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Existence and multiplicity results for elliptic equations involving the p-Laplacian-like

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Abstract. In this paper, existence results of positive solutions to a Neumann problem involving the p-Laplacian-like are established. Multiplicity results are also point out. The results of the equations discussed can be applied to a variety of different fields in applied mechanics.

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1. Introduction

In this paper, we consider the existence of positive solutions for the Neumann problem, originated from a capillary phenomena,

$$\begin{cases} -\mathrm{div}\left(\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right)+|u|^{p-2}u=\lambda f(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega, \end{cases} \qquad (N_\lambda^f)$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1, ν is the outer unit normal to $\partial\Omega$, $\lambda > 0$, $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $p > N$.

As for the p-Laplacian-like equation, the authors in [4] discussed the eigenvalue problem for the equation

$$-\mathrm{div}\left(\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right)=\lambda f(x,u), \;\; u\in W_0^{1,p}(\Omega),$$

and proved the existence of two eigenfunctions which have very different asymptotic behaviors.

Authors in [8] studied

$$\begin{cases} -\mathrm{div}\left(\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right)+a(x)|u|^{p-2}u=f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, $\lambda > 0$, $a \in C(\mathbb{R}^+,\mathbb{R})$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $1 < p < N$. They established the existence of one or infinitely many nontrivial solutions.

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Also, the study of $p(x)$-Laplacian-like equations has been received considerable attention in recent years (see for instance [1, 7, 11, 12, 13]).
In the present paper, motivated by [3], our analysis is mainly based on a recent critical point theorem of Bonanno [2], of whose two its consequences are here applied (see Theorems 2.1 and 2.2).

The plan of the paper is as follows. In Section 2, we introduce our abstract framework and main tool, while Section 3, is devoted to existence results of at least one solution. Precisely, our main result (Theorem 3.1) is proved and its consequences (Theorems 3.2, 3.3 and 3.4) are pointed out. Finally, in Section 4, multiplicity results are presented; precisely, an existence result of two solutions (Theorem 4.1), an existence result of three solutions (Theorem 4.2) and its consequence (Theorem 4.3) are pointed out, and lastly, a concrete example is given (Example 4.1).

2. Abstract framework

Our main tools are two consequences of a local minimum theorem [2, Theorem 3.1] which are recalled below.
For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\beta(r_1, r_2) = \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}, \tag{1}$$

$$\rho_2(r_1, r_2) = \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u)}{\Phi(v) - r_1}, \tag{2}$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, and

$$\rho(r) = \sup_{v\in\Phi^{-1}(]r,+\infty[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{\Phi(v) - r}, \tag{3}$$

for all $r \in \mathbb{R}$.

Theorem 2.1 ([2, Theorem 5.1]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2), \tag{4}$$

where β and ρ_2 are given by (1) and (2). Then, setting $T_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in]\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}[$ there is $u_{0,\lambda} \in \Phi(u)^{-1}(]r_1, r_2[)$ such that $T_\lambda(u_{0,\lambda}) \leq T_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $T'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 ([2, Theorem 5.3]). *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that*

$$\rho(r) > 0, \tag{5}$$

where ρ is given by (3), *and for each $\lambda > \frac{1}{\rho(r)}$ the function $T_\lambda := \Phi - \lambda\Psi$ is coercive. Then for each $\lambda > \frac{1}{\rho(r)}$ there is $u_{0,\lambda} \in \Phi^{-1}(]r, +\infty[)$ such that $T_\lambda(u_{0,\lambda}) \le T_\lambda(u)$ for all $u \in \Phi^{-1}(]r, +\infty[)$ and $T'_\lambda(u_{0,\lambda}) = 0$.*

Let X be the Sobolev space $W^{1,p}(\Omega)$ endowed with the usual norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega |u(x)|^p \right)^{\frac{1}{p}},$$

while on the space $C^0(\bar{\Omega})$ we consider the norm $\|u\|_\infty := \sup_{x\in\bar{\Omega}} |u(x)|$. Since $p > N$, X is compactly embedded in $C^0(\bar{\Omega})$, so that

$$k := \sup_{u\in X\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|}.$$

Clearly, $k^p|\Omega| \ge 1$.
If Ω is convex, an explicit upper bound for the constant k is

$$k \le 2^{\frac{p-1}{p}} \max\left\{ \left(\frac{1}{|\Omega|}\right)^{\frac{1}{p}}, \frac{\sigma}{N^{\frac{1}{p}}} \left(\frac{p-1}{p-N}|\Omega|\right)^{\frac{p-1}{p}} \frac{1}{|\Omega|} \right\},$$

where $\sigma = \operatorname{diam}(\Omega)$ and $|\Omega|$ is the Lebesgue measure of Ω (see [5]). Hence

$$|u(x)| \le k\|u\| \quad \text{for all } x \in \Omega \text{ and for all } u \in X. \tag{6}$$

We recall that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if
(a) the mapping $x \longmapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;
(b) the mapping $\xi \longmapsto f(x,\xi)$ is continuous for almost every $x \in \Omega$;
(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1(\Omega)$ such that

$$\sup_{|\xi|\le\rho} |f(x,\xi)| \le l_\rho(x)$$

for almost every $x \in \Omega$.

We say that a function $u \in X$ is a *weak solution* of problem (N^f_λ) if

$$\int_\Omega \left(|\nabla u(x)|^{p-2}\nabla u(x) + \frac{|\nabla u(x)|^{2p-2}\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^{2p}}} \right) \nabla v(x) dx$$
$$+ \int_\Omega |u(x)|^{p-2}u(x)\, v(x)\, dx - \lambda \int_\Omega f(x, u(x))v(x)\, dx = 0$$

holds for all $v \in X$.

3. Main results

In this section we present our main results. To be precise, we establish an existence result of at least one solution, Theorem 3.1, which is based on Theorem 2.1, and we point out some consequences, Theorems 3.2, 3.3 and 3.4. Finally, we present an other existence result of at least one solution, Theorem 3.5, which is based in turn on Theorem 2.2.

Given two non-negative constants c, d, with $c \ne k \sqrt[p]{(1 + d^p)|\Omega|}$, put

$$a_d(c) := \frac{\int_\Omega \max_{|\xi|\le c} F(x,\xi)dx - \int_\Omega F(x,d)dx}{c^p - k^p(1 + d^p)|\Omega|}.$$

Theorem 3.1. *Assume that there exist three constants c_1, c_2, d, with*

$$0 \le c_1 < k\sqrt[p]{(1+d^p)|\Omega|} < c_2,$$

such that $a_d(c_2) < a_d(c_1)$. Then, for each $\lambda \in \left]\frac{1}{pk^p a_d(c_1)}, \frac{1}{pk^p a_d(c_2)}\right[$, the problem (N_λ^f) admits at least one non-trivial weak solution $u_0 \in X$ such that

$$\frac{c_1}{k} < \|u_0\| < \frac{c_2}{k}.$$

Proof. Our aim is to apply Theorem 2.1 to our problem. To this end, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ by setting

$$\Phi(u) = \int_\Omega \frac{1}{p}\Big(|\nabla u(x)|^p + \sqrt{1+|\nabla u(x)|^{2p}} + |u(x)|^p\Big)\,dx,$$

$$\Psi(u) = \int_\Omega F(x,u(x))\,dx,$$

where $F(x,\xi) := \displaystyle\int_0^\xi f(x,t)\,dt$ for every $(x,\xi) \in \Omega \times \mathbb{R}$ and put

$$T_\lambda(u) := \Phi(u) - \lambda\Psi(u) \qquad \forall\, u \in X.$$

Clearly, $\Phi(u)$ and $\Psi(u)$ are well defined and continuously Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega \left(|\nabla u(x)|^{p-2}\nabla u(x) + \frac{|\nabla u(x)|^{2p-2}\nabla u(x)}{\sqrt{1+|\nabla u(x)|^{2p}}}\right)\nabla v(x)\,dx$$

$$+ \int_\Omega |u(x)|^{p-2}u(x)\,v(x)\,dx,$$

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)\,dx$$

for any $v \in X$.

Moreover, Φ is coercive and sequentially weakly lower semicontinuous and Ψ is sequentially weakly upper semicontinuous. Also, Φ' admits a continuous inverse on X^* (see [11]) and Ψ' is compact. Note that the critical points of the functional T_λ in X are exactly the weak solutions of problem (N_λ^f). We verify condition (4) of Theorem 2.1. To this end, put

$$r_1 := \Big(\frac{c_1}{k}\Big)^p, \qquad r_2 := \Big(\frac{c_2}{k}\Big)^p \qquad \text{and } \ u_0(x) = d \quad \text{for all } x \in \Omega.$$

Clearly, $u_0 \in X$ and one has

$$\Phi(u_0) = \frac{1}{p}(1+d^p)|\Omega|,$$

and

$$\Psi(u_0) = \int_\Omega F(x,u_0(x))dx = \int_\Omega F(x,d)dx.$$

From the condition $c_1 < k\sqrt[p]{(1+d^p)|\Omega|} < c_2$,, we obtain $r_1 < \Phi(u_0) < r_2$. Moreover, for all $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r_2[)$, from (6), one has $|u(x)| < c_2$ for all $x \in \Omega$, from which it follows

$$\Psi(u) = \int_\Omega F(x, u(x))dx \leq \int_\Omega \max_{|\xi|\leq c_2} F(x,\xi)dx,$$

for all $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r_2[)$. Hence,

$$\sup_{u\in\Phi^{-1}(]-\infty,r_2])} \Psi(u) \leq \int_\Omega \max_{|\xi|\leq c_2} F(x,\xi)\,dx.$$

Arguing as before, we obtain

$$\sup_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u) \leq \int_\Omega \max_{|\xi|\leq c_1} F(x,\xi)\,dx.$$

Therefore, one has

$$\begin{aligned}
\beta(r_1,r_2) &\leq \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) - \Psi(u_0)}{r_2 - \Phi(u_0)} \\
&\leq \frac{\int_\Omega \max_{|\xi|\leq c_2} F(x,\xi)\,dx - \int_\Omega F(x,d)\,dx}{\frac{1}{p}\left(\frac{c_2}{k}\right)^p - \frac{1}{p}(1+d^p)|\Omega|} \\
&= pk^p \frac{\int_\Omega \max_{|\xi|\leq c_2} F(x,\xi)\,dx - \int_\Omega F(x,d)\,dx}{c_2^p - k^p(1+d^p)|\Omega|} = pk^p a_d(c_2).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\rho_2(r_1,r_2) &\geq \frac{\Psi(u_0) - \sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(u_0) - r_1} \\
&\geq \frac{\int_\Omega F(x,d)\,dx - \int_\Omega \max_{|\xi|\leq c_1} F(x,\xi)\,dx}{\frac{1}{p}(1+d^p)|\Omega| - \frac{1}{p}\left(\frac{c_1}{k}\right)^p} \\
&= pk^p \frac{\int_\Omega F(x,d)\,dx - \int_\Omega \max_{|\xi|\leq c_1} F(x,\xi)\,dx}{k^p(1+d^p)|\Omega| - c_1^p} = pk^p a_d(c_1).
\end{aligned}$$

So, from our assumption, it follows that $\beta(r_1,r_2) < \rho_2(r_1,r_2)$. Therefore, from Theorem 2.1, for each $\lambda \in \left]\frac{1}{pk^p a_d(c_1)}, \frac{1}{pk^p a_d(c_2)}\right[$, the functional T_λ admits at least one critical point u_0 such that

$$r_1 < \Phi(u_0) < r_2,$$

that is,

$$\frac{c_1}{k} < \|u_0\| < \frac{c_2}{k},$$

and the proof is complete. □

Now, we point out the following consequence of Theorem 3.1.

Theorem 3.2. *Assume that there exist two positive constants c and d, with $k\sqrt[p]{(1+d^p)|\Omega|} < c$, such that*

$$\frac{\int_\Omega \max_{|\xi|\leq c} F(x,\xi)\,dx}{c^p} < \frac{1}{k^p}\frac{\int_\Omega F(x,d)\,dx}{(1+d^p)|\Omega|}. \tag{7}$$

Then, for each

$$\lambda \in \left]\frac{1}{p}\frac{(1+d^p)|\Omega|}{\int_\Omega F(x,d)\,dx}, \frac{1}{pk^p}\frac{c^p}{\int_\Omega \max_{|\xi|\leq c} F(x,\xi)\,dx}\right[,$$

the problem (N_λ^f) *admits at least one non-trivial weak solution* $\bar{u} \in X$ *such that* $|\bar{u}(x)| < c$ *for all* $x \in \Omega$.

Proof. The conclusion follows from Theorem 3.1, by taking $c_1 = 0$ and $c_2 = c$. Indeed, owing to assumption (7), one has

$$\begin{aligned} a_d(c) &< \frac{\left(1 - \frac{k^p(1+d^p)|\Omega|}{c^p}\right)\int_\Omega \max_{|\xi|\leq c} F(x,\xi)\,dx}{c^p - k^p(1+d^p)|\Omega|} \\ &< \frac{\int_\Omega F(x,d)\,dx}{k^p(1+d^p)|\Omega|} = a_d(0). \end{aligned}$$

Now, taking (6) into account, Theorem 3.1 ensures the conclusion. □

Now, we point out previous result when the nonlinear term has separable variables. To be precise, let $\alpha \in L^1(\Omega)$ such that $\alpha(x) \geq 0$ a.e. $x \in \Omega$, and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $G(t) := \displaystyle\int_0^t g(\xi)\,d\xi$ for all $t \in \mathbb{R}$. We have the following result as a direct consequence of Theorem 3.1.

Theorem 3.3. *Assume that* g *is non-negative and there exist two positive constants* c, d, *with* $k\sqrt[p]{(1+d^p)|\Omega|} < c$, *such that*

$$\frac{G(c)}{c^p} < \frac{1}{k^p}\frac{G(d)}{(1+d^p)|\Omega|}. \tag{8}$$

Then, for each

$$\lambda \in \left]\frac{(1+d^p)|\Omega|}{p\|\alpha\|_1 G(d)}, \frac{c^p}{pk^p\|\alpha\|_1 G(c)}\right[,$$

where $\|\alpha\|_1 := \int_\Omega |\alpha(x)|\,dx$, *the problem*

$$\begin{cases} -\mathrm{div}\left(\left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = \lambda\alpha(x)g(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

admits at least one positive weak solution $\bar{u} \in X$ *such that* $\bar{u}(x) < c$ *for all* $x \in \Omega$.

Proof. Put $f(x,\xi) = \alpha(x)g(\xi)$ for all $(x,\xi) \in \Omega \times \mathbb{R}$. Clearly, one has $F(x,t) = \alpha(x)G(t)$ for all $(x,t) \in \Omega \times \mathbb{R}$. Therefore, taking into account that G is a non-decreasing function, Theorem 3.2 ensures the existence of a non-zero weak solution $\bar{u}$. We claim that it is non-negative. In fact, arguing by a contradiction and setting $A = \{x \in \Omega : \bar{u}(x) < 0\}$ one has $A \neq \emptyset$. Put $\bar{u}^- = \min\{\bar{u}, 0\}$ one has $\bar{u}^- \in X$ (see, for instance, [6, Lemma 7.6]). So, taking into account that $\bar{u}$ is a weak solution and by choosing $v = \bar{u}^-$ one has

$$\begin{aligned} &\int_A \left(|\nabla\bar{u}(x)|^{p-2}\nabla\bar{u}(x) + \frac{|\nabla\bar{u}(x)|^{2p-2}\nabla\bar{u}(x)}{\sqrt{1+|\nabla\bar{u}(x)|^{2p}}}\right)\nabla\bar{u}(x)\,dx \\ &\qquad\qquad + \int_A |\bar{u}(x)|^{p-2}\bar{u}(x)\,\bar{u}(x)\,dx = \lambda\int_A f(x,\bar{u}(x))\bar{u}(x)dx \leq 0, \end{aligned}$$

that is, $\|\bar{u}\|_{W^{1,p}(A)} = 0$ which is an absurd. Hence, our claim is proved. Now, owing to the strong maximum principle (see, for instance, [10, Theorem 11.1]) the weak solution $\bar{u}$, being non-zero, is positive and the conclusion is achieved. $\square$

A further consequence of Theorem 3.1 is the following result.

Theorem 3.4. *Let $g : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that*

$$\lim_{\xi \to 0^+} \frac{g(\xi)}{\xi^{p-1}} = +\infty, \tag{10}$$

and put $\lambda^* = \frac{1}{pk^p\|\alpha\|_1} \sup_{c>0} \frac{c^p}{\int_0^c g(\xi)\,d\xi}$.

Then, for each $\lambda \in]0, \lambda^*[$, the problem (9) admits at least one positive weak solution.

Proof. Fix $\lambda \in]0, \lambda^*[$. Then, there is $c > 0$ such that $\lambda < \frac{1}{pk^p\|\alpha\|_1} \frac{c^p}{\int_0^c g(\xi)d\xi}$. From (10) there is a d such that $k\sqrt[p]{(1 + d^p)|\Omega|} < c$, and $\frac{p\|\alpha\|_1}{|\Omega|} \frac{\int_0^d g(\xi)\,d\xi}{(1+d^p)} > \frac{1}{\lambda}$. Hence, Theorem 3.3 ensures the conclusion. $\square$

Remark 3.1. Given $g : \mathbb{R} \to \mathbb{R}$ such that (10) holds (that is, without any assumption of sign). From (10) there is $\delta > 0$ such that $g(\xi) > 0$ for all $\xi \in]0, \delta[$. Then put $\bar{\lambda} = \frac{1}{pk^p\|\alpha\|_1} \sup_{c\in]0,\delta[} \frac{c^p}{\int_0^c g(\xi)d\xi}$. Clearly, $\bar{\lambda} \leq \lambda^*$, if g is non-negative. Now, fixed $\lambda \in]0, \bar{\lambda}[$ and arguing as in the proof of Theorem 3.4, there are $c \in]0, \delta[$ and d such that $k\sqrt[p]{(1 + d^p)|\Omega|} < c$, and $\frac{|\Omega|}{p\|\alpha\|_1} \frac{(1+d^p)}{\int_0^d g(\xi)\,d\xi} < \lambda < \frac{1}{pk^p\|\alpha\|_1} \frac{c^p}{\int_0^c g(\xi)\,d\xi}$. Hence, Theorem 3.3 ensures that, for each $\lambda \in]0, \bar{\lambda}[$, the problem (9) admits at least one positive weak solution $\bar{u}_\lambda$ such that $\bar{u}_\lambda(x) < \delta$ for all $x \in \Omega$. We also observe that in Theorem 3.4, condition (10) can be substituted by $\limsup_{\xi\to 0^+} \frac{G(\xi)}{\xi^p}$.

Finally, we also give an application of Theorem 2.2 which we will use in next section to obtain multiple solutions.

Theorem 3.5. *Assume that there exist two constants $\bar{c}, \bar{d}$, with $0 < \bar{c} < k\sqrt[p]{(1 + \bar{d}^p)|\Omega|}$, such that*

$$\int_\Omega \max_{|\xi|\leq\bar{c}} F(x, \xi)\,dx < \int_\Omega F(x, \bar{d})\,dx, \tag{11}$$

and

$$\limsup_{|\xi|\to+\infty} \frac{F(x, \xi)}{|\xi|^p} \leq 0 \quad \textit{uniformly in } X. \tag{12}$$

Then, for each $\lambda > \tilde{\lambda}$, where

$$\tilde{\lambda} = \frac{k^p(1 + \bar{d}^p)|\Omega| - \bar{c}^p}{pk^p\big(\int_\Omega F(x, \bar{d})\,dx - \int_\Omega \max_{|\xi|\leq\bar{c}} F(x, \xi)\,dx\big)},$$

the problem (N_λ^f) admits at least one non-trivial weak solution $\tilde{u}$ such that $\|\tilde{u}\| > \frac{\bar{c}}{k}$.

Proof. We take Φ and Ψ as in the proof of Theorem 3.1. They satisfy all regularity assumptions requested in Theorem 2.2. Moreover, by standard computations, condition (12) implies that $\Phi - \lambda\Psi$, $\lambda > 0$, is coercive. So, our aim is to verify condition (5) of Theorem 2.2. To this end, put

$$r = \frac{1}{p}\Big(\frac{\bar{c}}{k}\Big)^p \quad \text{and} \quad u_0(x) = \bar{d} \quad \text{for all } x \in \Omega.$$

Arguing as in the proof of Theorem 3.1 we obtain that

$$\rho(r) \geq \frac{pk^p\big(\int_\Omega F(x,\bar{d})\,dx - \int_\Omega \max_{|\xi|\leq \bar{c}} F(x,\xi)\,dx\big)}{k^p(1+\bar{d}^p)|\Omega| - \bar{c}^p}.$$

So, from our assumption it follows that $\rho(r) > 0$. Hence, from Theorem 2.2 for each $\lambda > \tilde{\lambda}$, the functional $\Phi - \lambda\Psi$ admits at least one local minimum $\tilde{u}$ such that $\|\tilde{u}\| > \frac{\bar{c}}{k}$ and our conclusion is achieved. □

4. Some consequences

The main aim of this section is to present multiplicity results. First, as consequence of Theorem 3.4, we have the following multiplicity result.

Theorem 4.1. *Assume that g is non-negative and*
(j) $\limsup_{\xi\to 0^+} \frac{G(\xi)}{\xi^p} = +\infty$,
(jj) *there are constants $\mu > p$ and $r > 0$ such that, for all $|\xi| \geq r$, one has*

$$0 < \mu G(\xi) < \xi g(\xi). \tag{13}$$

Then, for each $\lambda \in]0, \lambda^*[$, where $\lambda^* = \frac{1}{pk^p\|\alpha\|_1} \sup_{c>0} \frac{c^p}{G(c)}$ the problem (9) admits at least two non-negative weak solutions.

Proof. Fix $\lambda \in]0, \lambda^*[$. Owing to (j), Theorem 3.4 (see also Remark 3.1) ensures that the problem (9) admits at least one positive weak solution $\bar{u}$ which is local minimum of the functional $\Phi - \lambda\Psi$ as defined before. We can assume that $\bar{u}$ is a strict local minimum for $\Phi - \lambda\Psi$ in X. Therefore, there is $\rho > 0$ such that $\inf_{\|u-\bar{u}\|=\rho}(\Phi - \lambda\Psi)(u) > (\Phi - \lambda\Psi)(\bar{u})$. By standard computations from (jj) one has that $\Phi - \lambda\Psi$ is unbounded from below. So, there is u_2 such that $(\Phi - \lambda\Psi)(u_2) < (\Phi - \lambda\Psi)(\bar{u})$, for which $\Phi - \lambda\Psi$ satisfies the geometry of mountain pass. Again from (jj), by standard computations, $\Phi - \lambda\Psi$ satisfies the Palais-Smale condition. Hence, the classical theorem of Ambrosetti and Rabinowitz ensures a critical point u^* of $\Phi - \lambda\Psi$ such that $(\Phi - \lambda\Psi)(u^*) > (\Phi - \lambda\Psi)(\bar{u})$. So, $\bar{u}$ and u^* are two distinct weak solutions of (9) and the proof is complete. □

Next, as a consequence of Theorems 3.2, 3.5 the following theorem of the existence of three solutions is obtained and its consequence for the nonlinearity with separable variables is presented.

Theorem 4.2. *Assume that (12) holds. Moreover, assume that there exist four positive constants $c, d, \bar{c}, \bar{d}$ with*

$$k\sqrt[p]{(1+d^p)|\Omega|} < c \leq \bar{c} < k\sqrt[p]{(1+\bar{d}^p)|\Omega|},$$

such that (7), (11) and

$$\frac{\int_\Omega \max_{|\xi|\leq c} F(x,\xi)\,dx}{c^p} < \frac{\int_\Omega F(x,\bar{d})\,dx - \int_\Omega \max_{|\xi|\leq \bar{c}} F(x,\xi)\,dx}{k^p(1+\bar{d}^p)|\Omega| - \bar{c}^p} \tag{14}$$

are satisfied.

Then, for each $\lambda \in \Lambda = \Big]\max\Big\{\tilde{\lambda}, \frac{(1+d^p)|\Omega|}{p\int_\Omega F(x,d)dx}\Big\}, \frac{1}{pk^p}\frac{c^p}{\int_\Omega \max_{|\xi|\leq c} F(x,\xi)\,dx}\Big[$, the problem (N_λ^f) admits at least three weak solutions.

Proof. First, we observe that $\Lambda \neq \emptyset$ owing to (14). Next, fix $\lambda \in \Lambda$. Theorem 3.2 ensures a non-trivial weak solution $\bar{u}$ such that $\|\bar{u}\| < \frac{c}{k}$ which is a local minimum for the associated functional $\Phi - \lambda\Psi$, as well as Theorem 3.5 guarantees a non-trivial weak solution $\tilde{u}$ such that $\|\tilde{u}\| > \frac{\bar{c}}{k}$ which is a local minimum for $\Phi - \lambda\Psi$. Hence, the mountain pass theorem as given by Pucci and Serrin (see [9]) ensures the conclusion. □

Theorem 4.3. *Assume that g is a non-negative function such that*

$$\limsup_{\xi \to 0^+} \frac{G(\xi)}{\xi^p} = +\infty, \tag{15}$$

$$\limsup_{\xi \to +\infty} \frac{G(\xi)}{\xi^p} = 0. \tag{16}$$

Further, assume that there exist two positive constants $\bar{c}, \bar{d}$, with $\bar{c} < k\sqrt[p]{(1+\bar{d}^p)|\Omega|}$, such that

$$\frac{G(\bar{c})}{\bar{c}^p} < \frac{G(\bar{d})}{k^p(1+\bar{d}^p)|\Omega|}. \tag{17}$$

Then, for each $\lambda \in \left]\frac{1}{p\|\alpha\|_1}\frac{(1+\bar{d}^p)|\Omega|}{G(\bar{d})}, \frac{1}{pk^p\|\alpha\|_1}\frac{\bar{c}^p}{G(\bar{c})}\right[$, the problem (9) admits at least three weak non-negative solutions.

Proof. Clearly, (16) implies (12). Moreover, by choosing d small enough and $c = \bar{c}$, simple computations show that (15) implies (7). Finally, from (17) we get (11) and, arguing as in the proof of Theorem 3.2, also (14). Hence, Theorem 4.2 ensures the conclusion. □

Remark 4.1. If $g(0) \neq 0$ Theorem 4.1 ensures two positive weak solutions while Theorem 4.3 ensures three positive weak solutions (see proof of Theorem 3.3).

Finally, we present an example of a problem that admits two positive solutions owing to Theorem 4.1.

Example 4.1. owing to Theorem 4.1, for each $\lambda \in \left]0, \frac{1}{pk^p|\Omega|}\frac{2p+1}{2p+2}\right[$, the problem

$$\begin{cases} -\mathrm{div}\left(\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = \lambda(u^{2p}+1) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least two positive weak solutions. In fact,

$$\lim_{u\to 0^+} \frac{g(u)}{u^{p-1}} = \lim_{u \to 0^+} \frac{u^{2p}+1}{u^{p-1}} = +\infty,$$

and (13) is satisfied as a simple computation shows. Moreover, we have

$$\lambda^* = \frac{1}{pk^p\|\alpha\|_1} \sup_{c>0} \frac{c^p}{G(c)} = \frac{1}{pk^p|\Omega|} \sup_{c\in]0,+\infty[} \frac{c^p}{\frac{c^{2p+1}}{2p+1}+c} \geq \frac{1}{pk^p|\Omega|}\frac{2p+1}{2p+2}.$$

References

[1] G.A. Afrouzi, S. Shokooh, N.T. Chung, Existence and multiplicity of weak solutions for some $p(x)$-Laplacian-like problems via variational methods, *J. Applied Math. Info.* **35** (2017), 11–24.
[2] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.* **75** (2012), 2992–3007.
[3] G. Bonanno, A. Sciammetta, Existence and multiplicity results to Neumann problems for elliptic equations involving the p-Laplacian, *J. Math. Anal. Appl.* **390** (2012), 59–67.
[4] Z.C. Chen, T. Luo, The eigenvalue problem for p-Laplacian-Like equations, *Acta Math. Sinica* **64** (2003), 631–638.
[5] G. D'Aguì, A. Sciammetta, Infinitely many solutions to elliptic problems with variable exponent and nonhomogeneous Neumann conditions, *Nonlinear Anal.* **75** (2012), 5612–5619.
[6] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin, 1983.
[7] Y. Li, L. Li, Existence and Multiplicity of Solutions for $p(x)$-Laplacian Equations in $\mathbb{R}^N$, *Bull. Malays. Math. Sci. Soc.* **40** (2017), no. 4, 1455-1463.
[8] Z.X. Li, Y.T. Shen, Existence of nontrivial solutions for p-Laplacian-like equations, *Acta Mathematicae Applicatae Sinica, English Series* **27** (2011), 393-406.
[9] P. Pucci, J. Serrin, A mountain pass theorem, *J. Differential Equations* **63** (1985), 142–149.
[10] P. Pucci, J. Serrin, The strong maximum principle revisited, *J. Differential Equations* **196** (2004), 1–66.
[11] M. Manuela Rodrigues, Multiplicity of solutions on a nonlinear eigenvalue problem for $p(x)$-Laplacian-like operators, *Mediterr. J. Math.* **9** (2012), 211–223.
[12] S. Shokooh, A. Neirameh, Existence results of infinitely many weak solutions for $p(x)$-Laplacian-like operators, *U.P.B. Sci. Bull., Series A,* **78** (2016), 95–104.
[13] Q.M. Zhao, On the superlinear problems involving $p(x)$-Laplacian-like operators without AR-condition, *Nonlinear Anal. RWA* **21** (2015), 161–169.

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