Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 43(1), 2016, Pages 53–61
ISSN: 1223-6934

Dedicated to Marius Iosifescu
on the occasion of his 80th anniversary

On the escape probability of particles from a charged channel

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Abstract. We consider a mathematical model of the diffusing dynamics of particles in a channel under a potential field, and study a related optimal control problem, with the purpose of maximizing the probability of the particles escape from this channel.

2010 Mathematics Subject Classification. 49J20, 49K20.
Key words and phrases. Optimal control, variational methods.

1. Introduction

The motion of charged particles in channels under the action of the channel electric potential is a physical process which has wide applications in crystal theory, semiconductor design ([6], [8]), biological flows through cells and laboratory medical procedures ([1] [2], [3], [4], [5], [9]).

We treat the problem in one-dimensional approach, assuming that the channel is much more longer than wider and fixing as the space domain the interval $(0, L)$. The channel potential is denoted by $U(x)$ and the diffusion coefficient by $D(x)$. Let us consider that the particle is at position x_0 at the initial time $t = 0$. The diffusing dynamics of a particle in a potential field is described by the following equation for the function denoted $G(x, t; x_0)$,

$$\frac{\partial G}{\partial t} - \nabla \cdot D(x) [\nabla G + \beta G \nabla U] = 0 \text{ in } (0, L) \times (0, T),\ T > 0, \tag{1}$$

with the initial condition

$$G(x, 0; x_0) = G_0,$$

and with boundary conditions considered here of Robin type

$$\begin{aligned} D(x) [\nabla G + \beta G \nabla U] \cdot \nu &= k_0 G \text{ at } x = 0,\ t \in (0, T), \\ -D(x) [\nabla G + \beta G \nabla U] \cdot \nu &= k_L G \text{ at } x = L,\ t \in (0, T), \end{aligned} \tag{2}$$

where t is the time, ν is the outward unit vector to the boundary, and k_0 and k_L represent the rates of particle escape from the channel through the points $x = 0$ and $x = L$, respectively. Here, β is a constant depending on some physical quantities and it is supposed to be known, as well as U and D.

To explain the meaning of G we start from some considerations of stochastic theory. Let us consider the motion of the particle under an external force, perturbed by a

Received April 24, 2016.

Brownian process, and assert that it is described by the Itô stochastic differential equation

$$\begin{aligned} dX(t,s,\xi) &= v(X(t,s,\xi))dt + \sigma(X(t,s,\xi))dW(t),\ 0 \le s \le t \le T,\ \xi \in \mathbb{R}, \\ X(s) &= \xi, \end{aligned} \tag{3}$$

where X stands for the position at time t of the particle which was at position ξ at time s, $W(t)$ is the Brownian perturbation, v is called the drift and σ is the Brownian motion covariance (see for example [7]). If the stochastic process has a transition density denoted $G(x,t;\xi,s)$ defined for $0 \le s \le t$, $x,\xi \in \mathbb{R}$, one associates to equation (3) the parabolic equation on $(-\infty,+\infty)\times(0,+\infty)$, called the backward Kolmogorov equation, written for the function $(\xi,s) \to G(x,t;\xi,s)$ with x,t fixed

$$\frac{\partial G}{\partial s} - v(\xi,s)\frac{\partial G}{\partial \xi} - \frac{\sigma^2(\xi,s)}{2}\frac{\partial^2 G}{\partial \xi^2} = 0,\ \forall \xi \in \mathbb{R},\ 0 < s \le t, \tag{4}$$

(in the 1-D case) and the forward Kolmogorov equation for the function $(x,t) \to G(x,t;\xi,s)$ with ξ, s fixed

$$\frac{\partial G}{\partial t} - \frac{\partial}{\partial x^2}\left(\frac{\sigma^2(x,t)}{2}G\right) + \frac{\partial}{\partial x}\left(v(x,t)G\right) = 0,\ \forall x \in \mathbb{R},\ 0 < t, \tag{5}$$

where $\frac{\sigma^2(x,t)}{2}$ is the diffusion coefficient. The probability of finding the particle which was at ξ at time s in an interval I at time t is given by

$$P[X(t,s,\xi) \in I] = \int_I G(x,t;\xi,s)dx. \tag{6}$$

Now, we identify $\xi = x_0$ and see that (1) can be still rewritten as

$$\frac{\partial G}{\partial t} - \frac{\partial}{\partial x^2}\left(D(x)G\right) + \frac{\partial}{\partial x}\left(v(x)\,G\right) = 0 \text{ in } (0,L)\times(0,T).$$

Hence, $G(x,t;x_0)$ represents the transition density (from the position x_0 at time 0, to the position x at time t) for the Brownian motion with the diffusion coefficient $D(x)$ and drift $v(x) = \nabla D(x) - \beta D(x)\nabla U(x)$. If D is constant, then v reduces to $-\beta U'(x)$, or to a constant if U is linear.

If instead of G we write $(Ge^{\beta U})e^{-\beta U}$, by a straightforward computation (under hypotheses that allow all operations to make sense) we arrive at the system

$$\frac{\partial G}{\partial t} - \frac{\partial}{\partial x}\left[D(x)e^{-U(x)}\frac{\partial}{\partial x}\left(Ge^{U(x)}\right)\right] = 0 \text{ in } (0,L)\times(0,T), \tag{7}$$

$$G(x,0;x_0) = G_0 \text{ in } (0,L), \tag{8}$$

$$D(x)e^{-U(x)}\frac{\partial}{\partial x}\left(Ge^{U(x)}\right)\Big|_{x=0} = k_0 G(0,t;x_0) \text{ for } t \in (0,T), \tag{9}$$

$$-D(x)e^{-U(x)}\frac{\partial}{\partial x}\left(Ge^{U(x)}\right)\Big|_{x=L} = k_L G(L,t;x_0) \text{ for } t \in (0,T), \tag{10}$$

where we have denoted still by U the product βU. Later we shall indicate by $d(x)$ the product $D(x)e^{-\beta U(x)}$. We assume that there is no collision between particles.

We shall study an optimal control problem, consisting in the determination of the optimal initial distribution G_0, for which the particles succeed to escape from the channel within a given time interval, as it is of interest in real applications. To this end, we shall actually maximize the *escape probability* of the particles from the channel. We give a rigorous definition for this probability, then we prove the existence of the control and determine the optimality conditions.

2. The state system

Functional framework. Let us denote $\Omega = (0, L)$. We shall denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and norm respectively, in $L^2(\Omega)$ and indicate a partial derivative either by $\frac{\partial v}{\partial x}$ or v_x. We consider the following hypotheses:
(i) k_0, $k_L \in \mathbb{R}$, $k_0 > 0$, $k_L > 0$,
(ii) $D \in W^{1,\infty}(\Omega)$, $D(x) \geq D_m > 0$,
(iii) $U \in W^{1,\infty}(\Omega)$.

We shall approach the problem in the functional framework involving the spaces $V = H^1(\Omega) \subset L^2(\Omega) \subset V'$, where V' is the dual of V. V' is endowed with the scalar product

$$(v, \overline{v})_{V'} := v(A_\Delta^{-1}\overline{v}),\ \forall v, \overline{v} \in V', \tag{11}$$

where A_Δ^{-1} the inverse of the operator $A_\Delta : V \to V'$ defined by

$$\langle A_\Delta v, \psi\rangle_{V',V} := \int_0^L D(x)v_x\psi_x dx + k_0 v(0)\psi(0) + k_L v(L)\psi(L),\ \forall \phi \in V. \tag{12}$$

For simplicity, sometimes we shall not indicate all arguments for G or for some other functions.

We introduce the linear operator $A : V \to V'$ by

$$\langle Av, \phi\rangle_{V',V} = \int_0^L D(x)(v_x\phi_x + U_x v\phi_x)dx + k_0 v(0)\phi(0) + k_L v(L)\phi(L),\ \forall \phi \in V,$$

and the Cauchy problem

$$\begin{aligned} \frac{dG}{dt}(t) + AG(t) &= 0,\ \text{a.e. } t \in (0, T), \\ G(0) &= G_0. \end{aligned} \tag{13}$$

We remark that a strong solution to (13) is a solution (in the sense of Definition 1) to problem (7)-(10), so that, instead of this system we shall investigate the abstract Cauchy problem (13).

Definition 2.1. Let $G_0 \in L^2(\Omega)$. We call a *solution* to problem (13) a function $G \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; V) \cap W^{1,2}(0, T; V')$ which satisfies the equation

$$\begin{aligned} &\int_0^T \left\langle \frac{dG}{d\tau}(\tau), \phi(\tau)\right\rangle_{V',V} d\tau + \int_0^T \int_0^L (D(x)G_x\phi_x + D(x)U_x G\phi_x)\, dxd\tau \\ &+ \int_0^T (k_L G(L, \tau; x_0)\phi(L, \tau) + k_0 G(0, \tau; x_0)\phi(0, \tau))dxd\tau = 0, \end{aligned} \tag{14}$$

for any $\phi \in L^2(0, T; H^1(\Omega))$, and the initial condition $G(x, 0; x_0) = G_0$.

Proposition 2.1. *Let $G_0 \in L^2(\Omega)$. Then, problem* (13) *has a unique solution*

$$G \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; V) \cap W^{1,2}(0, T; V'), \tag{15}$$

which satisfies the estimates

$$\|G(t)\|^2 + \int_0^t \|G(\tau)\|_V^2 \, d\tau \leq \alpha_0 \|G_0\|^2, \ \forall t \in [0,T], \tag{16}$$

$$\|G(t)\|_{V'}^2 + \int_0^t \|G(\tau)\|^2 \, d\tau \leq \alpha_1 \|G_0\|_{V'}^2, \ \forall t \in [0,T]. \tag{17}$$

The solution tends asymptotically to zero for large time, i.e.,

$$\|G(t)\| \leq e^{-\frac{\gamma_\infty}{2}t} e^{\frac{U_M - U_m}{2}} \|G_0\|, \ \forall t > 0, \tag{18}$$

where α_0*,* α_1*,* γ_∞*,* U_M*,* U_m *are positive constants depending on* k_0*,* k_L*,* L*,* T *and the norm* $\|d\|_{W^{1,\infty}(\Omega)}$*. Moreover, if* $G_0 \geq 0$ *a.e.* $x \in \Omega$*, then* $G(t) \geq 0$ *a.e.* $x \in \Omega$*, for any* $t \in [0,T]$*.*

Proof. Let us take $G_0 \in L^2(\Omega)$. One easily notes that the operator A is bounded

$$\|Av\|_{V'} \leq c_M \|v\|_V, \tag{19}$$

and

$$\langle Av, v \rangle_{V',V} \geq -\frac{\lambda_0}{2} \|v\|^2 + \frac{c_m}{2} \|v\|_V^2, \tag{20}$$

with λ_0 and c_m constants depending on the problem parameters. Since A satisfies (19) and (20) it follows by Lions' theorem that the Cauchy problem (13) has a unique solution in the spaces (15). In order to prove (16) we multiply equation (13) by G and integrate over $\Omega \times (0,t)$. Using (20) we get

$$\frac{1}{2} \|G(t)\|^2 + \frac{c_m}{2} \int_0^t \|G(\tau)\|_V^2 \, d\tau \leq \frac{1}{2} \|G_0\|^2 + \frac{\lambda_0}{2} \int_0^t \|G(\tau)\|^2 \, d\tau.$$

Now we apply the Gronwall lemma and obtain (16). We multiply (13) by $G(t)$ scalarly in V' and integrate over $(0,t)$. We have

$$\frac{1}{2} \|G(t)\|_{V'}^2 + \frac{1}{2} \int_0^t \|G(\tau)\|^2 \, d\tau \leq \frac{1}{2} \|G_0\|_{V'}^2 + \frac{\lambda_1}{2} \int_0^t \|G(\tau)\|^2 \, d\tau,$$

with λ_1 another positive constant, and by the Gronwall lemma we get (17). From (13) and (19) we have

$$\int_0^t \left\| \frac{dG}{d\tau}(\tau) \right\|_{V'}^2 d\tau \leq c_M \alpha_0 \|G_0\|^2. \tag{21}$$

Due to the linearity of A we still obtain that (16)-(17) are satisfied by the difference of two solutions G and $\overline{G}$ corresponding to two different initial data, G_0 and $\overline{G_0}$ belonging to $L^2(\Omega)$.

Assume now that $G_0 \geq 0$. Let us multiply (13) by $G^-(t)$ and integrate over $\Omega \times (0,t)$. Using the Stampacchia lemma we obtain

$$\begin{aligned} & -\frac{1}{2} \left\|G^-(t)\right\|^2 + \frac{1}{2} \left\|G^-(0)\right\| - \int_0^t \int_0^L D(G_x^-)^2 dx d\tau \\ & - \int_0^t k_L \left(G^-(L,\tau)\right)^2 d\tau - \int_0^t k_0 \left(G^-(0,\tau)\right)^2 d\tau \\ = & \int_0^t \int_0^L D U_x G G_x^- \, dx d\tau. \end{aligned}$$

But $G_0^- = 0$ and eventually, we deduce that

$$\left\|G^-(t)\right\|^2 + c_m \int_0^t \left\|G^-(\tau)\right\|_V^2 d\tau \le \lambda_0 \int_0^t \left\|G^-(\tau)\right\|^2 d\tau.$$

Still by the Gronwall lemma we conclude that $\left\|G^-(t)\right\|^2 = 0$, $\forall t \in [0,T]$, hence $G(t) \ge 0$ for any $t \in [0,T]$.

We pass now to the proof of (18). For that we denote $U_M := \max_{x\in[0,L]} U(x)$, $U_m = \min_{x\in[0,L]} U(x)$, and take into account that if $u \in W^{1,\infty}(\Omega)$ and $\frac{dG}{dt} \in L^2(0,T;V')$ then we can define $u\frac{dG}{dt} \in L^2(0,T;V')$ by

$$u\frac{dG}{dt}(\phi) := \frac{dG}{dt}(\phi u),\ \forall \phi \in V.$$

We multiply (7) by $e^U G \in V$ and after straightforward computations we obtain

$$\frac{d}{dt}\left\|e^{\frac{U}{2}}G(t)\right\|^2 + \gamma_\infty \left\|e^{\frac{U}{2}}G(t)\right\|^2 \le 0, \tag{22}$$

with $\gamma_\infty := \frac{e^{U_m}\min\{D_m e^{-U_M}, k_0, k_L\}}{c_P}$. From here we deduce that the function $t \to e^{\gamma_\infty t}\left\|e^{\frac{U}{2}}G(t)\right\|^2$ is decreasing, whence we get (18), as claimed. □

3. The control problem

We are going to explain the control problem we shall deal with. This aims at computing G_0 such that to maximize the probability of the particle escape from the channel within the time interval $(0,T)$. Thus, the control acts in the initial condition. However, it will be not let free in $L^2(\Omega)$ but it will be restricted to have the integral over $(0,L)$ equal to 1,

$$\int_0^L G_0(x)dx = \int_0^L G(x,0;x_0)dx = 1. \tag{23}$$

First of all we shall define the escape probability (see also [9]).

We denote by $G(x,t;x_0)$ the solution to (7)-(10), or equivalently to (13).

By (7) we deduce that the probability (given by (6)) that the particle is in the channel $[0,L]$ at time t, satisfies the problem

$$\begin{aligned} -\frac{\partial}{\partial t}\left(\langle G(\cdot,t;x_0),1\rangle_{V',V}\right) &= k_0 G(0,t;x_0) + k_L G(L,t;x_0), \qquad (24)\\ \langle G(\cdot,0;x_0),1\rangle_{V',V} &= \int_0^L G(x,0;x_0)dx = 1, \end{aligned}$$

since by Proposition 2.1, $G \in L^2(0,T;V)$. We define

$$\begin{aligned} \mathcal{P}_{x_0}(t_1,t_2) &= -\int_{t_1}^{t_2} \left\langle \frac{\partial G}{\partial t}(\cdot,t;x_0),1\right\rangle_{V',V} dt \qquad (25)\\ &= \langle G(\cdot,t_1;x_0),1\rangle_{V',V} - \langle G(\cdot,t_2;x_0),1\rangle_{V',V}\\ &= \int_0^L G(x,t_1;x_0)dx - \int_0^L G(x,t_2;x_0)dx,\ \text{for } 0 \le t_1 < t_2 < \infty, \end{aligned}$$

and will show that it is a probability measure. By (24) we obtain

$$\mathcal{P}_{x_0}(t_1,t_2) = \int_{t_1}^{t_2} k_0 G(0,t;x_0)dt + \int_{t_1}^{t_2} k_L G(L,t;x_0)dt \tag{26}$$

and observe that by Proposition 2.1, equation (25) makes sense for $0 \le t_1 < t_2$ and (26) makes sense for $0 \le t_1 < t_2$.

For $t_1 = 0$ and $t_2 = T$ we have by (25) and (24) that

$$\begin{aligned} \mathcal{P}_{x_0}(0,T) &= 1 - \int_0^L G(x,T;x_0)dx \\ &= \int_0^T k_L G(L,t;x_0)dt + \int_0^T k_0 G(0,t;x_0)dt. \end{aligned} \tag{27}$$

Recalling Proposition 2.1, $\mathcal{P}_{x_0}(0,T)$ as defined in (27) exists. By (25), (23) and (18) we get that

$$\mathcal{P}_{x_0}(0,\infty) = 1 \tag{28}$$

and by (26) it follows that $\mathcal{P}_{x_0}(t_1,t_2) \ge 0$ (since $G(x,t;x_0) \ge 0$ for all $t \ge 0$). As a matter of fact, here we needed the constraint (23). Also, from

$$\langle G(t_1),1\rangle_{V',V} - \langle G(t_2),1\rangle_{V',V} = \int_{t_1}^{t_2} k_L G(L,\tau;x_0)d\tau + \int_{t_1}^{t_2} k_0 G(0,\tau;x_0)d\tau \ge 0$$

we see that

$$\langle G(t_1),1\rangle_{V',V} \ge \langle G(t_2),1\rangle_{V',V} \ \text{ for } 0 < t_1 < t_2. \tag{29}$$

Then, by (26), $\mathcal{P}_{x_0}(t_1,t_2) \le \mathcal{P}_{x_0}(0,t_2)$ for any $t_2 > t_1 \ge 0$. All these imply that $\mathcal{P}_{x_0}(t_1,t_2) \in [0,1]$, for $t_1 \ge 0$, and by all the other properties above it represents a probability. It may be interpreted that $\mathcal{P}_{x_0}(t_1,t_2)$ is the *probability of the particle presence in the channel in the interval* (t_1,t_2).

Using again (25) we deduce that for $T > 0$ we have

$$\mathcal{P}_{x_0}(T,\infty) = \int_0^L G(x,T;x_0)dx \tag{30}$$

and this is the *probability that the particle is still present in the channel at* $t \ge T$, as already specified before. We can still formulate it as the *probability that the particle has not escaped from the channel up to the time* T, or the *survival probability.* Therefore

$$\mathcal{P}_{x_0}(0,T) = 1 - \int_0^L G(x,T;x_0)dx \tag{31}$$

is the *probability that the particle has escaped from the channel within* $(0,T)$, through whatever exit ($x = 0$ or $x = L$), as defined in [1].

The control problem. We conclude that the problem of maximizing the escape probability of the particles from the channel up to the time T, means in fact to minimize the survival probability $\mathcal{P}_{x_0}(T,\infty) = \int_0^L G(x,T;x_0)dx$. Hence, we define the control problem as

$$\min_{G_0 \in \mathcal{U}} \left(\alpha \int_0^L G(x,T;x_0)dx + \frac{1}{2}\int_0^L G_0^2(x)dx \right), \tag{32}$$

subject to (13), where α is a fixed positive constant and

$$\mathcal{U} = \left\{ G_0 \in L^2(\Omega); \ \int_0^L G_0(x)dx = 1 \right\}. \tag{33}$$

So, we have an optimal control problem with a nonlocal constraint for the control.

Theorem 3.1. *Problem* (32) *has a solution* $G_0 \in \mathcal{U}$.

Proof. Let us denote

$$d_{\inf} := \min_{G_0 \in \mathcal{U}} \left(\alpha \int_0^L G(x,T;x_0)dx + \frac{1}{2} \int_0^L G_0^2(x)dx \right)$$

and consider a minimizing sequence $\{G_0^n\}_n \in \mathcal{U}$. Then

$$d_{\inf} \leq \alpha \int_0^L G^n(x,T;x_0)dx + \frac{1}{2} \int_0^L (G_0^n(x))^2 dx \leq d_{\inf} + \frac{1}{n}, \tag{34}$$

where $G^n(x,t;x_0)$ is the solution to (13) corresponding to G_0^n. By (34) we get that $\{G_0^n\}_n$ is bounded in $L^2(\Omega)$ and $\{G^n(\cdot,T;x_0)\}_n$ is bounded in $L^1(0,L)$ (recalling that it is positive by Proposition 2.1). Hence, we can select a subsequence such that, as $n \to \infty$,

$$\begin{aligned} G_0^n &\to G_0^*, \text{ weakly in } L^2(\Omega), \\ G^n(x,T;x_0) &\to \eta \text{ weakly in } L^1(\Omega). \end{aligned} \tag{35}$$

Then, by (15)-(17) we deduce that

$$G^n(\cdot,\cdot;x_0) \to G^*(\cdot,\cdot;x_0) \text{ weakly in } L^2(0,T;V) \cap W^{1,2}(0,T;V')$$

and strongly in $L^2(0,T;L^2(\Omega))$, by the Lions-Aubin lemma. Moreover, by Arzelà-Ascoli theorem we obtain that

$$G^n(\cdot,t;x_0) \to G^*(\cdot,t;x_0) \text{ strongly in } V', \text{ uniformly on subsets of } [0,T].$$

By (35) we obtain that $\eta = G^*(x,T;x_0) \in L^1(\Omega)$ and by the weakly lower semicontinuity of the cost functional we get at limit that

$$d_{\inf} = \alpha \int_0^L G^*(x,T;x_0)dx \frac{1}{2} \int_0^L G_0^{*2}(x)dx,$$

which ends the proof. □

The system in variations. Let us consider that $G_0^* \in \mathcal{U}$ is optimal for problem (32) and denote by G^*, the solution to (13) corresponding to G_0^*. We give a variation to G_0^* along the direction $\lambda > 0$

$$G_0^\lambda(x) := G_0^*(x) + \lambda w(x), \text{ where } w \in L^2(\Omega), \tag{36}$$

and denote

$$Y(x,t;x_0) := \lim_{\lambda \to 0} \frac{G^{G_0^\lambda}(x,t;x_0) - G^*(x,t;x_0)}{\lambda},$$

where $G^{G_0^\lambda}(x,t;x_0)$ is the solution to (13) corresponding to G_0^λ. Hence, the system in variations reads

$$\begin{aligned} \frac{dY}{dt} - \frac{\partial}{\partial x}\left[d(x)\frac{\partial}{\partial x}\left(Ye^{U(x)}\right)\right] &= 0, \text{ in } \Omega\times(0,T), \quad (37)\\ Y(x,0;x_0) &= w, \text{ in } \Omega,\\ d(x)\frac{\partial}{\partial x}\left(Ye^{U(x)}\right)\Big|_{x=0} &= k_0 Y(0,t;x_0) \text{ for } t\in(0,T),\\ -d(x)\frac{\partial}{\partial x}\left(Ye^{U(x)}\right)\Big|_{x=L} &= k_L Y(L,t;x_0) \text{ for } t\in(0,T). \end{aligned}$$

By Proposition 2.1, this system has a unique solution

$$Y\in C([0,T];L^2(\Omega))\cap L^2(0,T;V)\cap W^{1,2}(0,T;V').$$

The adjoint system. We introduce now the following adjoint system

$$\begin{aligned} \frac{\partial p}{\partial t} + e^{U(x)}\frac{\partial}{\partial x}(d(x)p_x) &= 0, \text{ in } \Omega\times(0,T), \quad (38)\\ p(x,T;x_0) &= \alpha, \text{ in } \Omega,\\ e^{U(0)}d(0)p_x(0,T;x_0) &= k_0 p(0,T;x_0), \ t\in(0,T),\\ -e^{U(L)}d(L)p_x(L,T;x_0) &= k_L p(L,T;x_0), \ t\in(0,T). \end{aligned}$$

The first equation in (38) can be written

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left(d(x)e^{U(x)}\frac{\partial p}{\partial x}\right) - d(x)U'(x)\frac{\partial p}{\partial x} = 0, \text{ in } \Omega\times(0,T)$$

and we can prove that (38) has a unique solution

$$p\in C([0,T];L^2(\Omega))\cap L^2(0,T;V)\cap W^{1,2}(0,T;V'),$$

by similar arguments as in Proposition 2.1.

Proposition 3.2. *Let us consider that $G_0^*\in U$ is optimal for problem* (34). *Then,*

$$G_0^*(x) = -p(x,0;x_0) + C, \quad (39)$$

where $p(x,0;x_0)$ is the solution to the adjoint system (38) *at $t=0$ and C is a constant that can be determined from* (23).

Proof. Let us consider that $G_0^*\in\mathcal{U}$ is optimal for problem (34).

The optimality condition calculated from (32) reads as

$$\alpha\int_0^L Y(x,T;x_0)dx + \int_0^L G_0^*(x)w(x)dx \geq 0, \text{ for all } w\in L^2(\Omega). \quad (40)$$

Next, we multiply (37) by $p(x,t;x_0)$ and integrate over Ω and $(0,T)$. By taking into account (38) we get that

$$\alpha\int_0^L Y(x,T;x_0)dx = \int_0^L p(x,0;x_0)w(x)dx. \quad (41)$$

By comparison with (40) we obtain

$$\int_0^L p(x,0;x_0)w(x)dx + \int_0^L G_0^*(x)w(x)dx \geq 0, \text{ for all } w\in L^2(\Omega).$$

Writing this relation for $-w$ we finally get

$$\int_0^L (p(x,0;x_0) + G_0^*(x))w(x)dx = 0, \text{ for all } w \in L^2(\Omega), \tag{42}$$

which implies (39), as claimed. This relation involves a constant due to (23) which is a restriction in the admissible set. Moreover, still by this restriction we can determine that $C = \frac{1}{L}\left(1 + \int_0^L p(x,0;x_0)dx\right)$. □

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