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Dedicated to Marius Iosifescu
on the occasion of his 80th anniversary

Minimax fractional programming problem with $(p, r) - \rho - (\eta, \theta)$-invex functions

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Abstract. In this paper, new classes of generalized $(p, r) - \rho - (\eta, \theta)$-invex functions are introduced i.e., $(p, r) - \rho - (\eta, \theta)$-quasi-invex and (strictly) $(p, r) - \rho - (\eta, \theta)$-pseudo-invex functions. We focus on minimax fractional programming problem and establish sufficient optimality conditions under the assumption of generalized $(p, r) - \rho - (\eta, \theta)$-invexity. Weak, strong and strict converse duality theorems are also derived for two type of dual models related to minimax fractional programming problem involving aforesaid generalized invex functions.

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1. Introduction

The optimality conditions and duality results for minimax fractional programming problems have been studied by many authors (see, for example, [1, 2, 3, 8, 11, 15, 25], and others). But in most of the studies, an assumption of convexity on the functions involving was made. Several classes of functions have been defined for the purpose of weakening the limitations of convexity. Among these, the concept of invexity [9] has received more attention. Recently, the notion of invexity has been extended in several directions. Some recent surveys and synthesis of results pertaining to various generalizations of invex functions and their applications along with extensive lists of relevant references are available in [7, 10, 13, 14, 20], and others.

Preda [21] introduced the concept of generalized (F, ρ)-convexity, an extension of F-convexity defined by Hanson and Mond [10] and generalized ρ-convexity defined by Vial [24], and he used the concept to obtain duality results for efficient solutions.

Schmitendorf [22] gave two sets of sufficient optimality conditions for minimax problem, under the conditions of convexity. Later, Tanimoto [23] derived duality theorems, under convexity assumptions on the functions involved, for the problems considered by Schmitendorf [22], which were extended for the fractional analogue of generalized minimax problem by Yadav and Mukherjee [25]. Liu and Wu [15] derived the sufficient optimality conditions and duality theorems for the generalized minimax fractional programming in the framework of (F, ρ)-convex functions. Ahmad [1] obtained sufficient optimality conditions and duality theorems for minimax fractional programming problem assuming the functions involved to be generalized convex.

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Liang *et al.* [14] introduced the concept of differentiable (F, α, ρ, d)-convex function and proved optimality theorems and duality results for a multiobjective fractional programming problem involving (F, α, ρ, d)-convexity assumptions.

Antczak [4] introduced to the optimization theory new classes of (p, r)-invex functions and studied some of its properties. Jayswal *et al.* [12] focus his study on multiobjective fractional programming problem and established sufficient optimality conditions and duality results under the assumptions of $(p, r) - \rho - (\eta, \theta)$-invex functions.

Motivated by the earlier works and seeing the importance of generalized convexity into the fields of optimization theory, in this paper we introduce new classes of generalized $(p, r) - \rho - (\eta, \theta)$-invex functions i.e. $(p, r) - \rho - (\eta, \theta)$-quasi-invex and (strictly) $(p, r) - \rho - (\eta, \theta)$-pseudo-invex functions and focus our study on minimax fractional programming problem. We establish sufficient optimality conditions and duality theorems for two types of dual problems under the aforesaid generalized invex functions.

The organization of the article is as follows. Some definitions and notation are given in Section 2. The sufficient optimality conditions are established in Section 3. By employing the sufficient conditions, we formulate two dual models and derive weak, strong and strict converse duality results in Sections 4 and 5. Finally, conclusions are given in Section 6.

2. Notation and preliminaries

Throughout the paper, let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the vector norm $\| \ \|$ and $\mathbb{R}^n_+$ be its non-negative orthant. We use the following conventions for vectors in $\mathbb{R}^n$:

$x \leqq y$ if and only if $x_i \leqq y_i$ for all $i = 1, 2, \ldots, n$;
$x \leq y$ if and only if $x_i \leqq y_i$, for all $i = 1, 2, \ldots, n$ and $x \neq y$;
$x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, \ldots, n$;
$x \nless y$ is the negation of $x < y$.

Let a non-empty set $X \subset \mathbb{R}^n$, a differentiable function $f : X \mapsto \mathbb{R}$, vector-valued functions $\eta, \theta : X \times X \mapsto \mathbb{R}^n$ and let p, r and θ be arbitrary real numbers.

Definition 2.1. [4] A differentiable function $f : X \to \mathbb{R}$ is said to be (strictly) (p, r)-invex with respect to η at $u \in X$ if and only if for each $x \in X$, one of the relations

$$\frac{1}{r}e^{rf(x)} \geqq \frac{1}{r}e^{rf(u)}[1 + \frac{r}{p}\nabla f(u)(e^{p\eta(x,u)} - 1)] \ (>) \text{ for } p \neq 0,\ r \neq 0,$$
$$\frac{1}{r}e^{rf(x)} \geqq \frac{1}{r}e^{rf(u)}[1 + r\nabla f(u)\eta(x, u)] \ (>) \text{ for } p = 0,\ r \neq 0,$$
$$f(x) - f(u) \geqq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - 1) \ (>) \text{ for } p \neq 0,\ r = 0,$$
$$f(x) - f(u) \geqq \nabla f(u)\eta(x, u) \ (>) \text{ for } p = 0,\ r = 0,$$

holds.

If the above inequalities are satisfied at any point $u \in X$, then f is said to be (strictly) (p, r)-invex with respect to η on X.

Definition 2.2. [19] A differentiable function $f : X \to \mathbb{R}$ is said to be $\rho-(\eta, \theta)$-invex with respect to vector-valued functions η and θ if and only if

$$f(x) - f(u) \geqq \eta^T(x, u)\nabla f(u) + \rho \left\|\theta(x, u)\right\|^2, \text{ for all } x, u \in X.$$

Definition 2.3. [17] A differentiable function $f : X \to \mathbb{R}$ is said to be $(p, r)-\rho-(\eta, \theta)$-invex at the point $u \in X$ with respect to vector-valued functions η and θ if and only if for each $x \in X$, one of the relations

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geqq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) + \rho \left\|\theta(x, u)\right\|^2 \text{ for } p \neq 0,\ r \neq 0,$$

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geqq \nabla f(u)\eta(x, u) + \rho \left\|\theta(x, u)\right\|^2 \text{ for } p = 0,\ r \neq 0,$$

$$f(x) - f(u) \geqq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) + \rho \left\|\theta(x, u)\right\|^2 \text{ for } p \neq 0,\ r = 0,$$

$$f(x) - f(u) \geqq \nabla f(u)\eta(x, u) + \rho \left\|\theta(x, u)\right\|^2 \text{ for } p = 0,\ r = 0,$$

holds.

Remark 2.1. If the above inequalities are satisfied at any point $u \in X$, then f is said to be (strictly) $(p, r) - \rho - (\eta, \theta)$-invex on X with respect to η and θ.

Remark 2.2. It should be noted that the exponentials appearing on the right-hand sides of inequalities above are understood to be taken componentwise and $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

Now we give an example of function which is $(p, r) - \rho - (\eta, \theta)$-invex but not (p, r)-invex [4].

Example 2.1. Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 \geqq 0\}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x) = \cos^2 x_1 - \cos^2 x_2$. Define

$$\bar{\theta}(x, u) = \begin{cases} 1, & if\ u = \pi/2, \\ 0, & if\ u \neq \pi/2, \end{cases}$$

$$\bar{\eta}(x, u) = \begin{cases} 0, & if\ u = \pi/2, \\ -\sin 2u, & if\ u \neq \pi/2, \end{cases}$$

where $\theta = (\bar{\theta}(x, u_1), \bar{\theta}(x, u_2))$ and $\eta = (\bar{\eta}(x, u_1), \bar{\eta}(x, u_2))$. At $u = (\pi/2, \pi/2)$, we have

$$\frac{1}{r}e^{rf(u)}[1 + r\nabla f(u)\eta(x, u)] = \frac{1}{r}.$$

Therefore the inequality

$$\frac{1}{r}e^{r(\cos^2 x_1 - \cos^2 x_2)} \geqq 1,$$

is not true. Because, if we take $x_1 = \pi/2,\ x_2 = 0$ and $r = 1$, then from the above inequality we get $1/e \geq 1$, which is not possible. Therefore, the function f is not $(0, 1)$-invex function (i.e. (p, r)-invex) with respect to η at $u = (\pi/2, \pi/2)$. Now, if $p = 0,\ r = 1$ and $\rho = -1/2$, then

$$\frac{1}{r}\left(e^{r(f(x)-f(u))} - 1\right) = e^{(\cos^2 x_1 - \cos^2 x_2)} - 1,$$

and

$$\nabla f(u)\eta(x,u) + \rho \|\theta(x,u)\|^2 = -1,$$

at the point $u = (\pi/2, \pi/2)$. Hence, the inequality

$$e^{\cos^2 x_1 - \cos^2 x_2} - 1 \geqq -1,$$

is always true. Therefore, from the Definition 2.3, we have f is $(0,1)-(-1/2)-(\eta,\theta)$-invex function (i.e.$(p,r) - \rho - (\eta,\theta)$-invex) with respect to η and θ at the point $u = (\pi/2, \pi/2)$.

The following example shows that there exists $(p,r)-\rho-(\eta,\theta)$-invex function but not $\rho-(\eta,\theta)$-invex [19].

Example 2.2. Let $X = [2,3] \subset \mathbb{R}$. Consider the function $f : X \to \mathbb{R}$ defined by

$$f(x) = x + \log \sqrt{x}.$$

Let $\eta : X \times X \to \mathbb{R}$ and $\theta : X \times X \to \mathbb{R}$ given by $\eta(x,u) = 1+u^2$ and $\theta(x,u) = x+u$, respectively.
For $p = 0,\ r = 1,\ \rho = -1/2$ and for all $x, u \in X$, we have

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) - \nabla f(u)\eta(x,u) - \rho \|\theta(x,u)\|^2 \geqq 0,$$

as can be seen from Figure 1.

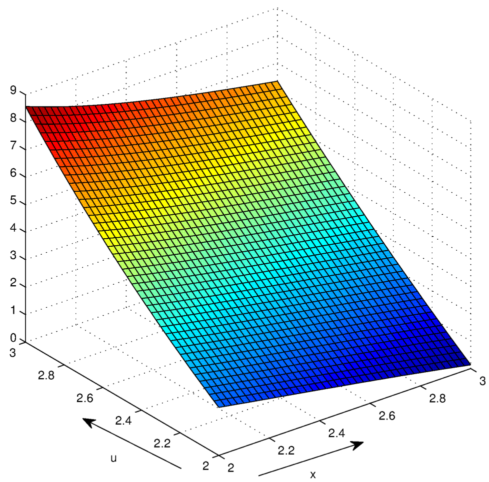

FIGURE 1

Therefore the function f defined above is $(0,1) - (-1/2) - (\eta,\theta)$-invex function (i.e.$(p,r) - \rho - (\eta,\theta)$-invex) with respect to η and θ.
But the function f it is not $\rho - (\eta,\theta)$-invex for all $x, u \in X$ (see Figure 2), because, if we take $x = 2$ and $u = 3$, then

$$f(x) - f(u) = -1.088045$$

and

$$\nabla f(u)\eta(x,u) + \rho \|\theta(x,u)\|^2 = -0.833.$$

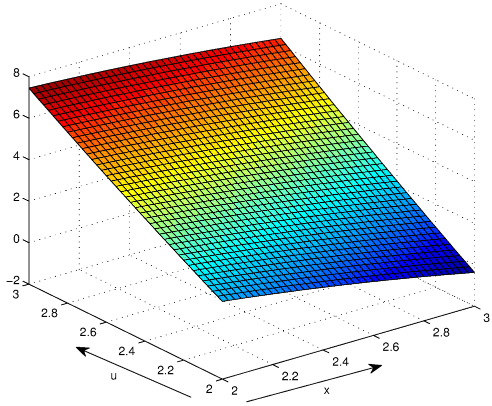

FIGURE 2

which shows that

$$f(x) - f(u) \ngeqq \nabla f(u)\eta(x,u) + \rho \|\theta(x,u)\|^2 .$$

Now we introduce the generalized $(p,r) - \rho - (\eta,\theta)$-invex functions as follows:

Definition 2.4. A differentiable function $f : X \mapsto \mathbb{R}$ is said to be $(p,r) - \rho - (\eta,\theta)$-quasi-invex at the point $u \in X$ with respect to vector-valued functions η and θ if and only if for each $x \in X$, one of the relations

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \leqq 0 \Rightarrow \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) \leqq -\rho \|\theta(x,u)\|^2 \text{ for } p \neq 0, r \neq 0,$$

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \leqq 0 \Rightarrow \nabla f(u)\eta(x,u) \leqq -\rho \|\theta(x,u)\|^2 \text{ for } p = 0, r \neq 0,$$

$$f(x) - f(u) \leqq 0 \Rightarrow \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) \leqq -\rho \|\theta(x,u)\|^2 \text{ for } p \neq 0, r = 0,$$

$$f(x) - f(u) \leqq 0 \Rightarrow \nabla f(u)\eta(x,u) \leqq -\rho \|\theta(x,u)\|^2 \text{ for } p = 0, r = 0,$$

holds.

If the above inequalities are satisfied at any point $u \in X$, then f is said to be $(p,r) - \rho - (\eta,\theta)$-quasi-invex on X with respect to η and θ.

Definition 2.5. A differentiable function $f : X \mapsto \mathbb{R}$ is said to be $(p,r) - \rho - (\eta,\theta)$-pseudo-invex at the point $u \in X$ with respect to vector-valued functions η and θ if and only if for each $x \in X$, one of the relations

$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) \geqq -\rho \|\theta(x,u)\|^2 \Rightarrow \frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geqq 0 \text{ for } p \neq 0, r \neq 0,$$

$$\nabla f(u)\eta(x,u) \geqq -\rho \|\theta(x,u)\|^2 \Rightarrow \frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geqq 0 \text{ for } p = 0, r \neq 0,$$

$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) \geqq -\rho \|\theta(x,u)\|^2 \Rightarrow f(x) - f(u) \geqq 0 \text{ for } p \neq 0, r = 0,$$

$$\nabla f(u)\eta(x,u) \geqq -\rho \|\theta(x,u)\|^2 \Rightarrow f(x) - f(u) \geqq 0 \text{ for } p = 0, r = 0,$$

holds.

If the above inequalities are satisfied at any point $u \in X$, then f is said to be $(p,r)-\rho-(\eta,\theta)$-pseudo-invex on X with respect to η and θ.

Definition 2.6. A differentiable function $f : X \mapsto \mathbb{R}$ is said to be strictly $(p,r)-\rho-(\eta,\theta)$-pseudo-invex at the point $u \in X$ with respect to vector-valued functions η and θ if and only if for each $x \in X$, one of the relations

$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) > -\rho \left\|\theta(x,u)\right\|^2 \Rightarrow \frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geqq 0 \text{ for } p \neq 0, r \neq 0,$$

$$\nabla f(u)\eta(x,u) > -\rho \left\|\theta(x,u)\right\|^2 \Rightarrow \frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geqq 0 \text{ for } p = 0, r \neq 0,$$

$$\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) > -\rho \left\|\theta(x,u)\right\|^2 \Rightarrow f(x) - f(u) \geqq 0 \text{ for } p \neq 0, r = 0,$$

$$\nabla f(u)\eta(x,u) > -\rho \left\|\theta(x,u)\right\|^2 \Rightarrow f(x) - f(u) \geqq 0 \text{ for } p = 0, r = 0,$$

holds.

If the above inequalities are satisfied at any point $u \in X$, then f is said to be strictly $(p,r)-\rho-(\eta,\theta)$-pseudo-invex on X with respect to η and θ.

In this paper, we consider the following minimax fractional programming problem:

$$\textbf{(P)} \qquad v^* = \min_{x\in S} \max_{1\leqq i\leqq p} \left[\frac{f_i(x)}{g_i(x)}\right]$$

where
(A1) $S = \{x \in \mathbb{R}^n;\ h_k(x) \leqq 0,\ k = 1,2,...,m\}$ is a non-empty and compact set;
(A2) $f_i,\ g_i : X \to \mathbb{R},\ i = 1,2,...,p$ and $h_k : X \to \mathbb{R},\ k = 1,2,...,m$ are differentiable functions, and X is a non-empty open subset of $\mathbb{R}^n$;
(A3) $f_i(x) \geqq 0,\ g_i(x) > 0,\ i = 1,2,...,p$ for all $x \in S$.

It is well known [5, 6] that the problem (P) is equivalent to the following problem (EP_v) for a given v.

$$(\mathbf{EP_v}) \qquad \min q,$$

subject to

$$f_i(x) - vg_i(x) \leqq q,\ i = 1,2,...,p, \tag{1}$$

$$h_k(x) \leqq 0,\ k = 1,2,...,m. \tag{2}$$

We shall use the following lemmas.

Lemma 2.1. [5] *If (x,v,q) is* (EP_v)*-feasible, then x is* (P)*-feasible. If x is* (P)*-feasible, then there exist v and q such that (x,v,q) is* (EP_v)*-feasible.*

Lemma 2.2. [5] *x^* is* (P)*-optimal with the corresponding optimal value of the* (P)*-objective equal to $v*$ if and only if (x^*,v^*,q^*) is* (EP_v)*-optimal with the corresponding optimal value of the* (EP_v)*-objective equal to zero; that is, $q^* = 0$.*

Following the same lines of Liu [16], we can write the necessary optimality conditions for (P) as follows:

Theorem 2.1. (Necessary optimality conditions). *Let x^* be an optimal solution of* (P) *with the optimal value of the* (P)*-objective equal to v^*. Let an appropriate* *constraints qualification* [18] *hold for* (EP_{v^*}). *Then, there exist* $q^* \in \mathbb{R},\ y^* \in \mathbb{R}^p,\ z^* \in \mathbb{R}^m$ *such that* (x^*, v^*, y^*, z^*) *satisfy the following conditions*

$$\sum_{i=1}^{p} y_i^* [\nabla f_i(x^*) - v^* \nabla g_i(x^*)] + \sum_{k=1}^{m} z_k^* \nabla h_k(x^*) = 0, \tag{3}$$

$$y_i^*(f_i(x^*) - v^* g_i(x^*)) = 0, \text{ for all } i = 1, 2, ..., p, \tag{4}$$

$$z_k^* h_k(x^*) = 0, \text{ for all } k = 1, 2, ..., m, \tag{5}$$

$$f_i(x^*) - v^* g_i(x^*) \leqq 0, \text{ for all } i = 1, 2, ..., p, \tag{6}$$

$$h_k(x^*) \leqq 0, \text{ for all } k = 1, 2, ..., m, \tag{7}$$

$$\sum_{i=1}^{p} y_i^* = 1, \tag{8}$$

$$q^* = 0, \tag{9}$$

$$q^* \in \mathbb{R},\ y^* \in \mathbb{R}^p,\ z^* \in \mathbb{R}^m,\ y \geq 0,\ z^* \geqq 0,\ v^* \geqq 0. \tag{10}$$

Remark 2.3. All the theorems in the subsequent parts of this paper will be proved only in the case when $p \neq 0, r \neq 0$. The proofs in other cases are easier than in this one since only changes arise from form of inequality. Moreover, without loss of generality, we shall assume that $r > 0$ (in the case when $r < 0$, the direction some of the inequalities in the proof of the theorems should be changed to the opposite one). In the subsequent part of the paper, we assume that $\rho,\ \rho_1$ and ρ_2 are all elements of $\mathbb{R}$.

3. Sufficient optimality conditions

In this section, we establish Karush-Kuhn-Tucker type sufficient optimality conditions under generalized $(p, r) - \rho - (\eta, \theta)$-invex functions defined in the previous section.

Theorem 3.1. (Sufficiency). *Let* $(x^*, v^*, q^*, y^*, z^*)$ *satisfy relations* (3) *to* (10). *Moreover, assume any one of the conditions below holds:*

(*a*) $A(x) = \sum_{i=1}^{p} y_i^* [f_i(x) - v^* g_i(x)] + \sum_{k=1}^{m} z_k^* h_k(x)$ *is* $(p, r) - \rho - (\eta, \theta)$*-invex at* x^* *with respect to* $\eta,\ \theta$ *and* $\rho \geqq 0$*;*

(*b*) $B(x) = \sum_{i=1}^{p} y_i^* [f_i(x) - v^* g_i(x)]$ *is* $(p, r) - \rho_1 - (\eta, \theta)$*-pseudo-invex at* x^* *and* $C(x) = \sum_{k=1}^{m} z_k^* h_k(x)$ *is* $(p, r) - \rho_2 - (\eta, \theta)$*-quasi-invex at* x^* *with respect to* $\eta,\ \theta$ *and* $\rho_1 + \rho_2 \geqq 0$*;*

(*c*) $B(x) = \sum_{i=1}^{p} y_i^* [f_i(x) - v^* g_i(x)]$ *is* $(p, r) - \rho_1 - (\eta, \theta)$*-quasi-invex at* x^* *and* $C(x) = \sum_{k=1}^{m} z_k^* h_k(x)$ *is strictly* $(p, r) - \rho_2 - (\eta, \theta)$*-pseudo-invex at* x^* *with respect to* $\eta,\ \theta$ *and* $\rho_1 + \rho_2 \geqq 0$,

for all (x, q) *that are* (EP_{v^*})*-feasible. Then* x^* *is* (P)*-optimal with the corresponding optimal value equal to* v^*.

Proof. Suppose contrary to the result that x^* is not (P)-optimal. Let v^* be the value of the objective function of Problem (P) for $x = x^*$. From Lemma 2.2, we conclude that (x^*, v^*, q^*) is not (EP_{v^*})-optimal with the corresponding optimal value of the (EP_{v^*})-objective equal to zero, that is,

$$q < q^*. \tag{11}$$

In (11), using (1), (2), (8) and (10) on LHS and using (4), (5) and (9) on RHS, we obtain

$$\sum_{i=1}^{p} y_i^*[f_i(x) - v_i^* g_i(x)] + \sum_{k=1}^{m} z_k^* h_k(x) < \sum_{i=1}^{p} y_i^*[f_i(x^*) - v_i^* g_i(x^*)] + \sum_{k=1}^{m} z_k^* h_k(x^*).$$

That is,

$$A(x) < A(x^*). \tag{12}$$

If condition (a) holds, then

$$\frac{1}{r}(e^{r(A(x)-A(x^*))} - 1) \geqq \frac{1}{p}\nabla A(x^*)(e^{p\eta(x,x^*)} - \mathbf{1}) + \rho \left\|\theta(x,x^*)\right\|^2.$$

The above inequality together with (12) gives

$$\frac{1}{p}\nabla A(x^*)(e^{p\eta(x,x^*)} - \mathbf{1}) + \rho \left\|\theta(x,x^*)\right\|^2 < 0. \tag{13}$$

Consequently, (3) and (13) yield

$$\rho \left\|\theta(x,x^*)\right\|^2 < 0,$$

which contradicts to the fact that $\rho \geqq 0$.
If condition (b) holds, from (2), (5) and (10), we have

$$\sum_{k=1}^{m} z_k^* h_k(x) \leqq \sum_{k=1}^{m} z_k^* h_k(x^*). \tag{14}$$

That is,

$$C(x) \leqq C(x^*), \tag{15}$$

which in turn implies that

$$\frac{1}{r}(e^{r(C(x)-C(x^*))} - 1) \leqq 0. \tag{16}$$

Using $(p,r) - \rho_2 - (\eta,\theta)$-quasi-invexity of C at x^* with respect to η and θ, we have

$$\frac{1}{p}\nabla C(x^*)(e^{p\eta(x,x^*)} - \mathbf{1}) \leqq -\rho_2 \left\|\theta(x,x^*)\right\|^2. \tag{17}$$

The above inequality together with equation (3) and the assumption $\rho_1 + \rho_2 \geqq 0$ gives

$$\frac{1}{p}\nabla B(x^*)(e^{p\eta(x,x^*)} - \mathbf{1}) \geqq -\rho_1 \left\|\theta(x,x^*)\right\|^2. \tag{18}$$

Now using $(p,r) - \rho_1 - (\eta,\theta)$-pseudo-invexity of B at x^* with respect to η and θ, we have

$$\frac{1}{r}(e^{r(B(x)-B(x^*))} - 1) \geqq 0, \tag{19}$$

which in turn implies that

$$B(x) \geqq B(x^*).$$

That is,

$$\sum_{i=1}^{p} y_i^*[f_i(x) - v_i^* g_i(x)] \geqq \sum_{i=1}^{p} y_i^*[f_i(x^*) - v_i^* g_i(x^*)]. \tag{20}$$

From (1), (4), (8) and the above inequality, we obtain

$$q \geqq \sum_{i=1}^{p} y_i^*[f_i(x) - v_i^* g_i(x)] \geqq \sum_{i=1}^{p} y_i^*[f_i(x^*) - v_i^* g_i(x^*)] = 0 = q^*.$$

That is,

$$q \geqq q^*.$$

This along with Lemma 2.2 yields that x^* is (P)-optimal with the corresponding optimal value equal to v^*.

The proof of hypothesis (c) follows along the lines similar to that of (b). This completes the proof. □

4. First duality model

With the help of (EP_v), we consider the following form of dual problem of Problem (P):

$$(\mathbf{DEP_v1}) \qquad \text{Maximize} \quad \sum_{i=1}^{p} y_i[f_i(u) - vg_i(u)] + \sum_{k=1}^{m} z_k h_k(u)$$

subject to

$$\sum_{i=1}^{p} y_i[\nabla f_i(u) - v\nabla g_i(u)] + \sum_{k=1}^{m} z_k \nabla h_k(u) = 0, \tag{21}$$

$$\sum_{i=1}^{p} y_i = 1, \tag{22}$$

$$u \in \mathbb{R}^n,\ y \in \mathbb{R}^p,\ z \in \mathbb{R}^m,\ y \geq 0,\ z \geqq 0,\ v \geqq 0. \tag{23}$$

Theorem 4.1. (Weak duality). *For a given v^*, let $(\hat{x}, \hat{q})$ be (EP_{v^*})-feasible, and let $(\bar{u}, \bar{y}, \bar{z})$ be $(\mathrm{DEP}_{v^*}1)$- feasible. Assume that $G(.) = \sum_{i=1}^{p} \bar{y}_i[f_i(.) - v^* g_i(.)] + \sum_{k=1}^{m} \bar{z}_k h_k(.)$ is $(p,r)-\rho-(\eta,\theta)$-invex at $\bar{u}$ with respect to η, θ and $\rho \geqq 0$. Then*

$$\inf(\mathrm{EP}_{v^*}) \ \geqq\ \sup(\mathrm{DEP}_{v^*}1).$$

Proof. Let $(\hat{x}, \hat{q})$ be (EP_{v^*})-feasible and let $(\bar{u}, \bar{y}, \bar{z})$ be $(\mathrm{DEP}_{v^*}1)$-feasible. Suppose, contrary to the result, i.e.,

$$\inf\ (\mathrm{EP}_{v^*}) < \sup\ (\mathrm{DEP}_{v^*}1).$$

Equivalently,

$$\hat{q} \ <\ \sum_{i=1}^{p} \bar{y}_i[f_i(\bar{u}) - v^* g_i(\bar{u})] + \sum_{k=1}^{m} \bar{z}_k h_k(\bar{u}). \tag{24}$$

In (24), using (1), (2), (22) and (23) on LHS, we have

$$\sum_{i=1}^{p} \bar{y}_i[f_i(\hat{x}) - v^* g_i(\hat{x})] + \sum_{k=1}^{m} \bar{z}_k h_k(\hat{x}) \ <\ \sum_{i=1}^{p} \bar{y}_i[f_i(\bar{u}) - v^* g_i(\bar{u})] + \sum_{k=1}^{m} \bar{z}_k h_k(\bar{u}).$$

That is,

$$G(\hat{x}) \ <\ G(\bar{u}),$$

which in turn implies that

$$\frac{1}{r}(e^{r(G(\hat{x}) - G(\bar{u}))} - 1) \ <\ 0. \tag{25}$$

The above inequality together with $(p,r)-\rho-(\eta,\theta)$-invexity of $G(.)$ at $\bar{u}$, implies that

$$\frac{1}{p}\nabla G(\bar{u})(e^{p\eta(\hat{x},\bar{u})}-\mathbf{1})+\rho\left\|\theta(\hat{x},\bar{u}))\right\|^2<0. \tag{26}$$

Consequently, (21) and (26) yield

$$\rho\left\|\theta(\hat{x},\bar{u}))\right\|^2<0,$$

which contradicts to the fact that $\rho \geqq 0$. This completes the proof. □

Theorem 4.2. (Strong duality). *Let*

$$v^*=\min_{x\in S}\max_{1\leqq i\leqq p}\left[\frac{f_i(x)}{g_i(x)}\right]$$

and let (x^*,q^*) be (EP$_{v^*}$)*-optimal, at which an appropriate constraint qualification holds* [18]*. Then, there exists (y^*,z^*) such that (x^*,y^*,z^*) is* (DEP$_{v^*}$1)*-feasible and the corresponding objective values of* (EP$_{v^*}$) *and* (DEP$_{v^*}$1) *are equal. If also the hypotheses of Theorem 4.1 are satisfied, then (x^*,q^*) and (x^*,y^*,z^*) are, respectively, global optimal for* (EP$_{v^*}$) *and* (DEP$_{v^*}$1) *with each objective value equal to zero.*

Proof. The proof follows along the lines of Bector *et al.* [5]. □

Theorem 4.3. (Strict converse duality). *Let*

$$v^*=\min_{x\in S}\max_{1\leqq i\leqq p}\left[\frac{f_i(x)}{g_i(x)}\right]$$

and let (x^*,q^*) be (EP$_{v^*}$)*-optimal, at which an appropriate constraint qualification holds* [18]*. Let $(\bar{u},\bar{y},\bar{z})$ be* (DEP$_{v^*}$1)*-optimal, and let*

$$\rho>0 \text{ and } G(.)=\sum_{i=1}^{p}\bar{y}_i[f_i(.)-v^*g_i(.)]+\sum_{k=1}^{m}\bar{z}_kh_k(.)$$

is $(p,r)-\rho-(\eta,\theta)$-invex at $\bar{u}$ for all (EP$_{v^*}$)*-feasible and* (DEP$_{v^*}$1)*-feasible solutions. Then $\bar{u}=x^*$; that is $(\bar{u},q^*)$ is* (EP$_{v^*}$)*-optimal, with each objective value equal to zero.*

Proof. Let $(\bar{u},\bar{y},\bar{z})$ be (DEP$_{v^*}$1)-optimal. Suppose on the contrary that $\bar{u}\neq x^*$. Since (x^*,q^*) is (EP$_{v^*}$)-optimal, there exist (y^*,z^*) such that (x^*,y^*,z^*) is (DEP$_{v^*}$1)-optimal and

$$\begin{aligned}q^*=0&=\sum_{i=1}^{p}y_i^*[f_i(x^*)-v^*g_i(x^*)]+\sum_{k=1}^{m}z_k^*h_k(x^*)\\&=\sum_{i=1}^{p}\bar{y}_i[f_i(\bar{u})-v^*g_i(\bar{u})]+\sum_{k=1}^{m}\bar{z}_kh_k(\bar{u}).\end{aligned} \tag{27}$$

That is,

$$G(x^*)\;=\;G(\bar{u}),$$

which in turn implies that

$$\frac{1}{r}(e^{r(G(x^*)-G(\bar{u}))}-1)\;=\;0.$$

The above inequality together with $(p,r)-\rho-(\eta,\theta)$-invexity of $G(.)$ at $\bar{u}$, implies that

$$\frac{1}{p}\nabla G(\bar{u})(e^{p\eta(x^*,\bar{u})}-\mathbf{1})+\rho\left\|\theta(x^*,\bar{u}))\right\|^2 \leqq 0. \tag{28}$$

Consequently, (21) and (28) yield

$$\rho\left\|\theta(x^*,\bar{u})\right\|^2 \leqq 0,$$

which contradicts to the fact that $\rho>0$. This completes the proof. □

Remark 4.1. If the function $G(.)$ in Theorem 4.3 is expressed by the sum of $B(.)$ and $C(.)$ as defined in previous section and if $B(.)$ is strictly $(p,r)-\rho-(\eta,\theta)$-invex and $C(.)$ is $(p,r)-\rho-(\eta,\theta)$-invex then the Theorem 4.3 is still hold.

5. Second duality model

In this section for a given v, we take the following form of dual problem:

$$(\mathbf{DEP}_v\mathbf{2}) \qquad \text{Maximize } \sum_{i=1}^{p} y_i[f_i(u)-vg_i(u)]$$

subject to

$$\sum_{i=1}^{p} y_i[\nabla f_i(u)-v\nabla g_i(u)]+\sum_{k=1}^{m} z_k\nabla h_k(u)=0, \tag{29}$$

$$\sum_{k=1}^{m} z_k h_k(u)\geqq 0, \tag{30}$$

$$\sum_{i=1}^{p} y_i=1, \tag{31}$$

$$u\in\mathbb{R}^n,\ y\in\mathbb{R}^p,\ z\in\mathbb{R}^m,\ y\geq 0,\ z\geqq 0,\ v\geqq 0. \tag{32}$$

Theorem 5.1. (Weak duality). *For a given v^*, let $(\hat{x},\hat{q})$ be* (EP_{v^*})*-feasible and let $(\bar{u},\bar{y},\bar{z})$ be* $(\mathrm{DEP}_{v^*}2)$*-feasible. Moreover, assume any one of the conditions below holds:*

(*a*) *$H(.)=\sum_{i=1}^{p}\bar{y}_i[f_i(.)-v^*g_i(.)]$ is $(p,r)-\rho_1-(\eta,\theta)$-pseudo-invex at $\bar{u}$ and $I(.)=\sum_{k=1}^{m}\bar{z}_kh_k(.)$ is $(p,r)-\rho_2-(\eta,\theta)$-quasi-invex at $\bar{u}$ with respect to η, θ and $\rho_1+\rho_2\geqq 0$;*

(*b*) *$H(.)=\sum_{i=1}^{p}\bar{y}_i[f_i(.)-v^*g_i(.)]$ is $(p,r)-\rho_1-(\eta,\theta)$-quasi-invex at $\bar{u}$ and $I(.)=\sum_{k=1}^{m}\bar{z}_kh_k(.)$ is strictly $(p,r)-\rho_2-(\eta,\theta)$-pseudo-invex at $\bar{u}$ with respect to η, θ and $\rho_1+\rho_2\geqq 0$,*

for all feasible solutions for (EP_v) *and* (DEP_v2)*. Then*

$$\inf(\mathrm{EP}_{v^*})\ \geqq\ \sup(\mathrm{DEP}_{v^*}2).$$

Proof. Let $(\hat{x},\hat{q})$ be (EP_{v^*})-feasible and let $(\bar{u},\bar{y},\bar{z})$ be $(\mathrm{DEP}_{v^*}2)$-feasible. From (2), (30) and (32), we have

$$\sum_{j=1}^{m}\bar{z}_kh_k(\hat{x})\leqq\sum_{j=1}^{m}\bar{z}_kh_k(\bar{u}). \tag{33}$$

On the other hand, suppose contrary to the result, i.e.,

$$\inf\ (EP_{v^*})<\sup\ (DEP_{v^*}2).$$

Equivalently,

$$\hat{q} \;<\; \sum_{i=1}^{p} \bar{y}_i[f_i(\bar{u}) - v^* g_i(\bar{u})]. \tag{34}$$

In (34), using (1), (31) and (32) on LHS, we have

$$\sum_{i=1}^{p} \bar{y}_i[f_i(\hat{x}) - v^* g_i(\hat{x})] < \sum_{i=1}^{p} \bar{y}_i[f_i(\bar{u}) - v^* g_i(\bar{u})].$$

That is,

$$H(\hat{x}) < H(\bar{u}),$$

which in turn implies that

$$\frac{1}{r}(e^{r(H(\hat{x})-H(\bar{u}))} - 1) < 0. \tag{35}$$

If hypothesis (a) holds, the above inequality together with $(p,r) - \rho_1 - (\eta,\theta)$-pseudo-invexity of $H(.)$ at $\bar{u}$, implies that

$$\frac{1}{p}\nabla H(\bar{u})(e^{p\eta(\hat{x},\bar{u})} - \mathbf{1}) < -\rho_1 \left\|\theta(\hat{x},\bar{u}))\right\|^2 . \tag{36}$$

Consequently, (29), (36) and the assumption $\rho_1 + \rho_2 \geqq 0$, yield

$$\frac{1}{p}\nabla I(\bar{u})(e^{p\eta(\hat{x},\bar{u})} - \mathbf{1}) > -\rho_2 \left\|\theta(\hat{x},\bar{u}))\right\|^2 . \tag{37}$$

The above inequality together with $(p,r) - \rho_2 - (\eta,\theta)$-quasi-invexity of $I(.)$ at $\bar{u}$, implies that

$$\frac{1}{r}(e^{r(I(\hat{x})-I(\bar{u}))} - 1) > 0, \tag{38}$$

which in turn implies that

$$I(\hat{x}) - I(\bar{u}) > 0. \tag{39}$$

That is,

$$\sum_{j=1}^{m} \bar{z}_k h_k(\hat{x}) > \sum_{j=1}^{m} \bar{z}_k h_k(\bar{u}),$$

which contradicts (33).
The proof of hypothesis (b) follows along the lines similar to that of (a). This completes the proof. □

Similarly, we can establish the following strong duality theorem and strict converse duality theorem.

Theorem 5.2. (Strong duality). *Let*

$$v^* = \min_{x\in S} \max_{1\leqq i\leqq p} \left[\frac{f_i(x)}{g_i(x)}\right]$$

and let (x^*, q^*) be (EP_{v^*})*-optimal, at which an appropriate constraint qualification holds* [18]*. Then, there exists (y^*, z^*) such that (x^*, y^*, z^*) is* $(\mathrm{DEP}_{v^*}2)$*-feasible and the corresponding objective values of* (EP_{v^*}) *and* $(\mathrm{DEP}_{v^*}2)$ *are equal. If also the hypotheses of Theorem 5.1 are satisfied, then (x^*, q^*) and (x^*, y^*, z^*) are, respectively, global optimal for* (EP_{v^*}) *and* $(\mathrm{DEP}_{v^*}2)$ *with each objective value equal to zero.*

Theorem 5.3. (Strict converse duality). *Let*

$$v^* = \min_{x\in S} \max_{1\leqq i\leqq p} \left[\frac{f_i(x)}{g_i(x)}\right]$$

and let (x^*, q^*) *be* (EP_{v^*})*-optimal, at which an appropriate constraint qualification holds* [18]. *Let* $(\bar{u}, \bar{y}, \bar{z})$ *be* $(\mathrm{DEP}_{v^*}2)$*-optimal. Moreover, assume any one of the conditions below holds:*

(a) $H(.) = \sum_{i=1}^{p} \bar{y}_i[f_i(.) - v^* g_i(.)]$ *is* $(p, r) - \rho_1 - (\eta, \theta)$*-quasi-invex at* $\bar{u}$ *and* $I(.) = \sum_{k=1}^{m} \bar{z}_k h_k(.)$ *is strictly* $(p, r) - \rho_2 - (\eta, \theta)$*-pseudo-invex at* $\bar{u}$ *with respect to* η, θ *and* $\rho_1 + \rho_2 \geqq 0$;

(b) $H(.) = \sum_{i=1}^{p} \bar{y}_i[f_i(.) - v^* g_i(.)]$ *is strictly* $(p, r) - \rho_1 - (\eta, \theta)$*-pseudo-invex at* $\bar{u}$ *and* $I(.) = \sum_{k=1}^{m} \bar{z}_k h_k(.)$ *is* $(p, r) - \rho_2 - (\eta, \theta)$*-quasi-invex at* $\bar{u}$ *with respect to* η, θ *and* $\rho_1 + \rho_2 \geqq 0$,

for all feasible solutions for (EP_{v^*}) *and* $(\mathrm{DEP}_{v^*}2)$. *Then* $\bar{u} = x^*$; *that is,* $(\bar{u}, q^*)$ *is* (EP_{v^*})*-optimal, with each objective value equal to zero.*

6. Conclusion

In this paper, we have defined the concept of generalized $(p, r) - \rho - (\eta, \theta)$-invex functions. An example is given to support this class of functions. Sufficient optimality conditions for minimax fractional programming problem have established under the $(p, r) - \rho - (\eta, \theta)$-invexity assumptions. Moreover, duality results for two types of dual models are derived under the aforesaid functions. The question arise whether sufficiency and duality theorems established in this paper are also holds under the assumption of $(p, r) - \rho - (\eta, \theta)$-invexity for a class of nondifferentiable minimax fractional programming problem:

$$\textbf{(NFP)} \qquad \min_{x\in\mathbb{R}^n} \sup_{y\in Y} \frac{f(x, y) + (x^t Bx)^{1/2}}{h(x, y) - (x^t Dx)^{1/2}},$$

$$\text{subject to} \qquad g(x) \leqq 0,\ x \in X,$$

where Y is a compact subset of $\mathbb{R}^m$, $f, h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are C^1-functions on $\mathbb{R}^n \times \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ is a C^1-function on $\mathbb{R}^n$; B and D are positive semi-definite matrices. We can consider a more general form of objective function, i.e.

$$\min_{x\in\mathbb{R}^n} \sup_{y\in Y} \frac{f(x, y) + s(x|D)}{h(x, y) + s(x|E)},$$

where D and E are compact convex set and for example $s(x|D)$ is the support function of D defined by

$$s(x|D) = \max\{x^T y | y \in D\}.$$

It will orient the future research of the authors.

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