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Dedicated to Marius Iosifescu
on the occasion of his 80th anniversary

How common sense can be misleading in ruin theory

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Abstract. Some graphical representations of ruin probability computed mainly for Erlang type claims suggested an idea that intuitively seems to be true: if the first claims are small then the chance to get ruined is also small. However, for other claims this does not hold, as is shown by counterexamples.

Key words and phrases. Ruin probability, nonhomogeneous claims, stochastic order.

1. The problem

In the insurance field, the evaluation of ruin probabilities is of great importance since it influences the future financial politics of any insurance company. In this sense some recursive formulae were obtained in [2], for the ruin probability evaluated at or before claim instants for the following risk model used to describe the evolution over time of the surplus of an insurance company

$$U_n = u - \sum_{i=1}^{n} (X_i - Y_i) = u - \sum_{i=1}^{n} \xi_i.$$

Here $(U_n)_n$ denotes the remaining capital after paying the n-th claim, u is the initial capital, $(X_n)_n$ are the claim sizes (CSs) assumed to be Erlang-distributed, independent and independent of $(Y_n)_n$, the nonnegative inter-claim revenues (ICRs) which themselves are assumed to be independent, identically distributed (i.i.d.), following an arbitrary distribution. Then $\xi_n = X_n - Y_n$ represents the loss increment between the $(n-1)$-th and the n-th claims. We recall [1], [8] that the ruin probability at or before the n-th claim is

$$\psi_n(u) = P\left(\min_{1\le j\le n} U_j < 0\right) = P\left(\max_{1\le j\le n} \sum_{i=1}^{j} \xi_i > u\right). \tag{1}$$

The novelty of Răducan et al. [4], [5] consists in assuming that the CSs are nonhomogeneous Erlang distributed, yielding a nonhomogeneous process.

Motivated by many numerical examples, Răducan et al. [4] stated the following conjecture relating the order in which the nonhomogeneous claims arrive to the magnitude of the corresponding ruin probability: if the claims arrive in the increasing stochastic order, then the ruin probability is smaller that if the same claims come under a different order. That seemed to be common sense: during the "small claims" period, the insurer's capital accumulates, hence it can face the "hard claims" period better than if a larger claim arrives sooner and decreases the insurer's capital.

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However, this conjecture was proved only in the particular case when the CSs are exponentially distributed with all parameters distinct, see [5] and [6].

In this paper, we deal with a case which is more restrictive than the stated conjecture, and more general in the same time: it is more restrictive because we consider only two claims, and more general because we let them follow any distribution supported on positive values. More precisely, we study if there is a relation between the magnitude of the ruin probability and the arrival order of the two claims when these claims satisfy some stochastic order. In this sense, in Section 2 we define a new stochastic relation which we call "ruin domination". This relation is not always transitive, as we point out using a counterexample.

2. Various stochastic orderings

Let $X_1,\ X_2$ be two nonnegative random variables (r.v), and let $F,\ G$ be their distribution functions (d.f). If the r.v are absolutely continuous either with respect to the Lebesgue measure, or with respect to the counting one, we will denote by p, q their densities. We recall the following stochastic orders (see, for instance [7]):

1. The usual stochastic domination: $X_1 \leqslant_{\rm st} X_2 \Leftrightarrow F \geqslant G$
2. The hazard rate domination: $X_1 \leqslant_{\rm hr} X_2 \Leftrightarrow \frac{p}{1-F} \geqslant \frac{q}{1-G}$
3. The likelihood ratio domination: $X_1 \leqslant_{\rm like} X_2 \Leftrightarrow \frac{p}{q}$ is decreasing.
4. The increasing convex order: $X_1 \leqslant_{\rm icx} X_2 \Leftrightarrow \mathbb{E}u(X_1) \leqslant \mathbb{E}u(X_2)$ for any non-decreasing convex u.

Remark 2.1. It is known that the likelihood ratio domination implies the hazard rate one, which in turn, implies the stochastic one; see, e.g., [7].

Back to the conjecture stated by [5] , if we deal with only two claims, it turns into the following claim:

Conjecture. Let X_1, X_2, Y_1, Y_2 be independent nonnegative r.v.s., withY_1 and Y_2 identically distributed. Let $\xi_i = X_i - Y_i,\ i = 1, 2,\ L_{1,2} = \max(\xi_1, \xi_1 + \xi_2)_+,\ L_{2,1} = \max(\xi_2, \xi_1 + \xi_2)_+$. If $X_1 \leqslant_{st} X_2$, then $L_{1,2} \leqslant_{\rm st} L_{2,1}$; or, in other words, if $\psi_{i,j}(t) = P(L_{i,j} > t), i \neq j \in \{1, 2\}, t > 0$, represents the two ruin probabilities, then the claim is that $\psi_{1,2} \leqslant \psi_{2,1}$. Or, in terms of survival probabilities denoted by $\phi_{i,j} = 1 - \psi_{i,j}$, the claim is that $\phi_{1,2} \geqslant \phi_{2,1}$.

We already know that this conjecture holds if the claims are exponentially distributed. More generally, the common sense says that if the ICRs are "the same", then it is better if the smaller claim X_1 comes first and the greater one X_2 comes next than if the greater claim comes first and the smaller one afterwards (i.e., in the first scenario one seems to be better prepared for the more dangerous claim than in the second one, provided that the incomes are identically distributed). Related to this situation, we define the following stochastic relation.

Definition 2.1. Let X_1, X_2 be two independent non-negative r.v.s. We say that X_2 dominates X_1 in the *ruin sense* if, for any i.i.d. non-negative r.v.s Y_1, Y_2, independent of X_1, X_2 it is true that $L_{1,2} \leqslant_{\rm st} L_{2,1}$ i.e., that

$$\max(0, X_1 - Y_1, X_1 + X_2 - Y_1 - Y_2) \leq_{\rm st} \max(0, X_2 - Y_2, X_1 + X_2 - Y_1 - Y_2). \quad (2)$$

We denote this relation by $X_1 \leqslant_{\text{ruin}} X_2$. If the domination holds only for $Y_1 = Y_2 = \text{const}$ we say that X_2 dominates X_1 in the *weak ruin domination* and write $X_1 \leqslant_{\text{wruin}} X_2$. Otherwise written: two claims X_1, X_2 are in the relation $X_1 \leqslant_{\text{ruin}} X_2$ if the scenario "first comes claim X_1, then X_2" is always better in stochastic order than "first comes claim X_2, then X_1" for any i.i.d. ICRs Y_1, Y_2, while $X_1 \leqslant_{\text{wruin}} X_2$ means that the first scenario is better only for constant ICRs.

Thus, our conjecture says that $X_1 \leqslant_{\text{st}} X_2 \Rightarrow X_1 \leqslant_{\text{ruin}} X_2$. The current study was prompted by the unpleasant surprise that the conjecture is false. Here are two counterexamples.

Example 2.1. Suppose that $X_1 \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ .3 & .3 & .3 & .1 \end{pmatrix}, X_2 \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ .3 & .2 & .2 & .3 \end{pmatrix}$, $Y_1 = Y_2 = 1$. The reader can check that $X_1 \leqslant_{\text{st}} X_2$ but it is not true that $L_{1,2} \leqslant_{\text{st}} L_{2,1}$. Precisely in the first scenario we have $\psi_{1,2}(0) = P(L_{1,2} > 0) = 1 - 0.36 = 0.64$ while in the second one $\psi_{2,1}(0) = P(L_{2,1} > 0) = 1 - 0.39 = 0.61$.

However, it is true that when the initial capital u is greater than 1, then $\psi_{1,2}(u) \leqslant \psi_{2,1}(u)$. The two scenarios are not stochastically comparable.

Example 2.2. Let $X_1 \sim \text{Uniform}(0,1),\ X_2 \sim \exp(1),\ Y_1 = Y_2 = \frac{1}{2}$. Clearly $X_1 \leqslant_{\text{st}} X_2$.

Then $\xi_1 = X_1 - \frac{1}{2}$, $\xi_2 = X_1 + X_2 - 1$, and for $t \geqslant 0$ the corresponding survival probabilities are.

$$
\begin{aligned}
\phi_{12}(t) &= P(L_{12} \le t) = P\left(X_1 \le \frac{1}{2} + t, X_1 + X_2 \le 1 + t\right) \\
&= P\left(X_1 \le \min\left(\frac{1}{2} + t, 1 + t - X_2\right)\right) \\
&= \mathbb{E}\left[P\left(X_1 \le \min\left(\frac{1}{2} + t, 1 + t - X_2\right) \mid X_2\right)\right] \\
&= \mathbb{E}\left[\min\left(1, \min\left(\frac{1}{2} + t, 1 + t - X_2\right)\right)_+\right] \\
&= \int_0^\infty \min\left(1, \frac{1}{2} + t, 1 + t - x\right)_+ e^{-x}\mathrm{d}x.
\end{aligned}
$$

Therefore $\varphi_{1,2}(t) = \begin{cases} \frac{1}{2} - \frac{1}{\sqrt{e}} + t + e^{-1-t} \text{ if } 0 \le t \le \frac{1}{2} \\ 1 - e^{-t} + e^{-1-t} \text{ if } t > \frac{1}{2} \end{cases}$. Next,

$$
\begin{aligned}
\varphi_{2,1}(t) &= P(L_{12} \le t) = P\left(X_2 \leqslant \frac{1}{2} + t, X_1 + X_2 \leqslant 1 + t\right) \\
&= \mathbb{E}\left[P\left(X_2 \leqslant \min(1, 1 + t - X_1) \mid X_1\right)\right] = 1 - \left(1 - \frac{1}{2\sqrt{e}}\right)e^{-t}.
\end{aligned}
$$

Thus $\varphi_{1,2}(0) = \frac{1}{2} - \frac{1}{\sqrt{e}} + e^{-1} = 0.26135$ while $\varphi_{2,1}(0) = \frac{1}{2\sqrt{e}} = 0.30327$. It is not true that $\varphi_{1,2} \geq \varphi_{2,1}$.

It turns out that even the weaker conjecture “$X_1 \leqslant_{\mathrm{st}} X_2 \Rightarrow X_1 \leqslant_{\mathrm{wruin}} X_2$” fails to be true. Even the weaker implication “ $X_1 \leqslant_{\mathrm{hr}} X_2 \Rightarrow X_1 \leqslant_{\mathrm{wruin}} X_2$” is false - as it is proved by the second example. However, in both examples it is true that the second scenario is worse in the increasing convex order (meaning that $L_{1,2} \leqslant_{\mathrm{icx}} L_{2,1}$) but maybe this is only by chance; we do not dare to add a new false guess here.

3. Ruin domination in general case

In order to see under which assumptions the ruin domination holds, we define the following notation: for two nonnegative r.v.s X_1, X_2 and $a, b, t \geqslant 0$ let

$$\Delta_{a,b} = P\left(X_1 \le a, X_1 + X_2 \le a + b\right) - P\left(X_2 \le a, X_1 + X_2 \le a + b\right), \tag{3}$$

and

$$D_{a,b}\left(t\right) = \frac{1}{2}\left(\Delta_{a+t,b} + \Delta_{b+t,a}\right). \tag{4}$$

Lemma 3.1. *i). If $a \geqslant 0$ then $\Delta_{a,0} = 0$ and $D_{a,0}\left(t\right) = \frac{1}{2}\left(\Delta_{t,a}\right)$.*
ii). Let $X_1,\ X_2$ be non-negative independent r.v.s. and let F_1, F_2 be their d.f.s. If they are absolutely continuous with respect to some measure μ, the densities are $dF_1\left(x\right) = f_1\left(x\right) d\mu\left(x\right),\ dF_2\left(x\right) = f_2\left(x\right) d\mu\left(x\right)$ and $\delta\left(x, y\right) = f_1\left(x\right) f_2\left(y\right) - f_1\left(y\right) f_2\left(x\right)$ then

$$\Delta_{a,b} = \int_0^a \int_0^{a+b-x} \delta\left(x,y\right) d\mu\left(y\right) d\mu\left(x\right) = \int_0^{\min\{a,b\}} \int_a^{a+b-x} \delta\left(x,y\right) d\mu\left(y\right) d\mu\left(x\right). \tag{5}$$

iii). If the d.f.s. $F_1 = \begin{pmatrix} 0 & 1 & 2 & \dots \\ p_0 & p_1 & p_2 & \dots \end{pmatrix}, F_2 = \begin{pmatrix} 0 & 1 & 2 & \dots \\ q_0 & q_1 & q_2 & \dots \end{pmatrix}$ *are discrete, then for any $a, b > 0$ positive integers,*

$$\Delta_{a,b} = \sum_{i=0}^{a} \sum_{j=0}^{a+b-i} \delta_{i,j} = \sum_{i=0}^{\min\{a,b\}} \sum_{j=a+1}^{a+b-i} \delta_{i,j}, \tag{6}$$

where $\delta_{i,j} = p_i q_j - p_j q_i$ and, by convention, an empty sum is 0.

Proof. i). For any $a \geqslant 0$ we have $\Delta_{a,0} = P\left(X_1 \leqslant a, X_1 + X_2 \leqslant a\right)$
$-P\left(X_2 \leqslant a, X_1 + X_2 \leqslant a\right) = P\left(X_1 + X_2 \leqslant a\right) - P\left(X_1 + X_2 \leqslant a\right) = 0$ and

$$D_{a,0}\left(t\right) = \frac{1}{2}\left(\Delta_{a+t,0} + \Delta_{t,a}\right) = \frac{\Delta_{t,a}}{2}$$

.

ii). The first equality is immediate. To prove the second one, we consider first the case $a \leqslant b$**,** when we split

$$\Delta_{a,b} = \int_0^a \int_0^a \delta\left(x,y\right) d\mu\left(y\right) d\mu\left(x\right) + \int_0^a \int_a^{a+b-x} \delta\left(x,y\right) d\mu\left(y\right) d\mu\left(x\right).$$

The first integral is 0 since if we change the order of integration followed by interchanging x and y, we get

$$\int_0^a \int_0^a \delta\left(y,x\right) d\mu\left(y\right) d\mu\left(x\right) = \int_0^a \int_0^a \delta\left(y,x\right) d\mu\left(x\right) d\mu\left(y\right)$$
$$= -\int_0^a \int_0^a \delta\left(y,x\right) d\mu\left(y\right) d\mu\left(x\right) = \frac{1}{2}\int_0^a \int_0^a \left(\delta\left(x,y\right) + \delta\left(y,x\right)\right) d\mu\left(x\right) d\mu\left(y\right) = 0.$$

When $a > b$ we split

$$\Delta_{a,b} = \int_0^b \int_0^a \delta(x,y)\, d\mu(y)\, d\mu(x) + \int_0^a \int_a^{a+b-x} \delta(x,y)\, d\mu(y)\, d\mu(x) - \int_b^a \int_{a+b-x}^a \delta(x,y)\, d\mu(y)\, d\mu(x).$$

With a similar reasoning, we obtain that

$$\int_0^a \int_a^{a+b-x} \delta(x,y)\, d\mu(y)\, d\mu(x) - \int_b^a \int_{a+b-x}^a \delta(x,y)\, d\mu(y)\, d\mu(x) = 0,$$

hence the equality from (5) also holds.
iii). Taking into consideration that $\delta_{i,j} + \delta_{j,i} = 0$**,** this formula results after a reasoning similar with the one in the continuous case ii). □

Remark 3.1. Let us notice that:
i) $D_{a,b}(t) = D_{b,a}(t), \forall a, b, t \geqslant 0$ (an obvious consequence of the definition),
ii) $D_{a,b}(t) \geqslant 0,\ \forall a, b, t \geqslant 0 \Leftrightarrow \Delta_{a,b} \geqslant 0,\ \forall a, b \geqslant 0$ (obvious from $D_{a,0}(t) = \frac{\Delta_{t,a}}{2}$).

Proposition 3.2. *Let $X_1,\ X_2$ be non-negative independent random variables. Then*
i) $X_1 \leqslant_{\text{wruin}} X_2$ if and only if $\Delta_{a,b} \geqslant 0$ for any $0 \leqslant b \leqslant a$.
ii) $X_1 \leqslant_{\text{ruin}} X_2$ if and only if $\int_0^\infty \int_0^\infty D_{a,b}(t)\, dH(a)\, dH(b) \geqslant 0$ for any probability distribution H on the halfline $[0, \infty)$ and $t \geqslant 0$.
iii) If $X_1,\ X_2$ are discrete, then $X_1 \leqslant_{\text{wruin}} X_2$ if and only if $\Delta_{m,n} \geqslant 0$ for all non-negative integers m, n such that $m \geqslant n - 1$.

Proof. i) $X_1 \leqslant_{\text{wruin}} X_2 \Leftrightarrow P(X_1 \leqslant x+t, X_1+X_2 \leqslant 2x+t) - P(X_2 \leqslant x+t, X_1+X_2 \leqslant 2x+t) \geqslant 0 \Leftrightarrow \Delta_{x+t,x} \geqslant 0$ for all $t, x \geqslant 0$. Put $a = x + t$ and $b = x$.
ii) $X_1 \leqslant_{\text{ruin}} X_2 \Leftrightarrow P(X_1 \leqslant Y_1+t, X_1+X_2 \leqslant Y_1+Y_2+t) \geqslant P(X_2 \leqslant Y_2+t, X_1+X_2 \leqslant Y_1+Y_2+t)$ for any i.i.d. nonnegative r.v.s. $Y_1,\ Y_2$, independent of $X_1,\ X_2$. Denoting by H the common distribution of $Y_1,\ Y_2$, we see that

$$\begin{aligned}
&P(X_1 \leqslant Y_1 + t, X_1 + X_2 \leqslant Y_1 + Y_2 + t) \\
&\quad = \mathbb{E}\left[P(X_1 \leqslant Y_1 + t, X_1 + X_2 \leqslant Y_1 + Y_2 + t \mid Y_1, Y_2)\right] \\
&\quad = \int_0^\infty \int_0^\infty P(X_1 \leqslant y_1 + t, X_1 + X_2 \leqslant y_1 + y_2 + t) dH(y_1) dH(y_2) \\
&\quad = \int_0^\infty \int_0^\infty P(X_1 \leqslant y_1 + t, X_1 + X_2 \leqslant y_1 + y_2 + t) dH(y_2) dH(y_1) \\
&\quad = \int_0^\infty \int_0^\infty P(X_1 \leqslant y_2 + t, X_1 + X_2 \leqslant y_2 + y_1 + t) dH(y_1) dH(y_2) \\
&\quad = P(X_1 \leqslant Y_2 + t, X_1 + X_2 \leqslant Y_1 + Y_2 + t).
\end{aligned}$$

Next

$$\begin{aligned}
&P(X_2 \leqslant Y_2 + t, X_1 + X_2 \leqslant Y_1 + Y_2 + t) \\
&\quad = \mathbb{E}\left[P(X_2 \leqslant Y_2 + t, X_1 + X_2 \leqslant Y_1 + Y_2 + t | Y_1, Y_2)\right] \\
&\quad = \int_0^\infty \int_0^\infty P(X_2 \leqslant y_2 + t, X_1 + X_2 \leqslant y_1 + y_2 + t) dH(y_1) dH(y_2).
\end{aligned}$$

Then

$$P(X_1 \leqslant Y_1 + t, X_1 + X_2 \leqslant Y_1 + Y_2 + t) - P(X_2 \leqslant Y_2 + t, X_1 + X_2 \leqslant Y_1 + Y_2 + t)$$
$$= \int_0^\infty \int_0^\infty [P(X_1 \leqslant y_2 + t, X_1 + X_2 \leqslant y_2 + y_1 + t) -$$
$$-P(X_2 \leqslant y_2 + t, X_1 + X_2 \leqslant y_1 + y_2 + t)] dH(y_1) dH(y_2) =$$
$$= \int_0^\infty \int_0^\infty \Delta_{t+y_2,y_1} dH(y_1) dH(y_2) = \int_0^\infty \int_0^\infty \Delta_{t+y_1,y_2} dH(y_2) dH(y_1)$$
$$= \frac{1}{2} \int_0^\infty \int_0^\infty (\Delta_{t+y_1,y_2} + \Delta_{t+y_2,y_1}) dH(y_1) dH(y_2) = \int_0^\infty \int_0^\infty D_{a,b}(t) dH(a) dH(b).$$

iii) From i) we know already that $X_1 \leqslant_{\text{wruin}} X_2 \Leftrightarrow \Delta_{x+t,x} \geqslant 0, \forall x, t \geqslant 0$. But in the discrete case $\Delta_{x+t,x} = \sum_{i=0}^{[x+t]} \sum_{j=0}^{[2x+t]-i} \delta_{i,j} \geqslant 0$, thus we have only to check that $m \geqslant n-1$ where $m := [x+t]$ and $m+n = [2x+t]$. Here $[x]$ is integer part of x. Thus the claim is that $[2a+t] - [a+t] \leqslant [a+t] + 1 \Leftrightarrow [2a+t] \leqslant 2[a+t] + 1$. This is easy and it is left to the reader $[a] \leqslant [a+t]$. □

Corollary 3.3. *Let X_1, X_2 be nonnegative independent r.v.s, absolutely continuous with respect to some σ-finite measure μ having densities p, q. Then a sufficient condition in order that the relation $X_1 \leqslant_{\text{ruin}} X_2$ hold is that $\delta(x,y) = p(x) q(y) - p(y) q(x) \geqslant 0$ for all $x \leqslant y$.*

Proof. According to (ii) from Proposition 3.2 $X_1 \leqslant_{\text{ruin}} X_2$ if and only if $\int_0^\infty \int_0^\infty D_{a,b}(t) dH(a) dH(b) \geqslant 0$. But $D_{a,b}(t) = \frac{1}{2}(\Delta_{a+t,b} + \Delta_{b+t,a})$. We denote $a' = a+t$ and $b' = b+t$. Using the second equality in (5), we note that

$$\Delta_{a',b} = \int_0^{\min(a',b)} \int_{a'}^{a'+b-x} \delta(x,y) d\mu(y) d\mu(x) \geqslant 0$$

since in this integral $x \leqslant a'$and $y \geqslant a'$, and we know that $x \leqslant y$ implies $\delta(x,y) \geqslant 0$. Similarly, $\Delta_{b',a} \geqslant 0$, which completes the proof. □

The following corollary is a consequence of the above one.

Corollary 3.4. *Let X_1, X_2 be non-negative independent r.v.s. absolutely continuous with respect to some σ- finite measure. Then $X_1 \leqslant_{\text{like}} X_2 \Rightarrow X_1 \leqslant_{\text{ruin}} X_2$.*

Remark 3.2. The " $\leqslant_{\text{like}}$ " relation is too strong, it seems to be far away from the relation " $\leqslant_{\text{ruin}}$ ". We consider another relation seemingly closer from the later one.

Definition 3.1. Denote by $X_1 \leqslant_{\text{Sruin}} X_2$ the condition $D_{a,b}(t) \geqslant 0$ for all $a, b \geqslant 0$ and $t \geqslant 0$.

Remark 3.3. According to remark 3.1 ii), one can also write$X_1 \leqslant_{\text{Sruin}} X_2 \Leftrightarrow \Delta_{a,b} \geqslant 0$ for all $a, b \geqslant 0$.

Proposition 3.5. *Let X_1, X_2 be non-negative independent r.v.s. absolutely continuous with respect to some σ- finite measure. The following implications hold:*

$$X_1 \leqslant_{\text{like}} X_2 \Rightarrow X_1 \leqslant_{\text{Sruin}} X_2 \Rightarrow X_1 \leqslant_{\text{ruin}} X_2 \Rightarrow X_1 \leqslant_{\text{wruin}} X_2$$

Proof. The implications are obvious. □

4. The comparison in the discret case

Here we suppose that

$$X_1 \sim F_1 = \begin{pmatrix} 0 & 1 & 2 & ... \\ p_0 & p_1 & p_2 & ... \end{pmatrix},\ X_2 \sim F_2 = \begin{pmatrix} 0 & 1 & 2 & ... \\ q_0 & q_1 & q_2 & ... \end{pmatrix},$$

hence $F_1(m) = \sum_{i=0}^{m} p_i,\ F_2(m) = \sum_{i=0}^{m} q_i$. We shall compare these four orderings in the most simple cases. The most convenient comparisons are those which involve only the quantities $\Delta_{i,j}$. They are easily to compare because, according to Proposition 3.2. we have the following **algorithm** :

$X_1 \leqslant_{\text{wruin}} X_2 \Leftrightarrow \Delta_{m,n} \geqslant 0$ for all positive integers m, n such that $m \geqslant n-1$.
$X_1 \leqslant_{\text{Sruin}} X_2 \Leftrightarrow \Delta_{m,n} \geqslant 0$ for all positive integers m, n.

Proposition 4.1. *Let X_1, X_2 be non-negative independent r.v.s. as above. Then*
(i). *If $X_1 \leqslant_{\text{wruin}} X_2$ then $\dfrac{p_0}{q_0} \geqslant \sup_{j\geqslant 1}\left(\dfrac{p_j}{q_j}\right)$.*
(ii). *If $X_1 \leqslant_{\text{Sruin}} X_2$ then $X_1 \leqslant_{\text{st}} X_2$.*
(iii). *If $X_1 \leqslant_{\text{wruin}} X_2$ and $X_2 \leqslant_{\text{wruin}} X_1$ then $X_1 \sim X_2$.*

Proof. (i). Note that for any $n \geqslant 0$ we have $\Delta_{n,1} = \delta_{0,n}$ and we know that $\Delta_{n,1} \geqslant 0$ or, which is the same $p_0 q_n \geqslant p_n q_0$.

(ii). Let $\Delta_{m,\infty} = \lim_{n\to\infty} \Delta_{m,n}$. As $\Delta_{m,n} \geqslant 0$ for all m, n it follows that $\Delta_{m,\infty} \geqslant 0$ for all m. But it is easy to see that $\Delta_{m,\infty} = F_1(m) - F_2(m)$.

(iii). According to (i). we see that $X_1 \leqslant_{\text{wruin}} X_2$ and $X_2 \leqslant_{\text{wruin}} X_1$ imply $\dfrac{p_0}{q_0} \geqslant \sup_{j\geq 1}\left(\dfrac{p_j}{q_j}\right)$ and $\dfrac{q_0}{p_0} \geqslant \sup_{j\geq 1}\left(\dfrac{q_j}{p_j}\right) \Leftrightarrow \dfrac{p_0}{q_0} \leqslant \inf_{j\geq 1}\left(\dfrac{p_j}{q_j}\right)$. Therefore $\sup_{j\geq 1}\left(\dfrac{q_j}{p_j}\right) \leqslant \dfrac{p_0}{q_0} \leqslant \inf_{j\geq 1}\left(\dfrac{p_j}{q_j}\right)$ from where $\dfrac{p_0}{q_0} = \dfrac{p_j}{q_j}$ for any $j \geqslant 1$. As $\sum_{i\geqslant 0} p_i = \sum_{i\geqslant 0} q_i = 1$ it follows that $p_i = q_i, \forall i \geqslant 0$. □

Here is a table with the first quantities $(\Delta_{m,n})_{m,n\geqslant 0}$. Here $\delta_{i,j:k}$ means $\delta_{i,j} + \delta_{i,j+1} + .. + \delta_{i,k}$.

Δ :

m/n	0	1	2	3	4
0	0	$\delta_{0,1}$	$\delta_{0,1:2}$	$\delta_{0,1:3}$	$\delta_{0,1:4}$
1	0	$\delta_{0,2}$	$\delta_{0,2:3} + \delta_{1,2}$	$\delta_{0,2:4} + \delta_{1,2:3}$	$\delta_{0,2:5} + \delta_{1,2:4}$
2	0	$\delta_{0,3}$	$\delta_{0,3:4} + \delta_{1,3}$	$\delta_{0,3:5} + \delta_{1,3:4} + \delta_{2,3}$	$\delta_{0,3:6} + \delta_{1,3:5} + \delta_{2,3:4}$
3	0	$\delta_{0,4}$	$\delta_{0,4:5} + \delta_{1,4}$	$\delta_{0,4:6} + \delta_{1,4:5} + \delta_{2,4}$	$\delta_{0,4:7} + \delta_{1,4:6} + \delta_{2,4:5} + \delta_{3,4}$

So, if

A. **Supp** (F_1) = **Supp** $(F_2) = \{0,1\}$, then the single interesting value of Δ is $\Delta_{0,1} = \delta_{0,1}$ and $\Delta_{1,1}$. Thus $X_1 \leqslant_{\text{wruin}} X_2$ iff $\delta_{0,1} \geqslant 0 \Leftrightarrow p_0 q_1 \geqslant p_1 q_0 \Leftrightarrow p_0(1-q_0) \geqslant q_0(1-p_0) \Leftrightarrow p_0 \geqslant q_0 \Leftrightarrow X_1 \leqslant_{\text{st}} X_2$.

Therefore we have

Corollary 4.2. *If **Supp** (F_1) = **Supp** $(F_2) = \{0,1\}$ then the relations " $\leqslant_{\text{st}}$ ", " $\leqslant_{\text{wruin}}$ ", " $\leqslant_{\text{ruin}}$ ", " $\leqslant_{\text{Sruin}}$ " and " $\leqslant_{\text{like}}$ " are the same.*

B. **Supp** $(F_1) =$ **Supp** $(F_2) = \{0,1,2\}$. In this case only the matrix

$$(\Delta_{i,j})_{0\leqslant i\leq 1, 0\leq j\leqslant 2} = \begin{pmatrix} 0 & \delta_{0,1} & \delta_{0,1:2} \\ 0 & \delta_{0,2} & \delta_{0,2:3} \end{pmatrix}$$

does matter. According to our algorithm $X_1 \leqslant_{\text{wruin}} X_2 \Leftrightarrow \delta_{0,1} \geqslant 0, \delta_{0,2} \geqslant 0, \delta_{0,2:3} + \delta_{1,2} \geqslant 0$. Adding the first two inequalities we also get $\delta_{0,1:2} \geqslant 0 \Leftrightarrow p_0(q_1+q_2) \geqslant q_0(p_1+p_2) \Leftrightarrow p_0(1-q_0) \geqslant q_0(1-p_0) \Leftrightarrow p_0 \geqslant q_0$. The last one is $p_0q_2 + p_1q_2 \geqslant q_0p_2 + q_1p_2 \Leftrightarrow q_2(p_0+p_1) \geqslant p_2(q_0+q_1) \Leftrightarrow q_2 \geqslant p_2 \Leftrightarrow p_0+p_1 \geqslant q_0+q_1$.

But $p_0 \geqslant q_0$, $p_0+p_1 \geqslant q_0+q_1$ means that $X_1 \leqslant_{\text{st}} X_2$. As the conditions for $X_1 \leqslant_{\text{Sruin}} X_2$ are the same, the conclusion is:

Corollary 4.3. *If* ***Supp*** $(F_1) =$ ***Supp*** $(F_2) = \{0,1,2\}$ *then*
i). $X_1 \leqslant_{\text{st}} X_2 \Leftrightarrow p_0 \geqslant q_0,\ p_0+p_1 \geqslant q_0+q_1 \Leftrightarrow p_0 \geqslant q_0,\ p_2 \leqslant q_2$.
ii). $X_1 \leqslant_{\text{wruin}} X_2 \Leftrightarrow \left(X_1 \leqslant_{\text{st}} X_2, \frac{p_0}{q_0} \geqslant \frac{p_1}{q_1}, \frac{p_0}{q_0} \geqslant \frac{p_2}{q_2}\right) \Leftrightarrow X_1 \leqslant_{\text{Sruin}} X_2 \Leftrightarrow X_1 \leqslant_{\text{ruin}} X_2$.
iii). $X_1 \leqslant_{\text{like}} X_2 \Leftrightarrow \frac{p_0}{q_0} \geqslant \frac{p_1}{q_1} \geqslant \frac{p_2}{q_2}$.

Remark 4.1. The answer to the question if the "Sruin" relation implies the "like" relation is negative, as we can see from the counterexample $X_1 \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ and $X_2 \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$ in which $X_1 \leqslant_{\text{Sruin}} X_2$ but it is not true that $X_1 \leqslant_{\text{like}} X_2$.

Remark 4.2. This is the most simple case when one can deny the implication $X_1 \leqslant_{\text{st}} X_2 \Rightarrow X_1 \leqslant_{\text{ruin}} X_2$. For instance, if $X_1 \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{3} & \frac{4}{9} & \frac{2}{9} \end{pmatrix}$ and $X_2 \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$, then $X_1 \leqslant_{\text{st}} X_2$ but it is not true that $X_1 \leqslant_{\text{ruin}} X_2$ since $\frac{p_0}{q_0} = 1 < \frac{p_1}{q_1} = \frac{4}{3}$.

C. **Supp** $(F_1) =$ **Supp** $(F_2) = \{0,1,2,3\}$.
The interesting part of Δ is

$$(\Delta_{i,j})_{0\leqslant i\leq 2, 0\leq j\leqslant 3} = \begin{pmatrix} 0 & \delta_{0,1} & \delta_{0,1:2} & \delta_{0,1:3} \\ 0 & \delta_{0,2} & \delta_{0,2:3}+\delta_{1,2} & \delta_{0,2:3}+\delta_{1,2:3} \\ 0 & \delta_{0,3} & \delta_{0,3}+\delta_{1,3} & \delta_{0,3}+\delta_{1,3}+\delta_{2,3} \end{pmatrix}.$$

Then$X_1 \leqslant_{\text{st}} X_2 \Leftrightarrow p_0 \geqslant q_0,\ p_0+p_1 \geqslant q_0+q_1,\ p_0+p_1+p_2 \geqslant q_0+q_1+q_2$ and $X_1 \leqslant_{\text{wruin}} X_2 \Leftrightarrow \delta_{0,1} \geqslant 0, \delta_{0,2} \geqslant 0, \delta_{0,3} \geqslant 0, \delta_{0,2:3}+\delta_{1,2} \geqslant 0, \delta_{0,3}+\delta_{1,3} \geqslant 0, \delta_{0,3}+\delta_{1,3}+\delta_{2,3} \geqslant 0$.Adding the first three inequalities, we obtain $p_0 \geqslant q_0$; the last one says that $p_3 \leqslant q_3$ or, which is the same, that $p_0+p_1+p_2 \geqslant q_0+q_1+q_2$.

The relation $X_1 \leqslant_{\text{sruin}} X_2$ adds to the inequalities for the weak comparison three more ones: $\delta_{0,1:2} \geqslant 0, \delta_{0,1:3} \geqslant 0$ and $\delta_{0,2:3} \geqslant 0, \delta_{1,2:3} \geqslant 0$. The first two of them are superfluous since they are implied by $\delta_{0,1} \geqslant 0, \delta_{0,2} \geqslant 0, \delta_{0,3} \geqslant 0$. The remaining one is not: written as $(p_0+p_1)(q_2+q_3) \geqslant (q_0+q_1)(p_2+p_3) \Leftrightarrow$
$(p_0+p_1)(1-(q_0+q_1)) \geqslant (q_0+q_1)(1-(p_0+p_1))$ it implies the stochastic order. To summarize:

Corollary 4.4. *If* ***Supp*** $(F_1) =$ ***Supp*** $(F_2) = \{0,1,2,3\}$ *then*
i). $X_1 \leqslant_{\text{st}} X_2 \Leftrightarrow p_0 \geqslant q_0,\ p_0+p_1 \geqslant q_0+q_1,\ p_0+p_1+p_2 \geqslant q_0+q_1+q_2$.
ii). $X_1 \leqslant_{\text{wruin}} X_2 \Leftrightarrow (\frac{p_0}{q_0} \geqslant \max\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right), p_0 \geqslant q_0, p_3 \leqslant q_3, \delta_{0,2:3}+\delta_{1,2} \geqslant 0,$

$\delta_{0,3} + \delta_{1,3} \geqslant 0)$.

iii). $X_1 \leqslant_{\text{Sruin}} X_2 \Leftrightarrow (X_1 \leqslant_{\text{wruin}} X_2)$ & $(X_1 \leqslant_{\text{st}} X_2)$.

iv). $X_1 \leqslant_{\text{like}} X_2 \Leftrightarrow \frac{p_0}{q_0} \geqslant \frac{p_1}{q_1} \geqslant \frac{p_2}{q_2} \geqslant \frac{p_3}{q_3}$.

The relation $X_1 \leqslant_{\text{ruin}} X_2$ should be between the last ones. It is possible to coincide with the relation $X_1 \leqslant_{\text{Sruin}} X_2$.

Remark 4.3. This is the first case when " $\leqslant_{\text{wruin}}$" does not imply the stochastic domination. For instance, if $X_1 \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ and $X_2 \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{11}{20} & \frac{3}{20} & \frac{6}{20} \end{pmatrix}$, the reader can check that $X_1 \leqslant_{\text{wruin}} X_2$ but it is not true that $X_1 \leqslant_{\text{st}} X_2$.

Remark 4.4. The notation " $\leqslant$ " attributed to ruin domination might not be the most proper one, since this is **not an ordering relation** as one can easily see from the following example: $X_1 \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$, $X_2 \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$, $X_3 \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & \frac{4}{7} & \frac{1}{7} & \frac{2}{7} \end{pmatrix}$ for which $X_1 \leqslant_{\text{wruin}} X_2 \leqslant_{\text{wruin}} X_3$ and yet it is not true that $X_1 \leqslant_{\text{wruin}} X_3$.

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