Recent advances in the metric theory of \(\theta\)-expansions

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Abstract. A survey of the metric theory of \(\theta\)-expansions discussed in [1, 5, 6, 7] is given. The limit properties of these expansions have been studied. Using a Wirsing-type approach to the Perron-Frobenius operator of the generalized Gauss map under its invariant measure we find a near-optimal solution to the Gauss-Kuzmin-Lévy problem for \(\theta\)-expansions.

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1. Introduction

In this paper we consider another expansion of reals different from the regular continued fraction expansion. In fact, one particular expansion discussed by Chakraborty and Rao in [1], which was studied in detail by Sebe and Lascu in [5, 6, 7], has raised to a new type of continued fractions, namely \(\theta\)-expansions.

1.1. Preliminary considerations. Fix an irrational \(\theta\in (0,1)\). Define on \((0,\theta)\) the transformation \(T_\theta\) by

\[
T_\theta(x) := \begin{cases} 
\frac{1}{x} - \theta \left\lfloor \frac{1}{x\theta} \right\rfloor & \text{if } x \in (0, \theta], \\
0 & \text{if } x = 0 
\end{cases}
\]

(1)

where \([\cdot]\) stands for integer part. For any \(x \in (0, \theta)\) put

\[
a_n(x) = a_1 \left(T_\theta^{n-1}(x)\right), \quad n \in \mathbb{N}_+
\]

(2)

with \(T_\theta^n(x) = x\) and

\[
a_1(x) = \begin{cases} 
\left\lfloor \frac{1}{x \theta} \right\rfloor & \text{if } x \neq 0, \\
\infty & \text{if } x = 0.
\end{cases}
\]

(3)

Then every \(x \in (0, \theta)\) has an infinite expansion

\[
x = \frac{1}{a_1 \theta + \frac{1}{a_2 \theta + \frac{1}{a_3 \theta + \ddots}}} = [a_1 \theta, a_2 \theta, a_3 \theta \ldots].
\]

(4)

We call (4) the \(\theta\)-expansion of \(x\). Such \(a_n\)'s are called continued fraction digits of \(x\) with respect to the expansion in (4).
The numbers $p_n(x)/q_n(x) = [a_1\theta, a_2\theta, \ldots, a_n\theta]$ are the $n$-th order convergents of $x \in [0, \theta]$. Then $p_n(x)/q_n(x) \to x$, $n \to \infty$. Here $p_n$’s and $q_n$’s satisfy for $n \in \mathbb{N}_+$ the following: $p_n(x) := a_n \theta p_{n-1}(x) + p_{n-2}(x)$, $q_n(x) := a_n \theta q_{n-1}(x) + q_{n-2}(x)$ and $p_{-1}(x) := 1$, $p_0(x) := 0$, $q_{-1}(x) := 0$, $q_0(x) := 1$.

In [1] it was shown that for $\theta^2 = 1/m$, $m \in \mathbb{N}_+$, the invariant probability measure of the transformation $T_\theta$ is

$$
\gamma_\theta(A) := \frac{1}{\log(1 + \theta^2)} \int_A \frac{\theta dx}{1 + \theta x}, \quad A \in \mathcal{B}_{[0,\theta]}.
$$

Hence, $\gamma_\theta(A) = \gamma_\theta(T_\theta^{-1}(A))$ for any $A \in \mathcal{B}_{[0,\theta]}$, the sequence $(a_n)_{n \in \mathbb{N}_+}$ is strictly stationary on $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta)$. Clearly, the case $\theta = 1$ corresponds to the regular continued fraction expansion, intensively studied in [3].

### 1.2. Some metric properties

Roughly speaking, the metrical theory of continued fraction expansions is about the sequence $(a_n)_{n \in \mathbb{N}_+}$ and related sequences. Let us fix $0 < \theta < 1$, $\theta^2 = 1/m$, $m \in \mathbb{N}_+$. Putting $\mathbb{N}_m = \{m, m+1, \ldots\}$, $m \in \mathbb{N}_+$, the digits $a_n$, $n \in \mathbb{N}_+$, take positive integer values in $\mathbb{N}_m$.

For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \ldots, i_n) \in \mathbb{N}_m^n$, define the fundamental interval associated with $i^{(n)}$ by

$$
I(i^{(n)}) = \{x \in [0,\theta] : a_k(x) = i_k \text{ for } k = 1, \ldots, n\},
$$

where $I(i^{(0)}) = [0,\theta]$. We will write $I(a_1, \ldots, a_n) = I(a^{(n)})$, $n \in \mathbb{N}_+$. If $n \geq 1$ and $i_n \in \mathbb{N}_m$, then we have $I(a_1, \ldots, a_n) = I(i^{(n)})$.

If $\lambda_\theta$ denote the Lebesgue measure on $[0,\theta]$, it can be shown that

$$
\lambda_\theta(T_\theta^n < x|a_1, \ldots, a_n) = \frac{(s_n\theta + 1)x}{\theta(s_n x + 1)}, \quad x \in [0,\theta], n \in \mathbb{N}_+
$$

where $s_n := q_{n-1}/q_n$, $n \in \mathbb{N}_+$ and $s_0 := 0$. Equation (7) is the Brodén-Borel-Lévy formula for $\theta$-expansions. It allows us to determine the probability distribution of $(a_n)_{n \in \mathbb{N}_+}$ under $\lambda_\theta$. Clearly, $\lambda_\theta(a_1 = i) = m/(i(i + 1))$, $i \in \mathbb{N}_m$, and $\lambda_\theta(a_{n+1} = i|a_1, \ldots, a_n) = P_i(s_n)$, where

$$
P_i(x) := \frac{x\theta + 1}{(x + i\theta)(x + (i + 1)\theta)}.
$$

We have already noticed that the sequence $(a_n)_{n \in \mathbb{N}_+}$ is strictly stationary on $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta)$. As such, a doubly infinite version of it, say $(\overline{a}_i)_{i \in \mathbb{Z}}$, should exist on a richer probability space. Indeed, such a version can be effectively constructed on $\left([0,\theta]^2, \mathcal{B}_{[0,\theta]}^2, \overline{\gamma}_\theta\right)$, where $\overline{\gamma}_\theta$ is the extended measure defined by

$$
\overline{\gamma}_\theta(B) := \frac{1}{\log(1 + \theta^2)} \int \int_B \frac{dxdy}{(1 + xy)^2}, \quad B \in \mathcal{B}_{[0,\theta]}^2.
$$

Put $\overline{a}_n(x, y) = a_n(x)$, $\overline{a}_0(x, y) = a_1(y)$, $\overline{a}_{-n}(x, y) = a_{n+1}(y)$, for any $n \in \mathbb{N}_+$ and $(x, y) \in [0,\theta]^2$. Then for any $l \in \mathbb{Z}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}_+$ the probability distribution of the random vector $(\overline{a}_l, \ldots, \overline{a}_{l+k})$ under $\overline{\gamma}_\theta$ is identical with that of the random vector $(a_n, \ldots, a_{n+k})$ under $\gamma_\theta$. In other words, $(\overline{a}_i)_{i \in \mathbb{Z}}$ is (under $\overline{\gamma}_\theta$) a doubly infinite version of $(a_n)_{n \in \mathbb{N}_+}$ (under $\gamma_\theta$).
The definition of \((\bar{a}_l)_{l \in \mathbb{Z}}\) is associated with the natural extension \(T_\theta\) of \(T_\theta\), which is a transformation of \([0, \theta]^2\) defined by

\[
T_\theta(x, y) := \left( T_\theta(x), \frac{1}{a_1(x) \theta + y} \right), \quad (x, y) \in [0, \theta]^2.
\]

This is a one-to-one transformation of \([0, \theta]^2\) with the inverse

\[
(T_\theta)^{-1}(x, y) = \left( \frac{1}{a_1(y) \theta + x}, T_\theta(y) \right), \quad (x, y) \in [0, \theta]^2.
\]

The extended measure \(\bar{\gamma}_\theta\) is \(T_\theta\)-invariant, that is, \(\bar{\gamma}_\theta = \bar{\gamma}_\theta (T_\theta)^{-1}\), and \(\bar{a}_{l+1}(x, y) = \bar{a}_l((T_\theta)^l(x, y))\), \(l \in \mathbb{Z}\), with \(\bar{a}_1(x, y) = a_1(x), (x, y) \in [0, \theta]^2\). Hence the sequence \((\bar{a}_l)_{l \in \mathbb{Z}}\) is strictly stationary on \(([0, \theta]^2, \mathcal{B}^2_{[0, \theta]}, \bar{\gamma}_\theta)\).

The dependence structure of \((\bar{a}_l)_{l \in \mathbb{Z}}\) follows from the fact that

\[
\bar{\gamma}_\theta([0, x] \times [0, \theta] | \bar{a}_0, \bar{a}_1, \ldots) = \frac{(a \theta + 1)x}{(ax + 1)\theta} \quad \bar{\gamma}_\theta\text{-a.s.},
\]

for any \(x \in [0, \theta]\), where \(a := [\bar{a}_0, \bar{a}_1, \ldots]\). Hence

\[
\bar{\gamma}_\theta(\bar{a}_1 = i | \bar{a}_0, \bar{a}_1, \ldots) = P_i(a) \quad \bar{\gamma}_\theta\text{-a.s.},
\]

for any \(i \in \mathbb{N}_m\), and by the strict stationarity of \((\bar{a}_l)_{l \in \mathbb{Z}}\) under \(\bar{\gamma}_\theta\) we also have

\[
\bar{\gamma}_\theta(\bar{a}_{l+1} = i | \bar{a}_l, \bar{a}_{l+1}, \ldots) = P_i(a) \quad \bar{\gamma}_\theta\text{-a.s.}
\]

with \(a = [\bar{a}_l, \bar{a}_{l+1}, \ldots]\) for \(l \in \mathbb{Z}\) and \(i \in \mathbb{N}_m\). We thus see that \((\bar{a}_l)_{l \in \mathbb{Z}}\) is an infinite order chain [2] on \(([0, \theta]^2, \mathcal{B}^2_{[0, \theta]}, \bar{\gamma}_\theta)\).

2. Solving Gauss’s problem

It is only recently [5, 6] that the limits and ergodic properties of these expansions have been studied. It should be stressed that the ergodic theorem does not yield rates of convergence for mixing properties; for this a Gauss-Kuzmin theorem is needed.

2.1. Limits properties. Let us consider the random system with complete connections RSCC [2]

\[
\{([0, \theta], \mathcal{B}_{[0, \theta]}), (\mathbb{N}_m, \mathcal{P}(\mathbb{N}_m)), u, P\},
\]

where \(u : [0, \theta] \times \mathbb{N}_m \to [0, \theta]\), \(u(s, i) = u_i(s) := 1/(s + i\theta)\) and the function \(P(s, i) = P_i(s) := (x\theta + 1)/((x + i\theta)(x + (i + 1)\theta))\) defines a transition probability from \(([0, \theta], \mathcal{B}_{[0, \theta]})\) to \((\mathbb{N}_m, \mathcal{P}(\mathbb{N}_m))\). Here \(\mathcal{P}(\mathbb{N}_m)\) denotes the power set of \(\mathbb{N}_m\). For any \(a \in [0, \theta]\) let \(s_{0,a} := a, s_{n,a} := 1/(a_n \theta + s_{n-1,a}), n \in \mathbb{N}_+,\) and consider the family \((\gamma_{\theta,a})_{a \in [0, \theta]}\) of probability measures on \(\mathcal{B}_{[0, \theta]}\) defined by their distribution functions \(\gamma_{\theta,a}([0, x]) := (a \theta + 1)x/((ax + 1)\theta)\). The sequence \((s_{n,a})_{n \in \mathbb{N}_+}\) is an \([0, \theta]\)-valued Markov chain on \(([0, \theta], \mathcal{B}_{[0, \theta]}, \gamma_{\theta,a})\) which starts at \(s_{0,a} := a\) and has the following transition mechanism: from state \(s \in [0, \theta]\) the possible transitions are to any state \(1/(s + i\theta)\) with corresponding transition probability \(P_i(s), i \in \mathbb{N}_m\). Thus the transition operator (Perron-Frobenius operator) \(U\) of all Markov chains \((s_{n,a})_{n \in \mathbb{N}_+}\) for any bounded complex-valued measurable function \(f\) on \([0, \theta]\), is given by

\[
U f(x) = \sum_{i \geq m} P_i(x) f(u_i(x)), \quad m \in \mathbb{N}_+
\]
where \( f \in L^1_{\gamma_\theta} := \{ f : [0, \theta] \to \mathbb{C} : f^\theta \gamma_\theta < \infty \} \). It was investigated in [5, 6] the Perron-Frobenius operator of the continued fraction transformation \( T_\theta \) under different probability measures on \( B_{[0, \theta]} \). The asymptotic behavior of this operator is derived in [6] and is given by

\[
\mu((T_\theta)^{-n}(A)) = \int_A U^n f(x) d\gamma_\theta(x),
\]

where \( f(x) := (\log(1 + \theta^2)) \frac{x\theta + 1}{\theta} h(x), \ x \in [0, \theta] \).

In the sequel the domain of \( U \) will be successively restricted to various Banach spaces. Recall that the variation \( \text{var}_A f \) of \( f \) on a subset \( A \) of \([0, \theta]\) is defined as

\[
sup \sum_{i=1}^{k-1} |f(t_i) - f(t_{i-1})| \text{ the supremum being taken over all } t_1 < \cdots < t_k \in A \text{ and } k \geq 2.
\]

If \( \text{var}_A f < \infty \), then \( f \) is called a function of bounded variation. A variation \( \nu(f) \) for \( L^\infty([0, \theta], B_{[0, \theta]}, \lambda_\theta) \), the collection of all classes of \( \lambda_\theta \)-essentially bounded measurable complex-valued \( \lambda_\theta \)-indistinguishable function on \([0, \theta]\) is defined as \( \nu(f) = \inf \text{var}_f \), the infimum being taken over all versions of \( f \). The set \( BEV([0, \theta]) \) is a Banach space under the norm \( ||f||_\nu := \nu(f) + ||f||_1 \), where \( || \cdot ||_1 \) is the usual \( L^1_{\lambda_\theta} \) norm \( ||f||_1 = \int_0^\theta |f| d\lambda_\theta \). For proofs and more details see [5, 6].

Whatever \( a \in [0, \theta] \) the Markov chain \( (s_{n,a})_{n \in \mathbb{N}} \) associated with the RSCC (15) has the transition operator \( U \), with the transition probability function

\[
Q(s, B) = \sum_{\{i \geq m \mid u_i(s) \in B\}} P_i(s), \ s \in [0, \theta], B \in B_{[0, \theta]}.
\]

Then \( Q^n(\cdot, \cdot) \) will denote the \( n \)-step transition probability function of the same Markov chain.

It was proved in [6] that the RSCC (15) is uniformly ergodic and its transition operator is regular with respect to the Banach space of Lipschitz functions.

Now for a probability measure \( \mu \) on \(([0, \theta], B_{[0, \theta]})\) we may determine the limit of the sequence \( (\mu(T_\theta^n < x))_{n \in \mathbb{N}^+} \) as \( n \to \infty \) and obtain the rate of this convergence, i.e.,

\[
\lim_{n \to \infty} \mu(T_\theta^n < x) = \frac{1}{\log(1 + \theta^2)} \log((m\theta + x)\theta), \quad x \in [0, \theta].
\]

### 2.2. A Gauss-Kuzmin-Lévy-type theorem.

The study of optimality of the convergence rate remains an open question. Using a Wirsing-type approach [8], in [7] it was obtained a better estimate of the convergence rate involved. The strategy was to restrict the domain of the Perron-Frobenius operator of \( T_\theta \) under its invariant measure \( \gamma_\theta \) to the Banach space of functions which have a continuous derivative on \([0, \theta]\). Define a linear operator \( V : C([0, \theta]) \to C([0, \theta]) \) by \( V g = -(U f)' \), \( g \in C([0, \theta]) \), where \( f' = g \). Since \( U \) is a Markov operator, \( V \) is well defined. It is easy to check that \( (U^n f)' = (-1)^nV^n f' \), \( n \in \mathbb{N}^+, f \in C^1([0, \theta]) \). Sebe proved in [7] that there are positive constants \( v_\theta < w_\theta < 1 \) and a real-valued function \( \varphi_\theta \in C([0, \theta]) \) defined by

\[
\varphi_\theta(x) = \frac{1}{\theta} \left[ \frac{e_\theta(m+1)\theta - a_\theta - 1}{(e_\theta + x(-e_\theta m\theta + a_\theta + 1))^2} - \frac{e_\theta(m-1)\theta - a_\theta - 1}{(e_\theta + x(-e_\theta(m-1)\theta + a_\theta + 1))^2} \right],
\]

\( x \in [0, \theta] \), where the coefficient \( e_\theta \) is chosen such that the equation

\[
E_\theta(x) = 2\theta(x+1)^2 - e_\theta^2 [(2m+1)(x+1) + e_\theta(m+1)\theta] = 0
\]
$x \in [0, \theta]$ has a unique solution $a_\theta \in [0, \theta]$. For this unique acceptable $a_\theta \in [0, \theta]$ we have $v_\theta \varphi_\theta \leq V \varphi_\theta \leq w_\theta \varphi_\theta$. Next, putting $\alpha_\theta = \min_{x \in [0, \theta]} \varphi_\theta(x)/(f_\theta)'(x)$ and $\beta_\theta = \max_{x \in [0, \theta]} \varphi_\theta(x)/(f_\theta)'(x)$ for any $f_\theta \in C^1([0, \theta])$ such that $(f_\theta)' > 0$, we get

$$\frac{\alpha_\theta}{\beta_\theta} v_\theta^n (f_\theta)' \leq V^n (f_\theta)' \leq \frac{\beta_\theta}{\alpha_\theta} w_\theta^n (f_\theta)' ,$$

for any $n \in \mathbb{N}_+$.

In Theorem 5.3 in [7] there are obtained upper and lower bounds of the convergence rate, respectively $O(w_n)$ and $O(v_n)$ as $n \to \infty$, which provide a near-optimal solution to the Gauss-Kuzmin-Lévy problem.

Let $\mu$ be a probability measure on $\mathcal{B}_{[0, \theta]}$ such that $\mu \ll \lambda_\theta$. For any $n \in \mathbb{N}$ put $F_\theta^n(x) = \mu(T_\theta^n < x)$, $x \in [0, \theta]$, where $T_\theta^n$ is the identity map. Let $f_\theta^n(x) = x^{\theta + 1}(F_\theta^n)'(x)$, $x \in [0, \theta]$, where $(F_\theta^n)' = d\mu/d\lambda_\theta$. Let us recall this theorem.

**Theorem 2.1.** Let $f_\theta^n \in C^1([0, \theta])$ such that $(f_\theta^n)' > 0$ and let $\mu$ be a probability measure on $\mathcal{B}_{[0, \theta]}$ such that $\mu \ll \lambda_\theta$. For any $n \in \mathbb{N}_+$ and $x \in [0, \theta]$ we have

$$(\log(1 + \theta^2))^2 \frac{\alpha_\theta}{2\theta \beta_\theta} \min_{x \in [0, \theta]} (f_\theta^n)'(x)v_\theta^n G_\theta(x)(\theta - G_\theta(x)) \leq |\mu(T_\theta^n < x) - G_\theta(x)|$$

$$\leq (\log(1 + \theta^2))^2 \frac{(1 + \theta^2)\beta_\theta}{2\theta \alpha_\theta} \max_{x \in [0, \theta]} (f_\theta^n)'(x)w_\theta^n G_\theta(x)(\theta - G_\theta(x))$$

where

$$G_\theta(x) = \frac{\log(1 + x\theta)}{\log(1 + \theta^2)} .$$

For example, for $m = 3$, the equation $E_\theta(x) = 0$, with $e_\theta = 0.67$, has as unique acceptable solution $a_\theta = 0.287897$. For this value of $a_\theta$ the function $\varphi_{a_\theta}/V \varphi_{a_\theta}$ attains its maximum equal to $7.389969626$ at $x = 0$ and $x = \theta$, and has a minimum $m(a_\theta) = (\varphi_{a_\theta}/V \varphi_{a_\theta})(0.256122) = 7.29924$. It follows that upper and lower bounds of the convergence rate are respectively $O(w^n_\theta)$ and $O(v^n_\theta)$ as $n \to \infty$, with $v_\theta > 0.135318553$ and $w_\theta < 0.137000564$.

Obviously, the determination of the exact convergence rate remains an open question. We may derive it using the same strategy as in [4].

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**References**


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