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On the controllability of a system of two Schrödinger equations

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Abstract. This article studies the controllability of a system coupling two linear Schrödinger equations with a boundary control at the extremity of one equation only. By using Fourier analysis we show that this system is null-controllable in any time $T > 0$.

Key words and phrases. Schrödinger equation, observability inequality, Ingham's inequality, biorthogonals.

1. Introduction

For $T > 0$, we consider the following system consisting on two coupled Schrödinger equations

$$\begin{cases} u_t(t,x) + iu_{xx}(t,x) - iv = 0 & (t,x) \in (0,T) \times (0,1) \\ v_t(t,x) + iv_{xx}(t,x) - iu = 0 & (t,x) \in (0,T) \times (0,1) \\ u(t,0) = 0,\ u(t,1) = f(t) & t \in (0,T) \\ v(t,0) = v(t,1) = 0 & t \in (0,T) \\ u(0,x) = u^0(x),\ v(0,x) = v^0(x) & x \in (0,1). \end{cases} \tag{1}$$

Equation (1) is said to be *null-controllable in time* $T > 0$ if, for every initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H} := H^{-1}(0,1) \times H^{-1}(0,1)$, there exists a control function $f \in L^2(0,T)$ such that the corresponding solution of (1) verifies

$$u(T,x) = v(T,x) = 0 \qquad (x \in (0,1)). \tag{2}$$

The controllability problem for the linear and nonlinear Schrödinger equation has been studied in the literature and has received a positive answer even in more general contexts (see, for instance, [3, 4, 5]). However, the study of system of such equations is less common.

The aim of this paper is to study the controllability of system (1). The controllability problem is reduced to an observability inequality for the adjoint system, which is proved by using techniques based on Fourier spectral analysis.

The main ingredient is the fact that the reunion of two families of exponential functions $(e^{i\lambda_n t})_{n\geq 1} \cup (e^{i\mu_n t})_{n\geq 1}$ verifies the Ingham's inequality, for any $T > 0$, in the case in which the asymptotic gaps $\liminf_{n\to\infty}(\lambda_{n+1} - \lambda_n)$, $\liminf_{n\to\infty}(\mu_{n+1} - \mu_n)$ are infinite, and the distance between each two exponents is bigger than $d > 0$.

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The rest of the paper is organized as follows. In Section 2 we present the extension of Ingham's Theorem. In Section 3 we prove the observability inequality corresponding to the adjoint system to (1).

2. An extension of Ingham's Theorem

In this section we prove an extension of Ingham's theorem. Let us introduce some notation. For $f \in L^1(\mathbb{R})$, the Fourier transform of f, denoted by $\widehat{f}$, is defined by

$$\widehat{f}(x) = \int_{\mathbb{R}} e^{-itx} f(t)dt \qquad (x \in \mathbb{R}),$$

and we recall the inversion formula

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(x) e^{itx} dx \qquad (t \in \mathbb{R}).$$

We recall that a sequence $(\theta_m)_{m\in\mathbb{Z}^*} \subset L^2(-T,T)$ is *biorthogonal to the family of exponential functions* $\left(e^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ *in* $L^2(-T,T)$ if

$$\int_{-T}^{T} \theta_m(t) e^{-i\lambda_n t} dt = \delta_{mn} \qquad (m, n \in \mathbb{Z}^*)\,. \tag{3}$$

Theorem 2.1. *Let $I, J \subset \mathbb{N}$. Let $\Lambda_1 = (\lambda_n)_{n\in I}$ and $\Lambda_2 = (\mu_m)_{m\in J}$ be two increasing sequences in $\mathbb{R}$ which satisfy the following properties*

$$\liminf_{n\to\infty} (\lambda_{n+1} - \lambda_n) = +\infty, \tag{4}$$

$$\liminf_{m\to\infty} (\mu_{m+1} - \mu_m) = +\infty, \tag{5}$$

and

$$\inf_{n,\in I, m\in J} (\lambda_n - \mu_m) = d > 0. \tag{6}$$

Then for every $T > 0$ there exists $C = C(T,d) > 0$ such that

$$\sum_{n\in I} |a_n|^2 + \sum_{m\in J} |b_m|^2 \le c \int_{-T}^{T} \left| \sum_{n\in I} a_n e^{i\lambda_n t} + \sum_{m\in J} b_m e^{i\mu_m t} \right|^2 dt,$$

for every sequences $(a_n)_{n\in I} \in \ell^2$ and $(b_m)_{m\in J} \in \ell^2$.

For simplicity we introduce the following spaces

$$E_{\Lambda_1} = \overline{\operatorname{Span}(e^{i\lambda_n t})_{n\in I}}^{\|\cdot\|_{L^2(-2T,2T)}} \qquad \text{and} \qquad E_{\Lambda_2} = \overline{\operatorname{Span}(e^{i\mu_m t})_{m\in J}}^{\|\cdot\|_{L^2(-2T,2T)}}.$$

As in [7, Proposition 8.4.1], we can evaluate the distance between an exponential function and a family of exponential functions. We have the following result proved in [6, Lemma 3.3].

Lemma 2.2. *Let $T > 0$, $I \subset \mathbb{N}$ and $\Lambda = (\nu_n)_{n\in I} \subset \mathbb{R}$ be a sequence with the property that there exists $C_1, C_2 > 0$ such that*

$$C_1 \sum_{n\in I} |a_n|^2 \le \int_{-T}^{T} \left| \sum_{n\in I} a_n e^{i\nu_n t} \right|^2 dt \le C_2 \sum_{n\in I} |a_n|^2, \tag{7}$$

for every sequence $(a_n)_{n\in I} \in l^2$. *Let* $\mu \in \mathbb{R}$ *be chosen such that*

$$\inf_{n\in I} |\mu - \nu_n| = d > 0. \tag{8}$$

Then, for every $\varepsilon \in (0, T]$ *and for every sequence* $(a_n)_{n\in I} \in l^2$, *we have*

$$\left\| e^{i\mu t} - \sum_{n\in I} a_n e^{i\nu_n t} \right\|_{L^2(-T-\varepsilon, T+\varepsilon)} \geq \sqrt{T+\varepsilon}\,\frac{\sigma\sqrt{C_1}}{2\sqrt{2C_2}}, \tag{9}$$

where $\sigma = \begin{cases} \frac{d^2\varepsilon^2}{\pi^2} & \text{if } d\varepsilon \in (0, \frac{\pi}{2}) \\ \frac{1}{4} & \text{if } d\varepsilon \geq \frac{\pi}{2}. \end{cases}$

We are now in position to prove the main result.

Proof of Theorem 2.1. From (4) and (5) and Ingham's Theorem (see [1, 2]) we deduce that, for every $T > 0$ there exists positive constants K_1, K_2, K_3 and K_4, which depends on T, such that

$$K_1 \sum_{n\in I} |a_n|^2 \leq \int_{-T}^{T} \left| \sum_{n\in I} a_n e^{i\lambda_n t} \right|^2 dt \leq K_2 \sum_{n\in I} |a_n|^2, \tag{10}$$

and

$$K_3 \sum_{m\in J} |b_m|^2 \leq \int_{-T}^{T} \left| \sum_{m\in J} b_m e^{i\mu_m t} \right|^2 dt \leq K_4 \sum_{m\in J} |b_m|^2, \tag{11}$$

for every sequences $(a_n)_{n\in I} \in \ell^2$ and $(b_m)_{m\in J} \in \ell^2$.

From (9) in Lemma 2.2 it follows that, for each $n \in I$, $e^{i\lambda_n t} \notin E_{\Lambda_2}$. By applying the Projection Theorem, we deduce that there exists a unique $P_{E_{\Lambda_2}} e^{i\lambda_n t} \in E_{\Lambda_2}$ with the property

$$\left(e^{i\lambda_n t} - P_{E_{\Lambda_2}} e^{i\lambda_n t}, v \right)_{L^2(-2T, 2T)} = 0 \qquad (v \in E_{\Lambda_2}). \tag{12}$$

For any $n \in I$, we define

$$F_n(t) = \begin{cases} \dfrac{e^{i\lambda_n t} - P_{E_{\Lambda_2}} e^{i\lambda_n t}}{\left\| e^{i\lambda_n t} - P_{E_{\Lambda_2}} e^{i\lambda_n t} \right\|^2_{L^2(-2T,2T)}} & \text{if } t \in (-2T, 2T) \\ 0 & \text{if } t \in (-\infty, -2T] \cup [2T, \infty), \end{cases} \tag{13}$$

sach that supp$F_n \subset [-2T, 2T]$, for every $n \in I$.

By using (12), for any $n \in I$, we have

$$\begin{aligned} \widehat{F}_n(\lambda_n) &= 1, \\ \widehat{F}_n(\mu_m) &= 0 \quad (m \in J). \end{aligned} \tag{14}$$

By using (11) and the fact that $\inf_{n\in I, m\in J} |\lambda_n - \mu_m| \geq d$ we can apply Lemma 2.2 on the interval $(-T, T)$ and for $\varepsilon = T$. Hence, for each $n \in I$ and $x \in \mathbb{R}$, we have that

$$\begin{aligned} \left| \widehat{F}_n(x) \right| &= \frac{\left| \int_{-2T}^{2T} e^{-itx} (e^{i\lambda_n t} - P_{E_{\Lambda_2}} e^{i\lambda_n t}) dt \right|}{\left\| e^{i\lambda_n t} - P_{E_{\Lambda_2}} e^{i\lambda_n t} \right\|^2_{L^2(-2T,2T)}} \leq \\ &\leq \frac{2\sqrt{T}}{\left\| e^{i\lambda_n t} - P_{E_{\Lambda_2}} e^{i\lambda_n t} \right\|_{L^2(-2T,2T)}} \leq \frac{2\sqrt{T}}{\sigma\sqrt{2T}\frac{\sqrt{K_3}}{2\sqrt{2K_4}}} = \frac{\sqrt{K_4}}{\sigma\sqrt{K_3}}, \end{aligned} \tag{15}$$

where $\sigma = \begin{cases} \frac{d^2T^2}{\pi^2} & \text{if } dT \in (0, 2\pi) \\ \frac{1}{4} & \text{if } dT \in [2\pi, \infty) \end{cases}$.

From [8, Theorem 2, p151] and (10) we deduce that there exists a biorthogonal sequence $(\Phi_n)_{n\in I}$ to the family of exponential functions $(e^{i\lambda_n t})_{n\in I}$ in $L^2(-T,T)$ with the following property

$$\|\Phi_n\|_{L^2(-T,T)} \le \frac{1}{\sqrt{K_1}} \qquad (n \in I). \tag{16}$$

For any $n \in I$ we consider the function

$$\widehat{G}_n(x) = \widehat{\Phi}_n(x)\widehat{F}_n(x). \tag{17}$$

We remark that $\widehat{G}_n$ is an entire function of exponential type less than $3T$, since $(\widehat{\Phi}_n)_{n\in I}$ are entire functions of exponential type less than T and $\widehat{F}_n$ is an entire function of exponential type less than $2T$.

From (14) and taking to account that the sequence $(\Phi_n)_{n\in I}$ is biorthogonal to the family $(e^{i\lambda_n t})_{n\in I}$ in $L^2(-T,T)$, it follows that

$$\begin{array}{ll} \widehat{G}_n(\lambda_n) = 1 & (n \in I), \\ \widehat{G}_n(\lambda_l) = 0 & (n, l \in I,\ n \neq l), \\ \widehat{G}_n(\mu_m) = 0 & (n \in I,\ m \in J). \end{array} \tag{18}$$

By using (15), (16) and (17) we obtain that, for any $n \in I$, we have

$$\|\widehat{G}_n\|_{L^\infty} \le \frac{\sqrt{2TK_4}}{\sigma\sqrt{K_1K_3}}. \tag{19}$$

Similarly we deduce that, for any $m \in J$, there exists H_m with the following properties

$$\operatorname{supp} H_m \subset [-3T, 3T] \qquad (m \in J), \tag{20}$$

$$\begin{array}{ll} \widehat{H}_m(\mu_m) = 1 & (m \in J), \\ \widehat{H}_m(\mu_k) = 0 & (m, k \in J,\ m \neq k), \\ \widehat{H}_m(\lambda_n) = 0 & (n \in I,\ m \in J), \end{array} \tag{21}$$

$$\|\widehat{H}_m\|_{L^\infty} \le \frac{\sqrt{2TK_2}}{\sigma\sqrt{K_1K_3}} \qquad (m \in J). \tag{22}$$

From (18) and (21) we deduce that the sequence $(G_n)_{n\in I} \cup (H_m)_{m\in J}$ is biorthogonal to the family $(e^{i\lambda_n t})_{n\in I} \cup (e^{i\mu_m t})_{m\in J}$ in $L^2(-3T, 3T)$.

Let $k_T = \frac{1}{T^2}(\chi_T * \chi_T)$, where χ_T represents the characteristic function $\chi_{[-T/2,T/2]}$. Evidently $\operatorname{supp}(k_T) \subset [-T,T]$. Also, we have

$$\widehat{k}_T(x) = \frac{1}{2\pi}\int_{\mathbb{R}} k_T(t)e^{-itx}\,dt = \frac{1}{T^2}\widehat{\chi}_T(x)\widehat{\chi}_T(x) = \frac{4}{T^2}\frac{\sin^2(\frac{xT}{2})}{x^2}, \tag{23}$$

and

$$|k_T(t)| \le |k_T(0)| = \frac{1}{T}. \tag{24}$$

We define

$$\begin{array}{ll} \rho_n(t) = e^{i\lambda_n t}k_T(t) & (n \in I), \\ \rho_m(t) = e^{i\mu_m t}k_T(t) & (m \in J), \end{array}$$

so that $\operatorname{supp}\rho_n, \operatorname{supp}\rho_m \subset [-T,T]$ and $\widehat{\rho}_n(x) = \widehat{k}_T(x-\lambda_n)$, $\widehat{\rho}_m(x) = \widehat{k}_T(x-\mu_m)$, for every $n \in I$ and $m \in J$.

For any $n \in I$ and $m \in J$, we define

$$\theta_n(t) = G_n(t) * \rho_n(t) \quad \text{and} \quad \theta_m(t) = H_m(t) * \rho_m(t), \tag{25}$$

such that $\operatorname{supp}\theta_n, \operatorname{supp}\theta_m \subset [-4T, 4T]$.

Thus, we deduce that the sequence $\theta_n(t)_{n\in I} \cup \theta_m(t)_{m\in J}$ is biorthogonal to the family $(e^{i\lambda_n t})_{n\in I} \cup (e^{i\mu_m t})_{m\in J}$ in $L^2(-4T,4T)$. From Cauchy-Schwarz inequality we have that

$$\begin{aligned}
&\sum_{n\in I}|a_n|^2 + \sum_{m\in J}|b_m|^2 \\
&= \int_{-4T}^{4T}\left(\sum_{n\in I} a_n e^{i\lambda_n t} + \sum_{m\in J} b_m e^{i\mu_m t}\right)\left(\sum_{n\in I}\overline{a}_n\theta_n(t) + \sum_{m\in J}\overline{b}_m\theta_m(t)\right)dt \\
&\leq \left\|\sum_{n\in I} a_n e^{i\lambda_n t} + \sum_{m\in J} b_m e^{i\mu_m t}\right\|_{L^2(-4T,4T)}\left\|\sum_{n\in I}\overline{a}_n\theta_n(t) + \sum_{m\in J}\overline{b}_m\theta_m(t)\right\|_{L^2(-4T,4T)},
\end{aligned} \tag{26}$$

for every sequences $(a_n)_{n\in I} \in \ell^2$ and $(b_m)_{m\in J} \in \ell^2$.

By using (25) and Plancherel's Theorem we obtain that

$$\begin{aligned}
&\left\|\sum_{n\in I}\overline{a}_n\theta_n(t) + \sum_{m\in J}\overline{b}_m\theta_m(t)\right\|^2_{L^2(-4T,4T)} = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\sum_{n\in I}\overline{a}_n\widehat{\theta}_n(x) + \sum_{m\in J}\overline{b}_m\widehat{\theta}_m(x)\right|^2 dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\left|\sum_{n\in I}\overline{a}_n\widehat{G}_n(x)\widehat{k}_T(x-\lambda_n) + \sum_{m\in J}\overline{b}_m\widehat{H}_m(x)\widehat{k}_T(x-\mu_m)\right|^2 dx \\
&\leq \frac{1}{2\pi}\max\left\{\|\widehat{G}_n\|^2_{L^\infty}, \|\widehat{H}_m\|^2_{L^\infty}\right\}\int_{\mathbb{R}}\left|\sum_{n\in I}\overline{a}_n\widehat{k}_T(x-\lambda_n) + \sum_{m\in J}\overline{b}_m\widehat{k}_T(x-\mu_m)\right|^2 dx \\
&= \max\left\{\|\widehat{G}_n\|^2_{L^\infty}, \|\widehat{H}_m\|^2_{L^\infty}\right\}\int_{-4T}^{4T}\left|\sum_{n\in I}\overline{a}_n k_T(t)e^{i\lambda_n t} + \sum_{m\in J}\overline{b}_m k_T(t)e^{i\mu_m t}\right|^2 dt \\
&\leq \max\left\{\|\widehat{G}_n\|^2_{L^\infty}, \|\widehat{H}_m\|^2_{L^\infty}\right\}\|k_T\|^2_{L^\infty}\int_{-4T}^{4T}\left|\sum_{n\in I}\overline{a}_n e^{i\lambda_n t} + \sum_{m\in J}\overline{b}_m e^{i\mu_m t}\right|^2 dt.
\end{aligned}$$

From the above inequality and (26) we obtain that

$$\begin{aligned}
&\sum_{n\in I}|a_n|^2 + \sum_{m\in J}|b_m|^2 \\
&\leq \max\left\{\|\widehat{G}_n\|_{L^\infty}, \|\widehat{H}_m\|_{L^\infty}\right\}\|k_T\|_{L^\infty}\left\|\sum_{n\in I} a_n e^{i\lambda_n t} + \sum_{m\in J} b_m e^{i\mu_m t}\right\|^2_{L^2(-4T,4T)}.
\end{aligned}$$

Consequently, the above inequality combined with (19), (22), and (24) implies that

$$\sum_{n\in I}|a_n|^2 + \sum_{m\in J}|b_m|^2 \leq \frac{\sqrt{2}\max\left\{\sqrt{K_2}, \sqrt{K_4}\right\}}{\sigma\sqrt{TK_1K_3}}\int_{-4T}^{4T}\left|\sum_{n\in I} a_n e^{i\lambda_n t} + \sum_{m\in J} b_m e^{i\mu_m t}\right|^2 dt,$$

and the proof is complete. □

3. Controllability result

The aim of this section is to prove the observability inequality which ensures the null-controllability property of the system (1). We recall that system (1) is null-controllable in $T > 0$ if and only if there exists a positive constant $C = C(T)$ such that

$$\|p^0\|^2_{H^{-1}(0,1)}+\|q^0\|^2_{H^{-1}(0,1)} \leq C\int_0^T |p_x(t,1)|^2\,dt \quad \left((p^0,q^0)\in H^{-1}(0,1)\times H^{-1}(0,1)\right), \tag{27}$$

where (p,q) are the solution of the adjoint backward problem

$$\begin{cases} p_t(t,x) - ip_{xx}(t,x) + iq(t,x) = 0 & (t,x)\in(0,T)\times(0,1)\\ q_t(t,x) - iq_{xx}(t,x) + ip(t,x) = 0 & (t,x)\in(0,T)\times(0,1)\\ p(t,0)=p(t,1)=0 & t\in(0,T)\\ q(t,0)=q(t,1)=0 & t\in(0,T)\\ p(T,x)=p^0(x),\ q(T,x)=q^0(x) & x\in(0,1). \end{cases} \tag{28}$$

Inequality (27) is called "observability" inequality and, as said before is equivalent to the controllability property of (1).

We have the following result:

Theorem 3.1. *There exists $C = C(T) > 0$ such that the observability inequality* (27) *holds.*

Proof. Let the initial data $\begin{pmatrix} p^0(x)\\ q^0(x)\end{pmatrix} \in H^{-1}(0,1)\times H^{-1}(0,1)$ with the Fourier expansion

$$\begin{pmatrix} p^0(x)\\ q^0(x)\end{pmatrix} = \begin{pmatrix} \sum\limits_{n\in\mathbb{N}^*}\frac{a_n}{n\pi}\sin(n\pi x)\\ \sum\limits_{n\in\mathbb{N}^*}\frac{b_n}{n\pi}\sin(n\pi x)\end{pmatrix}. \tag{29}$$

Then the solutions of the system (28) are given by

$$\begin{pmatrix} p(t,x)\\ q(t,x)\end{pmatrix} = \begin{pmatrix} \sum\limits_{n\in\mathbb{N}^*}\left(\frac{a_n+b_n}{2n\pi}e^{i\lambda_n^+(T-t)} + \frac{a_n-b_n}{2n\pi}e^{i\lambda_n^-(T-t)}\right)\sin(n\pi x)\\ \sum\limits_{n\in\mathbb{N}^*}\left(\frac{a_n+b_n}{2n\pi}e^{i\lambda_n^-(T-t)} - \frac{a_n-b_n}{2n\pi}e^{i\lambda_n^-(T-t)}\right)\sin(n\pi x)\end{pmatrix}, \tag{30}$$

where

$$\lambda_n^+ = n^2\pi^2+1 \quad\text{and}\quad \lambda_n^- = n^2\pi^2-1 \quad (n\in\mathbb{Z}^*).$$

From (29), we have that

$$\|p^0\|^2_{H^{-1}(0,1)}+\|q^0\|^2_{H^{-1}(0,1)} = \frac{1}{2}\sum_{n\in\mathbb{N}^*}|a_n|^2 + \frac{1}{2}\sum_{n\in\mathbb{N}^*}|b_n|^2, \tag{31}$$

Taking to account that

$$\liminf_{n\to\infty}(\lambda_{n+1}^+ - \lambda_n^+) = +\infty, \quad \liminf_{n\to\infty}(\lambda_{n+1}^- - \lambda_n^-) = +\infty \quad\text{and}\quad \inf_{n,m\geq 1}(\lambda_n^+ - \lambda_m^-) = 2,$$

we can apply Theorem 2.1 and we have that there exists $C = C(T) > 0$ such that

$$\int_0^T |p_x(t,1)|^2\,dt = \int_0^T \left| \sum_{n\in\mathbb{N}^*} \left(\frac{a_n+b_n}{2} e^{i\lambda_n^+(T-t)} + \frac{a_n-b_n}{2} e^{i\lambda_n^-(T-t)} \right) (-1)^n \right|^2 dt$$
$$\geq \frac{1}{C} \left(\sum_{n\in\mathbb{N}^*} \left| \frac{a_n+b_n}{2} \right|^2 + \sum_{n\in\mathbb{N}^*} \left| \frac{a_n-b_n}{2} \right|^2 \right) = \frac{1}{2C} \left(\sum_{n\in\mathbb{N}^*} |a_n|^2 + \sum_{n\in\mathbb{N}^*} |b_n|^2 \right). \tag{32}$$

From (31) and (32) we deduce that (27) holds and the proof ends. □

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