Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 43(1), 2016, Pages 88–93
ISSN: 1223-6934

Dedicated to Marius Iosifescu
on the occasion of his 80th anniversary

Recent advances in the metric theory of θ-expansions

Gabriela Ileana Sebe and Dan Lascu

Abstract. A survey of the metric theory of θ-expansions discussed in [1, 5, 6, 7] is given. The limit properties of these expansions have been studied. Using a Wirsing-type approach to the Perron-Frobenius operator of the generalized Gauss map under its invariant measure we find a near-optimal solution to the Gauss-Kuzmin-Lévy problem for θ-expansions.

2010 Mathematics Subject Classification. 49J20, 49K20.
Key words and phrases. Regular continued fractions, θ-expansions, invariant measure, Gauss-Kuzmin-Lévy problem, Perron-Frobenius operator.

1. Introduction

In this paper we consider another expansion of reals different from the regular continued fraction expansion. In fact, one particular expansion discussed by Chakraborty and Rao in [1], which was studied in detail by Sebe and Lascu in [5, 6, 7], has raised to a new type of continued fractions, namely θ-expansions.

1.1. Preliminary considerations. Fix an irrational $\theta \in (0,1)$. Define on $(0,\theta)$ the transformation T_θ by

$$T_\theta(x) := \begin{cases} \dfrac{1}{x} - \theta \left\lfloor \dfrac{1}{x\theta} \right\rfloor & \text{if } x \in (0,\theta], \\ 0 & \text{if } x = 0 \end{cases} \tag{1}$$

where $\lfloor \cdot \rfloor$ stands for integer part. For any $x \in (0,\theta)$ put

$$a_n(x) = a_1\left(T_\theta^{n-1}(x)\right), \quad n \in \mathbb{N}_+ \tag{2}$$

with $T_\theta^0(x) = x$ and

$$a_1(x) = \begin{cases} \lfloor \frac{1}{x\theta} \rfloor & \text{if } \quad x \neq 0, \\ \infty & \text{if } \quad x = 0. \end{cases} \tag{3}$$

Then every $x \in (0,\theta)$ has an infinite expansion

$$x = \cfrac{1}{a_1\theta + \cfrac{1}{a_2\theta + \cfrac{1}{a_3\theta + \ddots}}} = [a_1\theta, a_2\theta, a_3\theta \ldots]. \tag{4}$$

We call (4) the θ-expansion of x. Such a_n's are called *continued fraction digits* of x with respect to the expansion in (4).

Received April 24, 2016.

The numbers $p_n(x)/q_n(x) = [a_1\theta, a_2\theta, \ldots, a_n\theta]$ are the n-th order convergents of $x \in [0,\theta]$. Then $p_n(x)/q_n(x) \to x$, $n \to \infty$. Here p_n's and q_n's satisfy for $n \in \mathbb{N}_+$ the following: $p_n(x) := a_n\theta p_{n-1}(x) + p_{n-2}(x)$, $q_n(x) := a_n\theta q_{n-1}(x) + q_{n-2}(x)$ and $p_{-1}(x) := 1$, $p_0(x) := 0$, $q_{-1}(x) := 0$, $q_0(x) := 1$.

In [1] it was shown that for $\theta^2 = 1/m$, $m \in \mathbb{N}_+$, the invariant probability measure of the transformation T_θ is

$$\gamma_\theta(A) := \frac{1}{\log(1+\theta^2)} \int_A \frac{\theta dx}{1+\theta x}, \quad A \in \mathcal{B}_{[0,\theta]}. \tag{5}$$

Hence, $\gamma_\theta(A) = \gamma_\theta(T_\theta^{-1}(A))$ for any $A \in \mathcal{B}_{[0,\theta]}$, the sequence $(a_n)_{n\in\mathbb{N}_+}$ is strictly stationary on $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta)$. Clearly, the case $\theta = 1$ corresponds to the regular continued fraction expansion, intensively studied in [3].

1.2. Some metric properties. Roughly speaking, the metrical theory of continued fraction expansions is about the sequence $(a_n)_{n\in\mathbb{N}_+}$ and related sequences. Let us fix $0 < \theta < 1$, $\theta^2 = 1/m$, $m \in \mathbb{N}_+$. Putting $\mathbb{N}_m = \{m, m+1, ...\}$, $m \in \mathbb{N}_+$, the digits a_n, $n \in \mathbb{N}_+$, take positive integer values in $\mathbb{N}_m$.

For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \ldots, i_n) \in \mathbb{N}_m^n$, define the *fundamental interval associated with* $i^{(n)}$ by

$$I\left(i^{(n)}\right) = \{x \in [0,\theta] : a_k(x) = i_k \text{ for } k = 1, \ldots, n\}, \tag{6}$$

where $I\left(i^{(0)}\right) = [0,\theta]$. We will write $I(a_1, \ldots, a_n) = I\left(a^{(n)}\right)$, $n \in \mathbb{N}_+$. If $n \geq 1$ and $i_n \in \mathbb{N}_m$, then we have $I(a_1, \ldots, a_n) = I\left(i^{(n)}\right)$.

If λ_θ denote the Lebesgue measure on $[0,\theta]$, it can be shown that

$$\lambda_\theta(T_\theta^n < x | a_1, \ldots, a_n) = \frac{(s_n\theta + 1)x}{\theta(s_n x + 1)}, \quad x \in [0,\theta], n \in \mathbb{N}_+ \tag{7}$$

where $s_n := q_{n-1}/q_n$, $n \in \mathbb{N}_+$ and $s_0 := 0$. Equation (7) is the Brodén-Borel-Lévy formula for θ-expansions. It allows us to determine the probability distribution of $(a_n)_{n\in\mathbb{N}_+}$ under λ_θ. Clearly, $\lambda_\theta(a_1 = i) = m/(i(i+1))$, $i \in \mathbb{N}_m$, and $\lambda_\theta\left(a_{n+1} = i | a_1, \ldots, a_n\right) = P_i(s_n)$, where

$$P_i(x) := \frac{x\theta + 1}{(x + i\theta)(x + (i+1)\theta)}. \tag{8}$$

We have already noticed that the sequence $(a_n)_{n\in\mathbb{N}_+}$ is strictly stationary on $\left([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta\right)$. As such, a doubly infinite version of it, say $(\overline{a}_l)_{l\in\mathbb{Z}}$, should exist on a richer probability space. Indeed, such a version can be effectively constructed on $\left([0,\theta]^2, \mathcal{B}^2_{[0,\theta]}, \overline{\gamma_\theta}\right)$, where $\overline{\gamma_\theta}$ is the extended measure defined by

$$\overline{\gamma_\theta}(B) := \frac{1}{\log(1+\theta^2)} \iint_B \frac{\mathrm{d}x\mathrm{d}y}{(1+xy)^2}, \quad B \in \mathcal{B}^2_{[0,\theta]}. \tag{9}$$

Put $\overline{a}_n(x,y) = a_n(x)$, $\overline{a}_0(x,y) = a_1(y)$, $\overline{a}_{-n}(x,y) = a_{n+1}(y)$, for any $n \in \mathbb{N}_+$ and $(x,y) \in [0,\theta]^2$. Then for any $l \in \mathbb{Z}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}_+$ the probability distribution of the random vector $(\overline{a}_l, \ldots, \overline{a}_{l+k})$ under $\overline{\gamma_\theta}$ is identical with that of the random vector $(a_n, \ldots, a_{n+k})$ under γ_θ. In other words, $(\overline{a}_l)_{l\in\mathbb{Z}}$ is (under $\overline{\gamma_\theta}$) a doubly infinite version of $(a_n)_{n\in\mathbb{N}_+}$ (under γ_θ).

The definition of $(\overline{a}_l)_{l\in\mathbb{Z}}$ is associated with the natural extension $\overline{T_\theta}$ of T_θ, which is a transformation of $[0,\theta]^2$ defined by

$$\overline{T_\theta}(x,y) := \left(T_\theta(x),\ \frac{1}{a_1(x)\theta+y}\right), \quad (x,y)\in[0,\theta]^2. \tag{10}$$

This is a one-to-one transformation of $[0,\theta]^2$ with the inverse

$$(\overline{T_\theta})^{-1}(x,y) = \left(\frac{1}{a_1(y)\theta+x},\ T_\theta(y)\right), \quad (x,y)\in[0,\theta]^2\,. \tag{11}$$

The extended measure $\overline{\gamma_\theta}$ is $\overline{T_\theta}$-invariant, that is, $\overline{\gamma_\theta} = \overline{\gamma_\theta}\,\overline{T_\theta}^{-1}$, and $\overline{a}_{l+1}(x,y) = \overline{a}_1((\overline{T_\theta})^l(x,y))$, $l\in\mathbb{Z}$, with $\overline{a}_1(x,y)=a_1(x)$, $(x,y)\in[0,\theta]^2$. Hence the sequence $(\overline{a}_l)_{l\in\mathbb{Z}}$ is strictly stationary on $\left([0,\theta]^2, \mathcal{B}^2_{[0,\theta]}, \overline{\gamma_\theta}\right)$.

The dependence structure of $(\overline{a}_l)_{l\in\mathbb{Z}}$ follows from the fact that

$$\overline{\gamma_\theta}([0,x]\times[0,\theta]\,|\,\overline{a}_0,\overline{a}_{-1},\ldots) = \frac{(a\theta+1)x}{(ax+1)\theta} \quad \overline{\gamma_\theta}\text{-a.s.}, \tag{12}$$

for any $x\in[0,\theta]$, where $a := [\overline{a}_0\theta, \overline{a}_{-1}\theta, \ldots]$. Hence

$$\overline{\gamma_\theta}(\overline{a}_1 = i|\,\overline{a}_0,\overline{a}_{-1},\ldots) = P_i(a) \quad \overline{\gamma_\theta}\text{-a.s.}, \tag{13}$$

for any $i\in\mathbb{N}_m$, and by the strict stationarity of $(\overline{a}_l)_{l\in\mathbb{Z}}$ under $\overline{\gamma_\theta}$ we also have

$$\overline{\gamma_\theta}(\overline{a}_{l+1} = i\,|\,\overline{a}_l,\overline{a}_{l-1},\ldots) = P_i(a) \quad \overline{\gamma}_\theta\text{-a.s.} \tag{14}$$

with $a=[\overline{a}_l\theta,\overline{a}_{l-1}\theta,\ldots]$ for $l\in\mathbb{Z}$ and $i\in\mathbb{N}_m$. We thus see that $(\overline{a}_l)_{l\in\mathbb{Z}}$ is an infinite order chain [2] on $\left([0,\theta]^2, \mathcal{B}^2_{[0,\theta]}, \overline{\gamma_\theta}\right)$.

2. Solving Gauss's problem

It is only recently [5, 6] that the limits and ergodic properties of these expansions have been studied. It should be stressed that the ergodic theorem does not yield rates of convergence for mixing properties; for this a Gauss-Kuzmin theorem is needed.

2.1. Limits properties. Let us consider the random system with complete connections RSCC [2]

$$\left\{\left([0,\theta],\mathcal{B}_{[0,\theta]}\right), \left(\mathbb{N}_m, \mathcal{P}(\mathbb{N}_m)\right), u, P\right\}, \tag{15}$$

where $u : [0,\theta]\times\mathbb{N}_m \to [0,\theta]$, $u(s,i)=u_i(s) := 1/(s+i\theta)$ and the function $P(s,i)=P_i(s) := (x\theta+1)/((x+i\theta)(x+(i+1)\theta))$ defines a transition probability from $([0,\theta],\mathcal{B}_{[0,\theta]})$ to $(\mathbb{N}_m, \mathcal{P}((\mathbb{N}_m)))$. Here $\mathcal{P}(\mathbb{N}_m)$ denotes the power set of $\mathbb{N}_m$. For any $a\in[0,\theta]$ let $s_{0,a} := a$, $s_{n,a} := 1/(a_n\theta + s_{n-1,a})$, $n\in\mathbb{N}_+$, and consider the family $(\gamma_{\theta,a})_{a\in[0,\theta]}$ of probability measures on $\mathcal{B}_{[0,\theta]}$ defined by their distribution functions $\gamma_{\theta,a}([0,x]) := (a\theta+1)x/((ax+1)\theta)$. The sequence $(s_{n,a})_{n\in\mathbb{N}_+}$ is an $[0,\theta]$-valued Markov chain on $([0,\theta],\mathcal{B}_{[0,\theta]},\gamma_{\theta,a})$ which starts at $s_{0,a} := a$ and has the following transition mechanism: from state $s\in[0,\theta]$ the possible transitions are to any state $1/(s+i\theta)$ with corresponding transition probability $P_i(s)$, $i\in\mathbb{N}_m$. Thus the transition operator (Perron-Frobenius operator) U of all Markov chains $(s_{n,a})_{n\in\mathbb{N}_+}$ for any bounded complex-valued measurable function f on $[0,\theta]$, is given by

$$Uf(x) = \sum_{i\geq m} P_i(x)\,f(u_i(x)), \quad m\in\mathbb{N}_+ \tag{16}$$

where $f \in L^1_{\gamma_\theta} := \{f : [0,\theta] \to \mathbb{C} : \int_0^\theta |f| \mathrm{d}\gamma_\theta < \infty\}$. It was investigated in [5, 6] the Perron-Frobenius operator of the continued fraction transformation T_θ under different probability measures on $\mathcal{B}_{[0,\theta]}$. The asymptotic behavior of this operator is derived in [6] and is given by

$$\mu\left((T_\theta)^{-n}(A)\right) = \int_A U^n f(x) \mathrm{d}\gamma_\theta(x), \tag{17}$$

where $f(x) := (\log(1+\theta^2)) \frac{x\theta+1}{\theta} h(x)$, $x \in [0,\theta]$.

In the sequel the domain of U will be successively restricted to various Banach spaces. Recall that the variation $\mathrm{var}_A f$ of f on a subset A of $[0,\theta]$ is defined as $\sup \sum_{i=1}^{k-1} |f(t_i) - f(t_{i-1})|$ the supremum being taken over all $t_1 < \cdots < t_k \in A$ and $k \geq 2$. If $\mathrm{var} f = \mathrm{var}_{[0,\theta]} f < \infty$, then f is called a function of bounded variation. A variation $\nu(f)$ for $L^\infty([0,\theta], \mathcal{B}_{[0,\theta]}, \lambda_\theta)$, the collection of all classes of λ_θ - essentially bounded measurable complex-valued λ_θ-indistinguishable function on $[0,\theta]$ is defined as $\nu(f) = \inf \mathrm{var} f$, the infimum being taken over all versions of f. The set $BEV([0,\theta])$ is a Banach space under the norm $\|f\|_\nu := \nu(f) + \|f\|_1$, where $\|\cdot\|_1$ is the usual $L^1_{\lambda_\theta}$ norm $\|f\|_1 = \int_0^\theta |f| \mathrm{d}\lambda_\theta$. For proofs and more details see [5, 6].

Whatever $a \in [0,\theta]$ the Markov chain $(s_{n,a})_{n\in\mathbb{N}}$ associated with the RSCC (15) has the transition operator U, with the transition probability function

$$Q(s,B) = \sum_{\{\, i \geq m | u_i(s) \in B\}} P_i(s), \quad s \in [0,\theta], B \in \mathcal{B}_{[0,\theta]}. \tag{18}$$

Then $Q^n(\cdot,\cdot)$ will denote the n-step transition probability function of the same Markov chain.

It was proved in [6] that the RSCC (15) is uniformly ergodic and its transition operator is regular with respect to the Banach space of Lipschitz functions.

Now for a probability measure μ on $\left([0,\theta], \mathcal{B}_{[0,\theta]}\right)$ we may determine the limit of the sequence $(\mu(T_\theta^n < x))_{n\in\mathbb{N}_+}$ as $n \to \infty$ and obtain the rate of this convergence, i.e.,

$$\lim_{n\to\infty} \mu(T_\theta^n < x) = \frac{1}{\log(1+\theta^2)} \log((m\theta + x)\theta), \quad x \in [0,\theta]. \tag{19}$$

2.2. A Gauss-Kuzmin-Lévy-type theorem. The study of optimality of the convergence rate remains an open question. Using a Wirsing-type approach [8], in [7] it was obtained a better estimate of the convergence rate involved. The strategy was to restrict the domain of the Perron-Frobenius operator of T_θ under its invariant measure γ_θ to the Banach space of functions which have a continuous derivative on $[0,\theta]$. Define a linear operator $V : C([0,\theta]) \to C([0,\theta])$ by $Vg = -(Uf)'$, $g \in C([0,\theta])$, where $f' = g$. Since U is a Markov operator, V is well defined. It is easy to check that $(U^n f)' = (-1)^n V^n f'$, $n \in \mathbb{N}_+$, $f \in C^1([0,\theta])$. Sebe proved in [7] that there are positive constants $v_\theta < w_\theta < 1$ and a real-valued function $\varphi_\theta \in C([0,\theta])$ defined by

$$\varphi_\theta(x) = \frac{1}{\theta}\left[\frac{e_\theta(m+1)\theta - a_\theta - 1}{(e_\theta + x(-e_\theta m\theta + a_\theta + 1))^2} - \frac{e_\theta(m-1)\theta - a_\theta - 1}{(e_\theta + x(-e_\theta(m-1)\theta + a_\theta + 1))^2}\right],$$

$x \in [0,\theta]$, where the coefficient e_θ is chosen such that the equation

$$E_\theta(x) = 2\theta(x+1)^4 - e_\theta^3\left[(2m+1)(x+1) + e_\theta(m+1)\theta\right] = 0$$

$x \in [0,\theta]$ has a unique solution $a_\theta \in [0,\theta]$. For this unique acceptable $a_\theta \in [0,\theta]$ we have $v_\theta \varphi_\theta \leq V\varphi_\theta \leq w_\theta \varphi_\theta$. Next, putting $\alpha_\theta = \min_{x\in[0,\theta]} \varphi_\theta(x)/(f_\theta)'(x)$ and $\beta_\theta = \max_{x\in[0,\theta]} \varphi_\theta(x)/(f_\theta)'(x)$ for any $f_\theta \in C^1([0,\theta])$ such that $(f_\theta)' > 0$, we get

$$\frac{\alpha_\theta}{\beta_\theta} v_\theta^n (f_\theta)' \leq V^n (f_\theta)' \leq \frac{\beta_\theta}{\alpha_\theta} w_\theta^n (f_\theta)',$$

$n \in \mathbb{N}_+$.

In Theorem 5.3 in [7] there are obtained upper and lower bounds of the convergence rate, respectively $\mathcal{O}(w_n)$ and $\mathcal{O}(v_n)$ as $n \to \infty$, which provide a near-optimal solution to the Gauss-Kuzmin-Lévy problem.

Let μ be a probability measure on $\mathcal{B}_{[0,\theta]}$ such that $\mu \ll \lambda_\theta$. For any $n \in \mathbb{N}$ put $F_\theta^n(x) = \mu(T_\theta^n < x)$, $x \in [0,\theta]$, where T_θ^0 is the identity map. Let $f_\theta^0(x) = \frac{x\theta+1}{\theta}(F_\theta^0)'(x)$, $x \in [0,\theta]$, where $(F_\theta^0)' = d\mu/d\lambda_\theta$. Let us recall this theorem.

Theorem 2.1. *Let $f_\theta^0 \in C^1([0,\theta])$ such that $(f_\theta^0)' > 0$ and let μ be a probability measure on $\mathcal{B}_{[0,\theta]}$ such that $\mu \ll \lambda_\theta$. For any $n \in \mathbb{N}_+$ and $x \in [0,\theta]$ we have*

$$(\log(1+\theta^2))^2 \frac{\alpha_\theta}{2\theta\beta_\theta} \min_{x\in[0,\theta]} (f_\theta^0)'(x) v_\theta^n G_\theta(x)(\theta - G_\theta(x)) \leq |\mu(T_\theta^n < x) - G_\theta(x)|$$

$$\leq (\log(1+\theta^2))^2 \frac{(1+\theta^2)\beta_\theta}{2\theta\alpha_\theta} \max_{x\in[0,\theta]} (f_\theta^0)'(x) w_\theta^n G_\theta(x)(\theta - G_\theta(x))$$

where

$$G_\theta(x) = \frac{\log(1+x\theta)}{\log(1+\theta^2)}.$$

For example, for $m = 3$, the equation $E_\theta(x) = 0$, with $e_\theta = 0.67$, has as unique acceptable solution $a_\theta = 0.287897$. For this value of a_θ the function $\varphi_{a_\theta}/V\varphi_{a_\theta}$ attains its maximum equal to 7.389969626 at $x = 0$ and $x = \theta$, and has a minimum $m(a_\theta) = (\varphi_{a_\theta}/V\varphi_{a_\theta})(0.256122) = 7.29924$. It follows that upper and lower bounds of the convergence rate are respectively $O(w_\theta^n)$ and $O(v_\theta^n)$ as $n \to \infty$, with $v_\theta > 0.135318553$ and $w_\theta < 0.137000564$.

Obviously, the determination of the exact convergence rate remains an open question. We may derive it using the same strategy as in [4].

Acknowledgments

The authors would like to express their sincere thanks and unlimited gratitude to Professor Marius Iosifescu for his entire scientific guidance.

References

[1] S. Chakraborty, B.V. Rao, θ-expansions and the generalized Gauss map, In (K. Athreya, M. Majumdar, M. Puri, E. Waymire (eds.)), *Probability, Statistics, and Their Applications: Papers in Honor of Rabi Bhattacharya*, Institute of Mathematical Statistics, Lecture Notes-Monograph Series **41** (2003), 49–64.

[2] M. Iosifescu, S. Grigorescu, *Dependence With Complete Connections and its Applications*, Cambridge Tracts in Mathematics **96**, Cambridge Univ. Press, Cambridge, 1990; [(2009): second printing slightly corrected].

[3] M. Iosifescu, C. Kraaikamp, *Metrical Theory of Continued Fractions*, Kluwer Academic Publishers, Dordrecht, 2002.

[4] M. Iosifescu, G.I. Sebe, An Exact Convergence Rate in a Gauss-Kuzmin-Lévy Problem for Some Continued Fraction Expansion. In *Mathematical Analysis and Applications*, 90–109, AIP Conf. Proc. **835**, Amer. Inst. Physics, Melville, New York, 2006.

[5] D. Lascu, F. Nicolae, *Ergodic properties of θ-expansions and a Gauss-Kuzmin-type problem*, arXiv preprint arXiv: 1405.3882 (2014), 1–19.

[6] G.I. Sebe, D. Lascu, A Gauss-Kuzmin theorem and related questions for θ-expansions, *Journal of Function Spaces* (2014) **2014**, Article ID 980461, 12 pages.

[7] G.I. Sebe, A near-optimal solution to the Gauss-Kuzmin-Lévy problem for θ-expansions. *Submitted.*

[8] E. Wirsing, On the theorem of Gauss-Kuzmin-Lévy and a Frobenius-type theorem for function spaces, *Acta Arithmetica* (1974) **24**, 506–528.

(Gabriela Ileana Sebe) Politehnica University of Bucharest, Faculty of Applied Sciences, Splaiul Independentei 313, 060042, Bucharest, Romania
Institute of Mathematical Statistics and Applied Mathematics, Calea 13 Septembrie, nr. 13, Bucharest, Romania
E-mail address: igsebe@yahoo.com

(Dan Lascu) Mircea cel Batran Naval Academy, Fulgerului nr. 1, 900218 Constanta, Romania
E-mail address: lascudan@gmail.com