$, both, belong to V. Hence $a*y \in V(a*x) \subseteq U$. On the other hand, by (B4),

$$(x*y)*[(y*a)*(x*a)] = (y*a)*[(x*y)*(x*a)] \geq (y*a)*(y*a) = 1,$$

so $(x*y) \leq (y*a)*(a*x)$. As $x*y \in V$, we get that $(y*a)*(x*a) \in V$. In a similar fashion, one can prove that $(x*a)*(y*a) \in V$. Hence $y*a \in V(x*a) \subseteq U$. Since $a*y$ and $y*a$, both, belong to U, we get that $y \in U(a)$. Thus we could show that $U(a) \in \mathcal{T}$, for each $a \in X$. Now it is easy to prove that B is a base for $\mathcal{T}$. In continue we will prove that $*$ is continuous in first and second variable. Let $y*x \in V(y*x) \in B$. We show that $y*V(x) \subseteq V(y*x)$ and $V(y)*x \subseteq V(y*x)$. If $a \in V(x)$, then since

$$a*x \leq (y*a)*(y*x), \quad x*a \leq (y*x)*(y*a),$$

we get that $(y*a)*(y*x) \in V$ and $(y*x)*(y*a) \in V$. Hence $y*a \in V(y*x)$ and so $y*V(x) \subseteq V(y*x)$. If $b \in V(y)$, since

$$b*y \leq (y*x)*(b*x), \quad y*b \leq (b*x)*(y*x),$$

we get that $b*x \in V(x*y)$. Hence $V(y)*x \subseteq V(y*x)$. □

Example 3.2. (i) An algebra $X = \{1, a, b, c\}$ defined by the table:

*	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

is a BE-algebra. BE-filters have the form $\{1,a\}, \{1,b\}, \{1,a,c\}$ and $\{1,b,c\}$. Let $U = \{1,b\}$ and $V = \{1,b,c\}$. Then V is not an filter because $c*a = 1 \in V$ but $a \notin V$. Also $V[1], V[b], U[c]$, all, are the subsets of V and the sets $U[1], U[b]$, both, are the subsets of U. Hence $\Omega = \{U, V\}$ satisfies in Theorem 3.10. Therefore, $\mathcal{T} = \{W : \forall x \in W,\ \exists G \in \Omega\ s.t\ G[x] \subseteq W\}$ is a topology on X such that $(X, *, 1, \mathcal{T})$ is a left topological BE-algebra.
(ii) It is easy to verify that $X = [1, \infty)$ by

$$x*y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

is a BE-algebra. Let $F_n = [1, n]$, for any $n \geq 1$. Then:

$$F_n(x) = \begin{cases} F_n & \text{if } x \leq n \\ \{x\} & \text{if } x > n \end{cases}$$

Now if $\Omega = \{F_n : n \geq 1\}$, then Ω satisfies in Theorem 3.11 and so there is a topology $\mathcal{T}$ on X such that $(X, *, 1, \mathcal{T})$ is a semitopological BE-algebra.

4. Separation axioms on topological commutative BE-algebra

Theorem 4.1. *Let α be an infinite cardinal number. Then there is a topological commutative self distributive BE-algebra of order α.*

Proof. Let X be a set with cardinal number α. Consider $X^0 = \{x_0 = 1, x_1, x_2, ...\}$ as a countable subset of X and define the operation $*$ on X^0 by

$$x_i * x_j = \begin{cases} 1 & \text{if } i = j \\ x_j & \text{if } i \neq j. \end{cases}$$

Then $(X^0, *, 1)$ is a commutative BE-algebra. The set $F_n = \{1, x_1, ..., x_n\}$, for any $n \geq 1$ is a normal filter of X^0. Since $B_0 = \{F_n : n \geq 1\}$ is a BE-filter of normal filters in X^0. By Theorem 3.1, there is a nontrivial topology $\mathcal{T}^0$ on X^0 such that $(X^0, *, \mathcal{T}^0)$ is a topological BE-algebra. Now define the binary operation $\circ$ on X by

$$x \circ y = \begin{cases} x * y & \text{if } x, y \in X^0 \\ y & \text{if } x \in X^0, y \notin X^0 \\ y & \text{if } x \notin X^0, y \in X^0 \\ 1 & \text{if } x = y \notin X^0 \\ y & \text{if } x \neq y, x, y \notin X^0. \end{cases}$$

It is routine to check that $(X, \circ, 1)$ is a commutative self distributive BE-algebra of order α and the set $B = \mathcal{T}^0 \cup \{\{x\} : x \notin X^0\}$ is a subbase for a topology $\mathcal{T}$ on X. Since $\{1\} \notin \mathcal{T}$, $\mathcal{T}$ is a nontrivial topology on X. In the following cases we will show that $(X, \circ, \mathcal{T})$ is a topological BE-algebra. For this, let $x \circ y \in U \in B$.

Case 1. If $x, y \in X^0$, then $x \circ y = x * y \in U \in \mathcal{T}^0$. Since $*$ is continuous in $(X^0, \mathcal{T}^0)$, there are $V, W \in \mathcal{T}^0$ containing x, y, respectively, such that $V * W \subseteq U$. Hence $V \circ W \subseteq U$ which implies that $\circ$ is continuous in $(X, \mathcal{T})$.

Case 2. If $x \in X^0$ and $y \notin X^0$, then X^0 and $\{y\}$ are two elements of $\mathcal{T}$ such that $x \in X^0$, $y \in \{y\}$ and $X^0 \circ \{y\} = \{y\} \subseteq U$.

Case 3. If $x \notin X^0$ and $y \in X^0$, then $x \circ y = y \in U$. Now $\{x\}$ and X^0, both, belong to $\mathcal{T}$ and $x \in \{x\}$, $y \in X^0$ and $\{x\} \circ X^0 = \{y\} \subseteq U$.

Case 4. If $x = y \notin X^0$, then $x \circ y = 1 \in U$. Then $\{x\}$ is an open set in $\mathcal{T}$ which contains x, y and $\{x\} \circ \{x\} = \{1\} \subseteq U$.

Case 5. If $x \notin X^0$, $y \notin X^0$ and $x \neq y$. then $\{x\}, \{y\} \in \mathcal{T}$ and $\{x\} \circ \{y\} \subseteq U$.

The cases $1, 2, 3, 4, 5$ show that operation $\circ$ is continuous in $(X, \mathcal{T})$. □

Theorem 4.2. *Let $\mathcal{T}$ be a topology on commutative BE-algebra $(X, *, 1)$. If for any $a \in X$ the map $l_a : X \hookrightarrow X$, by $l_a(x) = a * x$, is an open map, then $(X, \mathcal{T})$ is a T_0 space.*

Proof. Let $x \neq y \in X$ and U be an open neighborhood of 1. Then $U * x$ and $U * y$ are two open neighborhoods of x and y, respectively. If $x \in U * y$ and $y \in U * x$, then for some $a, b \in X$, by $(B1)$, $y \leq a * y = x$ and $x \leq b * x = y$. Hence $x = y$ which is a contradiction. Therefore, $x \notin U * y$ or $y \notin U * x$. This shows that $(X, \mathcal{T})$ is a T_0 space. □

Theorem 4.3. *Let $(X, *, 1, \mathcal{T})$ be a right (left) topological commutative BE-algebra. Then $(X, \mathcal{T})$ is a T_0 space iff, for any $x \neq 1$, there is a $U \in \mathcal{T}$ such that $x \in U$ and $1 \notin U$.*

Proof. Suppose for any $x \neq 1$, there is a $U \in \mathcal{T}$ such that $x \in U$ and $1 \notin U$. We prove that $(X, \mathcal{T})$ is a T_0 space. Given $x \neq y \in X$. Since X is commutative BE-algebra, $x * y \neq 1$ or $y * x \neq 1$. Suppose $x * y \neq 1$, then there exists a $U \in \mathcal{T}$ such that $x * y \in U$ and $1 \notin U$. Since $*$ is continuous in first variable, there an open set V containing x

such that $V * y \subseteq U$. y is not in V because if $y \in V$, then $1 = y * y \in V * y \subseteq U$, which is a contradiction. Hence $(X, \mathcal{T})$ is a T_0 space. The proof of converse is clear. □

Theorem 4.4. *If α is an infinite cardinal number, then there is a T_0 topological commutative BE-algebra of order α which its topology is nontrivial.*

Proof. Let $(X^0, *, \mathcal{T}^0)$ and $(X, \circ, \mathcal{T})$ be topological commutative BE-algebras in Theorem 4.1. It is clear that $\mathcal{T}$ is nontrivial. Let $x \in X \setminus \{1\}$. If $x \in X^0$, then for some $n \geq 1$, $x \notin F_n$. Hence $x \in F_n(x) \in \mathcal{T}$ and $1 \notin F_n(x)$. If $x \notin X^0$, then $x \in \{x\} \in \mathcal{T}$ and $1 \notin \{x\}$. Now by Theorem 4.3, $(X, \circ, \mathcal{T})$ is a T_0 topological commutative BE-algebra of order α, with nontrivial topology $\mathcal{T}$. □

Theorem 4.5. *Let $(X, *, 1, \mathcal{T})$ be a semitopological commutative BE-algebras. Then $(X, \mathcal{T})$ is a T_1 space if and only if for any $x \neq 1$, there are two open neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$.*

Proof. If $(X, \mathcal{T})$ is T_1, then the proof is obvious. Conversely, let for any $x \neq 1$, there are two open neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$. We prove that $(X, \mathcal{T})$ is a T_1 space. Given $x \neq y$, then $x * y \neq 1$ or $y * x \neq 1$. Without the lost of the generality, assume that $x * y \neq 1$. Then there are two open neighborhoods U and V of $x * y$ and 1, respectively, such that $x * y \notin V$ and $1 \notin U$. Since $*$ is continuous in each variable separately, there are W and W_1 belong to $\mathcal{T}$ such that $x \in W$, $y \in W_1$ and $W * y \subseteq U$ and $x * W_1 \subseteq U$. But $x \notin W_1$ because if $x \in W_1$, then $1 = x * x \in x * W_1 \subseteq U$, a contradiction. Similarly, $y \notin W$. Therefore, $(X, \mathcal{T})$ is a T_1 space. □

Theorem 4.6. *Let $(X, *, 1, \mathcal{T})$ be a semitopological commutative BE-algebras. Then $(X, \mathcal{T})$ is a T_1 space if and only if it is T_0 space.*

Proof. Let $(X, \mathcal{T})$ be a T_0 space and $x \neq y$. Then $x * y \neq 1$ or $y * x \neq 1$. Without the lost of the generality suppose $x * y \neq 1$. Then there is a $U \in \mathcal{T}$ such that $x * y \in U$ and $1 \notin U$ or $1 \in U$ and $x * y \notin U$. First assume that $x * y \in U$ and $1 \notin U$. Since $(X, *, \mathcal{T})$ is semitopological BE-algebra, there are two open neighborhoods V and W of x and y, respectinvely, such that $V * y \subseteq U$ and $x * W \subseteq U$. But $x \notin W$ because if $x \in W$, then $1 = x * x \in x * W \subseteq U$, a contradiction. Similarly, $y \notin V$. Now if $1 \in U$ and $x * y \notin U$, then since $x * x = y * y = 1 \in U$, there are open sets V and W such that $x \in V$, $y \in W$ and $V * y \subseteq U$ and $x * W \subseteq U$. If $x \in W$, then $x * y \in x * W \subseteq U$, a conteradiction. Similarly, $y \notin V$. Therefore, $(X, \mathcal{T})$ is a T_1 space. If $(X, \mathcal{T})$ is T_1, clearly it is T_0. □

Corollary 4.7. *If α is an infinite cardinal number, then there is a T_1 topological commutative BE-algebra of order α which its topology is nontrivial.*

Proof. By Theorems 4.4 and 4.6, the proof is clear. □

Theorem 4.8. *Let $(X, *, 1, \mathcal{T})$ be a topological commutative BE-algebra. Then $(X, \mathcal{T})$ is Hausdorff if and only if for each $x \neq 1$, there are two disjoint open neighborhoods U and V of x and 1, respectively.*

Proof. If $(X, \mathcal{T})$ is Hausdorff, the proof is clear. Conversely, let for each $x \neq 1$, there are two disjoint open neighborhoods U and V of x and 1, respectively. We prove that $(X, \mathcal{T})$ is Hausdorff. For this, take $x \neq y$. Then $x * y \neq 1$ or $y * x \neq 1$. Without the lost of the generality, we suppose that $x * y \neq 1$. Then there are two disjoint open neighborhoods U and V of $x * y$ and 1, respectively. Since $*$ is continuous, there are two open sets W and W_1 such that $x \in W$, $y \in W_1$ and $W * W_1 \subseteq U$. If $z \in W \cap W_1$, then $1 = z * z \in W * W_1 \subseteq U$, which is a contradiction. Hence $W \cap W_1 = \phi$. Therefore, $(X, \mathcal{T})$ is Hausdorff. □

Theorem 4.9. *Let $(X, *, 1, \mathcal{T})$ be a topological commutative BE-algebras. Then $(X, \mathcal{T})$ is a T_1 space if and only if it is Hausdorff space.*

Proof. Let $(X, *, 1, \mathcal{T})$ be T_1 topological BE-algebra. Given $x \neq 1$. Then there are $U, V \in \mathcal{T}$ such that $x \in U$, $1 \in V$ and $x \notin V$ and $1 \notin U$. There are two open neighborhoods W and W_1 such that $x \in W$, $1 \in W_1$ and $W * W_1 \subseteq U$. If $z \in W \cap W_1$, then $1 = z * z \in W * W_1 \subseteq U$, which is a contradiction. Hence $W \cap W_1 = \phi$. By Theorem 4.8, $(X, \mathcal{T})$ is Hausdorff. Conversely is obvious. □

Corollary 4.10. *If α is an infinite cardinal number, then there is a Hausdorff topological commutative BE-algebra of order α which its topology is nontrivial.*

Proof. By Theorems 4.4, 4.6 and 4.9, the proof is clear. □

Theorem 4.11. *Let $\mathcal{N}$ be a fundamental system of neighborhoods of 1 in topological commutative BE-algebra $(X, *, 1, \mathcal{T})$. The following conditions are equivalent.*
(i) $(X, \mathcal{T})$ is T_0 space,
(ii) $(X, \mathcal{T})$ is T_1 space,
(iii) $(X, \mathcal{T})$ is Hausdorff space,
(iv) $\cap\mathcal{N} = \{1\}$.

Proof. By Theorems 4.6, 4.9, $(i), (ii), (iii)$ are equivalent. We prove that (ii) and (iv) are equivalent. If $(X, \mathcal{T})$ is T_1 space and $x \neq 1$, then by Theorem 4.5, there is a $U \in \mathcal{N}$ such that $x \notin U$, hence $x \notin \cap\mathcal{N}$. Conversely, let $\cap\mathcal{N} = \{1\}$ and $x \neq 1$. Then there is a $V \in \mathcal{N}$ such that $x \notin V$. We show that $U(x)$ is an open set. For this let $a \in U(x)$. Then $x * a$ and $a * x$ belong to U. Since $(X, *, \mathcal{T})$ is semitopological BE-algebra, there is an open set W containing x such that $W * x$ and $x * W$ are two subsets of V. Hence $a \in W \subseteq U(x)$. Then $U(x)$ is an open neighborhood of x. But $1 \notin U(x)$ because $x \notin U$. Thus U and $U(x)$ are two open sets containing $1, x$, respectively, such that $x \notin U$ and $1 \notin U(x)$. By Theorem 4.5, $(X, \mathcal{T})$ is T_1 space. □

Theorem 4.12. *Topological commutative BE-algebra $(X, *, 1, \mathcal{T})$ is Uryshon space if and only if for any $x \neq 1$, there are two open sets U and V containing x and 1, respectively, such that $\overline{U} \cap \overline{V} = \phi$.*

Proof. Let for any $x \neq 1$, there are two open sets U and V containing x and 1, respectively, such that $\overline{U} \cap \overline{V} = \phi$. We prove that $(X, \mathcal{T})$ is Uryshon space. For this, suppose $x \neq y$. Then we can assume that $x * y \neq 1$. Take two open sets U and V such that $x * y \in U$ and $1 \in V$ and $\overline{U} \cap \overline{V} = \phi$. Since $*$ is continuous, there are open neighborhoods W and W_1 of x and y, respectively, $W * W_1 \subseteq U$. If $z \in \overline{W} \cap \overline{W_1}$, then there are two nets $\{x_i : i \in I\}$ and $\{y_i : i \in I\}$ in W and W_1, respectively, which converges to z. Now $\{x_i * y_i : i \in I\}$ is a net in U which converges to 1. Hence $1 \in \overline{U} \cap \overline{V} = \phi$, a contradiction. This proves that $(X, \mathcal{T})$ is Uryshon space. Conversely, is clear. □

Theorem 4.13. *Topological commutative BE-algebra* $(X, *, 1, \mathcal{T})$ *is Uryshon space if and only if it is Hausdorff.*

Proof. Let $(X, \mathcal{T})$ be Hausdorff space and $x \neq 1$. Then there are two disjoint open neighborhoods U and V of x and 1, respectively, there are two open sets W and W_1 such that $x \in W$ and $1 \in W_1$ and $W * W_1 \subseteq U$. We prove that $\overline{W} \cap \overline{W_1} = \phi$. Let $z \in \overline{W} \cap \overline{W_1}$. Then there are two nets $\{x_i : i \in I\} \subseteq W$ and $\{y_i : i \in I\} \subseteq W_1$, which both converges to z. Thus $\{x_i * y_i : i \in I\}$ is a net in U converges to 1 which implies that $1 \in \overline{U}$. Since V is an open neighborhood of 1, $V \cap U \neq \phi$, a contradiction. Therefore, $(X, \mathcal{T})$ is Uryshon space. If $(X, \mathcal{T})$ is Uryshon space, clearly it is Hausdorff. □

Corollary 4.14. *If α is an infinite cardinal number, then there is a Uryshon topological commutative BE-algebra of order α which its topology is nontrivial.*

Proof. By Theorem 4.13 and Corollary 4.10, the proof is obvious. □

Conclusion

In this note, we have studied (semi) topological BE-algebras. In Theorems 3.5 and 3.6, we have built a topological self distributive BE-algebra of order α, for every cardinal number α. In section 4 we have investigated separation axioms T_0, T_1, Hausdorff and Uryshon spaces on topological commutative BE-algebras and showed that they are equivalent on topological commutative BE-algebras. In particular, in Corollary 4.10, we have proved that for each infinite cardinal number α there exists at least a Hausdorff topological commutative BE-algebra of order α with nontrivial topology. For the future, we suggest to study:
(1) regularity, normality and metrizability on (semi) topological BE-algebras,
(2) topological pseudo BE-algebras and separation axioms on them,
(3) quasi-uniformities and uniformities on BE-algebras and to investigate quasi-uniformizable BE-algebras.

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