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Dedicated to Marius Iosifescu
on the occasion of his 80th anniversary

Uniformly integrable potential operators and the existence of quasi-stationary distributions

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Abstract. We consider irreducible Markov processes having lifetimes with finite means. We show-using the theory developed in [8]-that the condition of uniform integrability of the potential operators implies the existence of quasi-stationary distributions. We also show that this condition (weaker than the usually assumed compactness of operators) is not necessary for the existence of quasi-stationary distributions. As an auxiliary result we prove the existence in this context of a probability excessive irreducibility measure.

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1. Introduction

The long-term behaviour of certain Markov processes expected to die out but having some sort of equilibrium for long periods until their death has generated the theory of quasi-stationary distributions. There is an enormous literature on quasi-stationary distributions and a multitude of approaches. Among these is the one based upon the spectral analysis of compact operators associated with the process, as done for instance in [7], [4], [3] for killed brownian motion and in [6], [9] for certain killed diffusion processes.

The aim of this paper is to show that the weaker condition of uniform integrability of the potentials of a general Markov process is sufficient for the existence of quasi-stationary distributions for that process.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, X_t, P^x)$ be a right Markov processes with state space $(E, \mathcal{E})$, semigroup $(P_t)_{t\geq 0}$, resolvent $(U^\alpha)_{\alpha\geq 0}$ and lifetime ζ. Denote as usual $P^\nu(\cdot) = \int \nu(\mathrm{d}x) P^x(\cdot)$.

Definition 1.1. A *quasi-stationary distribution* (in short a QSD) for X is a probability measure ν on $(E, \mathcal{E})$ satisfying

$$P^\nu(X_t \in A, t < \zeta) = \nu(A) P^\nu(t < \zeta)$$

for any $t > 0$, $A \in \mathcal{E}$.

In order to eventually associate a QSD with the Markov process X, the latter will be subject to certain conditions that we now introduce:

(i) *m-irreducibility* with respect to a measure m satisfying $mU^1 \ll m$, i.e.

$$m(A) > 0 \Rightarrow U^1(x, A) > 0 \text{ for every } x \in E.$$

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(ii) $P_t 1(x) > 0$ for every $t > 0,\ x \in E$.
(iii) $P^x(\zeta) < \infty$ for every $x \in E$.

Comments on the hypotheses.

The condition of m-irreducibility is traditionally imposed on the process when looking for QSDs. According to it whenever A is an absorbing set we have either $A = \emptyset$ or $m(A^c) = 0$. Applying this property to $A_t = \{x : P_t 1(x) = 0\}$ for $t > 0$ (which is absorbing since $x \to P_t 1(x)$ is excessive) we get that either $P_t 1(x) > 0$ for each x, or $P_t 1 = 0$ a.e.m, the latter being excluded by the long term analysis we have in mind. Similarly, $x \to P^x(\zeta)$ being excessive we either have $P^x(\zeta) = \infty$ for every $x \in E$, or $P^x(\zeta) < \infty$ a.e.m which leads to the assumption (iii).

We would also like to note that under (i)-(iii) a QSD is always a dissipative excessive measure (in the terminology of [5]). It follows from the result in [2] that it is in fact a potential measure.

In the next section we summarize the properties of uniformly integrable operators to be used in the sequel, show the existence in the context of an excessive irreducibility probability measure and prove that the uniform integrability of the potential operators implies the existence of QSDs.

In the last section we show that the potentials of regular step processes are not uniformly integrable, nevertheless these processes may have QSDs.

2. Uniformly integrable operators

Uniformly integrable operators were introduced by Wu in [8] and extensively used in the framework of large deviations for Markov processes. As the notion of uniform integrability of operators is a relaxation of compactness yet preserving important properties of compact operators it was to be expected that it may be successfully used in all results involving these properties.

Let μ be a probability measure on the space $(E, \mathcal{E})$ and let

$$\mathcal{E}_\mu^+ := \{f \in \mathcal{E}^+ : \mu(f) > 0\}$$

$$L_+^p(\mu) := \mathcal{E}_\mu^+ \cap L^p(\mu))$$

$$B^p(L)\ \big(\text{resp. } B_+^p(L)\big) := \big\{f \in L^p(\mu)\ \big(\text{resp. } L_+^p(\mu)\big) : \| f \|_p \le L\big\}$$

Definition 2.1. Let $p \in [1, \infty)$. A bounded linear operetor π is said to be *uniformly integrable* in $L^{(p)}(\mu)$ (in short *p-U.I.*) if $\{| \pi f |^p : f \in B^p(1)\}$ is uniformly integrable.

All properties concerning uniformly integrable operators that will be needed are recalled in:

Theorem 2.1. (i) *Let $p \in (1, \infty)$ and let p' be the conjugate exponent of p. Assume that π is bounded and nonnegative in $L^{(p)}(\mu)$ (i.e. $\pi f \ge 0$ for any $f \in L_+^p(\mu)$). Then π is p-U.I. iff π' (the dual of π) is p'-U.I..*
(ii) *When π is given by a sub-Markov kernel, $\mu\pi \le \mu$ and π is p-U.I., then π is q-U.I. for any $q \in (1, \infty)$.*

(iii) *Assume that π is p-U.I. and positive. If $R_{\mu}^{(p)}(\pi) > 0$ ($R_{\mu}^{(p)}(\pi)$ being the spectral radius of π in $L^p(\mu)$), then there exists $\varphi \in L_+^p(\mu)$, $\| \varphi \|_p = 1$ such that*

$$\pi\varphi = R_{\mu}^{(p)}(\pi)\varphi.$$

(iv) *If the p-U.I. positive operator π is realized by an irreducible kernel N (i.e. for any $A \in E_+$, $\sum_{n=0}^{\infty} N^{(n)}(x, A) > 0$ $\forall x \in E$), then $R_{\mu}^{(p)}(\pi) > 0$.*

(v) *Let π' be another bounded operator on $L^p(\mu)$. If π is p-U.I., then so is $\pi\pi'$.*

These properties may be found in [8] as follows: (i) is a consequence of Proposition 1.2 (c) and Remark 1.3 (b); (ii) follows from Proposition 1.2 (c) and Remark 1.3 (a); (iii) is Theorem 3.2; (iv) is Theorem 3.11 and (v) is Theorem 1.1 (d).

The theory of uniformly integrable operators is particularly nice when the measure μ is both an excessive ($\mu P_t \leq \mu$ for every $t \geq 0$) and an irreducibility measure for the given Markov process. Whence the interest of the following result:

Theorem 2.2. *Let X be a Markov process subject to conditions* (i)-(iii) *in section 1. Then there exists a probability μ which is an irreducibility measure as well as an excessive measure for X.*

Proof. The hypothesis (iii) on X implies the existence of $M \in (0, \infty)$ such that $m(U1_E \leq M) > 0$.

The theory of irreducible Markov processes is based upon the so called "small sets"; accordingly we may assume that there exists a set C satisfying $m(C) > 0$, $C \subseteq \{x : U(x, E) \leq M\}$ and

$$U^1(x, \cdot) \geq a\nu(\cdot) \text{ for any } x \in C$$

with ν a probability measure on $(E, \mathcal{E})$ such that $\nu(C) > 0$ and $a > 0$.

For $0 < \alpha < 1$ define

$$\mu_\alpha(B) := \frac{\nu(1_C U^\alpha 1_B)}{\nu(1_C U^\alpha 1_C)}.$$

For any $\alpha \in (0, 1)$ we have that μ_α is an α-excessive measure, $\mu_\alpha \geq aM^{-1}\nu$ and $\mu_\alpha(E) \leq M(a\nu(C))^{-1}$. Therefore we may define

$$\widetilde{\mu}_\alpha := \inf_{\beta \in (0,\alpha]} \mu_\beta$$

which in turn is α-excessive, $\widetilde{\mu}_\alpha \geq aM^{-1}\nu$ and $\widetilde{\mu}_\alpha(E) \leq M(a\nu(C))^{-1}$.

Next we define the measure

$$\widetilde{\mu} :=\uparrow \lim_{\alpha \downarrow 0} \widetilde{\mu}_\alpha$$

satisfying $\widetilde{\mu}(C) \geq aM^{-1}\nu$, $\widetilde{\mu}(E) \leq M(a\nu(C))^{-1}$ and

$$\widetilde{\mu}(P_t 1_A) = \lim_{\alpha \downarrow 0} \widetilde{\mu}_\alpha P_t(A) \leq \lim_{\alpha \downarrow 0} e^{\alpha t} \widetilde{\mu}_\alpha(A) = \widetilde{\mu}(A).$$

Also, since successively the measures μ_α, $\widetilde{\mu}_\alpha$ are irreducibility measures so is $\widetilde{\mu}$. The measure $\mu(A) := \widetilde{\mu}(A)[\widetilde{\mu}(E)]^{-1}$ is the requested one. □

The main result of this paper is the following:

Theorem 2.3. *Let X be a Markov process satisfying the conditions* (i)-(iii) *in section 1 and let μ be a probability excessive irreducibility measure for X. Assume that for some $\alpha > 0$ and some $p \in (1, \infty)$ the operator U^α is uniformly integrable in $L^p(\mu)$. Then there exists a quasi-stationary distributions η of X.*

Proof. Due to the resolvent equations, Theorem 2.1(v) entails the $p - U.I.$ of U^β for any $\beta > 0$. For simplicity we shall work with U^1 in the sequel.

By Theorem 2.1(ii) there is also an independence of $p \in (1, \infty)$ so we shall assume that U^1 is 2-U.I..

Next Theorem 2.1(iv) ensures that R the spectral radius of U^1 in $L^2(\mu)$ is strictly positive. Therefore Theorem 2.1(iii) applied to the operator U^1 and its adjoint $\widehat{U}^1$in $L^2(\mu)$ implies the existence of $\varphi,\ \widehat{\varphi} \in L^2_+(\mu)$ such that

$$\begin{aligned} \| \varphi \|_{L^2(\mu)} &= \| \widehat{\varphi} \|_{L^2(\mu)} = 1 \\ U^1\varphi &= R\varphi \\ \widehat{U}^1\widehat{\varphi} &= R\widehat{\varphi} \end{aligned}$$

Define now the probability

$$\eta(f) := \frac{\mu(\widehat{\varphi} f)}{\mu(\widehat{\varphi})}$$

which satisfies

$$\begin{aligned} \eta\left(U^1 f\right) &= \tfrac{\mu\left(\widehat{\varphi} U^1 f\right)}{\mu(\widehat{\varphi})} = \tfrac{\mu\left(f \widehat{U}^1 \widehat{\varphi}\right)}{\mu(\widehat{\varphi})} = R\eta(f) = \\ &= \eta(f)\,\eta\left(U^1 1\right) \end{aligned}$$

Using this and the resolvent equation we get for any $\alpha > 0$

$$\eta(U^\alpha f) = \eta(f)\frac{\eta\left(U^1 1\right)}{1 + (\alpha - 1)\,\eta(U^1 1)} = \eta(f)\,\eta(U^\alpha 1).$$

The inequality $\alpha\eta(U^\alpha f) \leq \eta(f)$ for any $\alpha > 0$ implies that η is an excessive measure and hence the mapping $t \to \eta(P_t f)$ is right continuous for any nonnegative, bounded $f \in \mathcal{E}$ and therefore the uniqueness of the Laplace transform gives $\eta(P_t f) = \eta(f)\,\eta(P_t 1)$ for any $\dot{t} \geq 0$ which means that η is indeed a quasi-stationary distribution for X. □

3. The case of regular step processes

The basic data of these processes are: a function $\lambda \in \mathcal{E}$ satisfying $0 < \lambda(x) < \infty$ for all $x \in E$ and a Markov kernel Q such that $Q(x, \{x\}) = 0$ for every $x \in E$. Then one defines $\tau_1 := \inf\{t > 0 : X_t \neq X_0\}$ and one assumes that $P^x(\tau_1 > t) = \exp[-\lambda(x)\,t]$ for every $x \in E, t > 0$. The sequence inductively defined by $\tau_0 := 0,\ \tau_{n+1} := \tau_n + \tau_1 \circ \theta_{\tau_n},\ n \geq 0$ is such that $\zeta = \lim_{n\to\infty} \tau_n$, the process takes the value X_{τ_n} on the interval $[\tau_n, \tau_{n+1})$ and $\{X_{\tau_n}\}_n$ is a Markov chain with transition kernel Q. By means of the kernels

$$Q_\alpha(x, A) := \frac{\lambda(x)}{\alpha + \lambda(x)} Q(x, A) \qquad \alpha \geq 0,\ x \in E$$

one expresses the potential kernels of the process X as

$$U^{\alpha}f(x)=\sum_{n=0}^{\infty}Q_{\alpha}^{(k)}\frac{f}{\alpha+\lambda}(x)=\frac{f}{\alpha+\lambda}(x)+Q_{\alpha}U^{\alpha}f(x).$$

Accordingly the process is m-irreducible iff the chain with the kernel Q is m-irreducible. This process always satisfies condition (ii) since $P_t1(x)\geq P^x(\tau_1>t)=\exp[-\lambda(x)t]$. For (iii) a sufficient condition may be set up as $Q\frac{1}{\lambda}\leq a\frac{1}{\lambda}$ with $a\in(0,1)$. which implies $Q_{\alpha}\frac{1}{\alpha+\lambda}\leq a\frac{1}{\alpha+\lambda}$ for every $\alpha\geq 0$. Note that this requires λ to be unbounded, which should be imposed anyway in our context since under a bounded λ we would have $P^x(\zeta=\infty)=1$ for every $x\in E$ ([1], ch.1, Exercise (12.13)).

We now claim that for any $\alpha>0$, $p\in(1,\infty)$ the operator U^{α} is not uniformly integrable in $L^p(\mu)$, μ being as before the excessive probability measure equivalent to m. Assume by way of contradiction that U^{α} is p-$U.I.$ Since for any non-negative function $f\in\mathcal{E}$, $U^{\alpha}f$ is α-excessive we would have $Q_{\alpha}U^{\alpha}f\leq f$ (by Theorem 5.9, ch.II in [1]) and this would imply the uniform inegrabilty in $L^p(\mu)$ of the operator $T_{\alpha}f=\frac{f}{\alpha+\lambda}$ and this is known not to be the case.

Nevertheless a regular step process may have QSDs as shown by many examples of processes with a counatble state space. In the general case we set out a class of regular step processes having this property. Namely, suppose that Q_1 has a QSD η and that moreover $Q_1$is η-symmetric. Then η turns out to be a QSD for X as well.

First we have $\eta\left(Q_1^{(0)}\frac{f}{1+\lambda}\right)=\eta\left(\frac{f}{1+\lambda}\right)=\eta(f)-\eta\left(\frac{\lambda f}{1+\lambda}\right)=\eta(f)-\eta(fQ_11)$

$$=\eta(f)-\eta(Q_1f)=\eta(f)-\eta(f)\eta(Q_11)=\eta(f)\eta\left(\frac{1}{1+\lambda}\right).$$

Next assume that for some $k\geq 1$ we have $\eta\left(Q_1^{(k)}\frac{f}{1+\lambda}\right)=\eta(f)\eta\left(\left(Q_1^{(k)}\frac{1}{1+\lambda}\right)\right)$. Then using this and the fact that η is a QSD for Q_1 we get that

$$\begin{aligned}\eta\left(Q_1^{(k+1)}\frac{f}{1+\lambda}\right)&=\eta\left(Q_1^{(k)}\frac{f}{1+\lambda}\right)\eta(Q_11)\\&=\eta(f)\eta\left(\left(Q_1^{(k)}\frac{1}{1+\lambda}\right)\right)\eta(Q_11)\\&=\eta(f)\eta\left(\left(Q_1^{(k+1)}\frac{1}{1+\lambda}\right)\right)\end{aligned}$$

and thus we get $\eta\left(U^1f\right)=\eta(f)\eta\left(U^11\right)$, which by the proof of Theorem 2.2 implies that η is a QSD for X as claimed.

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