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Some new inequalities of Jensen's type for operator s-convex functions

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Abstract. In this paper some new inequalities, associated with discrete and integral Jensen's inequalities, for operator s-convex functions are given. It is also shown that these inequalities, under some special conditions, are refinements of classic Jensen's inequalities for s-convex functions.

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1. Introduction

Let f be a continuous convex function defined on an interval $I \subseteq \mathbb{R}$. In 1905, Jensen showed that for any finite subset $\{x_j \mid j \in J\}$ of I, and for any family of non-negative scalars $\{\lambda_j \mid j \in J\}$ in $\mathbb{R}$, with $\sum_{j\in J} \lambda_j = 1$,

$$f\Big(\sum_{j\in J} \lambda_j x_j\Big) \le \sum_{j\in J} \lambda_j f(x_j) \tag{1}$$

(see [15, 16]). One can also pass from the discrete case to a more general case and obtain an integral formula extending (1). Let (X, Σ, μ) be a probability measure space; i.e. Σ be a σ-algebra of subsets of a non-empty set X and $\mu : \Sigma \to [0, 1]$ be a probability measure. Then for an integrable function $\varphi : X \to \mathbb{R}$, with $\varphi(X) \subseteq I$, we have

$$f\Big(\int_X \varphi d\mu\Big) \le \int_X (f \circ \varphi)\, d\mu \tag{2}$$

The above inequalities, widely known as Jensen's inequalities, can be generalized to more general contexts. (see [1, 4, 6, 7, 9, 12, 13, 17, 18]).

If f is a real continuous function on some interval I in $\mathbb{R}$ and A is a bounded self-adjoint operator on a Hilbert space with spectrum in I, we can use spectral theory to define a bounded self-adjoint operator

$$f(A) = \int_{-\infty}^{+\infty} f(\lambda) dE_A(\lambda), \tag{3}$$

where E_A is the spectral measure of A. A continuous function $f : I \to \mathbb{R}$ is said to be operator convex if

$$f\big(\alpha A + (1 - \alpha)B\big) \le \alpha f(A) + (1 - \alpha) f(B) \tag{4}$$

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for any $\alpha \in [0,1]$ and every pair of self-adjoint operators A and B on an infinite-dimensional Hilbert space $\mathcal{H}$ with spectra in I.

In [13], Hansen and Pedersen proved that if f is a continuous, operator convex function defined on an interval I, then for each natural number n, Jensen' s operator inequality

$$f\Big(\sum_{i=1}^{n} a_i^* x_i a_i\Big) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i \tag{5}$$

holds for every n-tuple $(x_1, ..., x_n)$ of bounded, self-adjoint operators on an arbitrary Hilbert space $\mathcal{H}$ with spectra contained in I and every n-tuple $(a_1, ..., a_n)$ of operators on $\mathcal{H}$ with $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{H}}$.

In [9], a Bochner integral formation of Jensen's inequality is presented for self-adjoint matrix-valued functions and measures. Farenick and Zhou proved that for a matrix-valued probability measure space (X, Σ, μ), if I is an open interval such that $[\alpha, \beta] \subseteq I$ and if φ is a measurable map from X to the real vector space of all self-adjoint operators on $\mathbb{C}^n$ for which the spectra $\sigma(\varphi(x)) \subseteq [\alpha, \beta]$ for every $x \in X$, then

$$f\Big(\int_X \varphi d\mu\Big) \leq \int_X (f \circ \varphi) d\mu \tag{6}$$

for every operator convex function $f : I \to \mathbb{R}$.

The aim of this paper is to generalize Jensen's inequality to operator s-convex functions.

2. Preliminaries

First, we review the operator order in $B(\mathcal{H})$ and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators $A, B \in B(\mathcal{H})$ we write $A \leq B$(or $B \geq A$) if $\langle Ah, h\rangle \leq \langle Bh, h\rangle$ for every vector $h \in \mathcal{H}$, we call it the operator order. Now, let A be a bounded selfadjoint linear operator on a complex Hilbert space $(H; \langle ., .\rangle)$ and $C(\sigma(A))$ the C^* -algebra of all continuous complex-valued functions on the spectrum of A. The Gelfand map establishes a $*$-isometrically isomorphism ψ between $C(\sigma(A))$ and the C^*-algebra $C^*(A)$ generated by A and the identity operator $1_{\mathcal{H}}$ on $\mathcal{H}$ as follows (see for instance [10, p.3]): For $f, g \in C(\sigma(A))$ and $\alpha, \beta \in \mathbb{C}$

(i) $\psi(\alpha f + \beta g) = \alpha\psi(f) + \beta\psi(g)$;

(ii) $\psi(fg) = \psi(f)\psi(g)$ and $\psi(f^*) = \psi(f)^*$;

(iii) $\|\psi(f)\| = \|f\| := \sup_{t\in\sigma(A)} |f(t)|$;

(iv) $\psi(f_0) = 1_{\mathcal{H}}$ and $\psi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \sigma(A)$.

If f is a continuous complex-valued functions on $\sigma(A)$, the element $\psi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator A.

If A is a bounded selfadjoint operator and f is a real-valued continuous function on $\sigma(A)$, then $f(t) \geq 0$ for any $t \in \sigma(A)$ implies that $f(A) \geq 0$, i.e., $f(A)$ is a positive operator on $\mathcal{H}$. Moreover, if both f and g are real-valued functions on $\sigma(A)$ such that $f(t) \leq g(t)$ for any $t \in \sigma(A)$, then $f(A) \leq g(A)$ in the operator order in $B(\mathcal{H})$.

A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$f((1-\lambda)A+\lambda B) \leq (\geq)(1-\lambda)f(A)+\lambda f(B) \tag{7}$$

in the operator order in $B(\mathcal{H})$, for all $\lambda \in [0,1]$ and for every bounded self-adjoint operators A and B in $B(\mathcal{H})$ whose spectra are contained in I. When considering n-by-n symmetric or hermitian matrices, an operator convex function is called a matrix convex function of order n.

J. Bendat and S. Sherman in [3] proved the following lemma:

Lemma 2.1. *A function is an operator convex function in (a,b) if and only if it is a matrix convex function in (a,b) for all finite orders n.*

As examples of such functions, we give the following examples, another proof of them and further examples can be found in [10].

Example 2.1. (i) The convex function $f(t)=\alpha t^2+\beta t+\gamma$ $(\alpha \geq 0, \beta, \gamma \in \mathbb{R})$ is operator convex on every interval. To see it, for all self-adjoint operators A and B:

$$\begin{aligned}\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right) &= \alpha\left(\frac{A^2+B^2}{2}-\left(\frac{A+B}{2}\right)^2\right)\\ &+\beta\left(\frac{A+B}{2}-\frac{A+B}{2}\right)+(\gamma-\gamma)\\ &=\frac{\alpha}{4}(A^2+B^2-AB-BA)=\frac{\alpha}{4}(A-B)^2 \geq 0.\end{aligned}$$

(ii) The convex function $f(t)=t^3$ on $[0,\infty)$ is not operator convex. In fact, if we put

$$A=\left[\begin{array}{cc}3 & -1\\ -1 & 1\end{array}\right] \ \& \ B=\left[\begin{array}{cc}1 & 0\\ 0 & 0\end{array}\right],$$

then

$$\frac{A^3+B^3}{2}-\left(\frac{A+B}{2}\right)^3=\frac{1}{8}\left[\begin{array}{cc}67 & -34\\ -34 & 17\end{array}\right] \ngeq 0.$$

In this section, we also mention some useful lemmas, which is frequently applied in the next sections (see [10]).

Lemma 2.2. *Let $\mathcal{H}$ be a Hilbert space. If $A \in B_h(\mathcal{H})$ is selfadjoint and U is unitary, i.e. $U^*U=UU^*=1_{\mathcal{H}}$, then $f(U^*AU)=U^*f(A)U$ for every $f \in C(\sigma(A))$.*

Lemma 2.3. *Let $\mathcal{H}$ be a Hilbert space, I be an interval in $\mathbb{R}$ and $A_j \in B_h(\mathcal{H})$, $(j=1,2,...,n)$, be selfadjoint with $\sigma(A_j) \subseteq I$. If f is a real value function on I, then*

$$f\begin{pmatrix}A_1 & & & 0\\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n\end{pmatrix}=\begin{pmatrix}f(A_1) & & & 0\\ & f(A_2) & & \\ & & \ddots & \\ 0 & & & f(A_n)\end{pmatrix}$$

3. Some discrete inequalities of Jensen's type for operator s-convex functions

In this section we establish an operator Jensen's inequality and some new inequalities of Jensen's type, for operator s-convex functions. Then, we compere these inequalities. First, we recall some definitions and properties of s-convex and operator s-convex functions.

Definition 3.1. i) A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

holds for all $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. The class of s-convex functions in the first sense is usually denoted with K_s^1.

ii) A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$. The class of s-convex functions in the second sense is usually denoted with K_s^2. It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

It is proved in [14] that if $s \in (0, 1)$ then $f \in K_s^2$ implies $f([0, \infty)) \in [0, \infty)$, i.e., they proved that all functions from $K_s^2, s \in (0, 1)$, are nonnegative.

In [5], Chen proved that $f : [0, \infty) \to \mathbb{R}$ is an s-convex function in the second sense if and only if for every $x_1, ...x_n \geq 0$ and no-negative real number $\alpha_1, ...\alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$, we have

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i^s f(x_i). \tag{8}$$

In the special case, if $\alpha_i = \frac{1}{n}$ for every $i \in \{1, 2, ..., n\}$, we have

$$f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \leq \frac{1}{n^s}\sum_{i=1}^n f(x_i), \tag{9}$$

for every s-convex function f. Note that (8) is Jensen's inequality for s-convex function f.

We denoted by $B(\mathcal{H})^+$ the set of all positive operators in $B(\mathcal{H})$ and

$$C(\mathcal{H}) := \{A \in B(\mathcal{H})^+ : AB + BA \geq 0, \text{for all} B \in B(\mathcal{H})^+\}$$

It is obvious that $C(\mathcal{H})$ is a closed convex cone in $B(\mathcal{H})$.

Definition 3.2. Let I be an interval in $[0, \infty)$ and K be a convex subset of $B(\mathcal{H})^+$. A continuous function $f : I \to \mathbb{R}$ is said to be operator s-convex on I for operators in K if

$$f((1-\lambda)A + \lambda B) \leq (1-\lambda)^s f(A) + \lambda^s f(B) \tag{10}$$

in the operator order in $B(\mathcal{H})$, for all $\lambda \in [0, 1]$ and for every positive operators A and B in K whose spectra are contained in I and for some fixed $s \in (0, 1]$. For $K = B(\mathcal{H})^+$ we say f is operator s-convex on I. It is easy to show that every operator s-convex function on I is s-convex in the second sense on I.

When considering n-by-n real symmetric or hermitian matrices with non-negative eigenvalues, an operator s-convex function is called a matrix s-convex function of order n.

The following lemma will turn out to be useful in the proof of results (see [2, Lemma 4.1]).

Lemma 3.1. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of self-adjoint operators in $B(\mathcal{H})$ converging to some $A \in B_h(\mathcal{H})$ (in the norm topology of $B(\mathcal{H})$). Suppose I is an open interval of $\mathbb{R}$ which encompasses $\sigma(A)$ and $\sigma(A_n)$ for every $n \in \mathbb{N}$. If $f : I \to \mathbb{R}$ is a continuous function, then $f(A_n) \to f(A)$.*

Lemma 3.2. *A function is an operator s-convex function on I if and only if it is a matrix s-convex function on I for all finite orders n.*

Proof. It is obvious that an operator s-convex function on I is a matrix s-convex function on I for all finite orders n. To see the converse, without loss of generality, assume that $f(x)$ defined in interval I containing the origin. Let A be a bounded positive operator in a Hilbert space $\mathcal{H}$, with spectrum in I, represented in a special coordinate system $(x_1, x_2, ..., x_n, ...)$ of $\mathcal{H}$ by the matrix $(a_{ik})_{i,k=1}^{\infty}$. Define A_n by the matrix

$$A_n = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

Now the strong limit, $\lim_{n\to\infty} A_n = A$, i.e. $\|A_n x - Ax\| \to 0$ for each x in $\mathcal{H}$. Since f is a matrix s-convex function on I for all finite order n, f is continuous on I. So, by Lemma 3.1, we have $\lim_{n\to\infty} f(A_n) = f(A)$. Let A and B be two bounded positive operators with spectrum in I. Then $f((1-\lambda)A_n + \lambda B_n) \le (1-\lambda)^s f(A_n) + \lambda^s f(B_n)$, since s-convexity of the nth order is assumed. Now, letting $n \to \infty$, we obtain

$$\begin{aligned} f((1-\lambda)A + \lambda B) &\leftarrow f((1-\lambda)A_n + \lambda B_n) \\ &\le (1-\lambda)^s f(A_n) + \lambda^s f(B_n) \to (1-\lambda)^s f(A) + \lambda^s f(B) \end{aligned}$$

or $f((1-\lambda)A + \lambda B) \le (1-\lambda)^s f(A) + \lambda^s f(B)$, since $f(x)$ converges uniformly in the closed convex hull of the combined spectra of A and B contained in I. It proves that $f(x)$ is an operator s-convex function. $\square$

Lemma 3.3. *If f is operator s-convex on $[0,\infty)$ for operators in K, then $f(A)$ is positive for every $A \in K$.*

One can see the proof of this lemma in [11].
In [19], Moslehian and Najafi proved the following theorem for positive operators as follows:

Theorem 3.4. *Let $A, B \in B(\mathcal{H})^+$. Then $AB + BA$ is positive if and only if $f(A + B) \le f(A) + f(B)$ for all non-negative operator monotone functions f on $[0,\infty)$.*

As an example of operator s-convex function, we give the following example.

Example 3.1. Since for every positive operators $A, B \in C(\mathcal{H})$, $AB + BA \ge 0$, utilizing Theorem 3.4 we get

$$((1-t)A + tB)^s \le (1-t)^s A^s + t^s B^s.$$

Therefore the continuous function $f(t) = t^s$ $(0 < s \leq 1)$ is operator s-convex on $[0, \infty)$ for operators in $C(\mathcal{H})$.

Proposition 3.5. ***(Jensen's inequality)*** *Let $\mathcal{H}$ be a Hilbert space, I be an interval in $[0, \infty)$ and f be a real valued continuous function on I. The following statements are equivalent:*

(i) f is an operator s-convex function on I,

(ii) For every $A_1, ..., A_n \in B(\mathcal{H})^+$ with spectra contained in I, and non-negative real number $\lambda_1, ..., \lambda_n$ with $\sum_{j=1}^n \lambda_j = 1$, we have

$$f\left(\sum_{j=1}^n \lambda_j A_j\right) \leq \sum_{j=1}^n \lambda_j^s f(A_j). \tag{11}$$

Proof. Suppose that f is an operator s-convex function on I. We will prove (ii) by induction on $n \in \mathbb{N}, n \geq 2$. For $n = 2$, the inequality is obvious by definition. Suppose that the above inequality is valid for n. For natural number $n+1$, let $A_1, ..., A_{n+1} \in B(\mathcal{H})^+$ with spectra contained in I, and $\lambda_1, ..., \lambda_{n+1} \in \mathbb{R}^+$ with $\sum_{j=1}^{n+1} \lambda_j = 1$. Then

$$\begin{aligned} f\left(\sum_{j=1}^{n+1} \lambda_j A_j\right) &= f\left((1-\lambda_{n+1}) \sum_{j=1}^n \frac{\lambda_j}{1-\lambda_{n+1}} A_j + \lambda_{n+1} A_{n+1}\right) \\ &\leq (1-\lambda_{n+1})^s f\left(\sum_{j=1}^n \frac{\lambda_j}{1-\lambda_{n+1}} A_j\right) + \lambda_{n+1}^s f(A_{n+1}) \\ &\leq (1-\lambda_{n+1})^s \sum_{j=1}^n \left(\frac{\lambda_j}{1-\lambda_{n+1}}\right)^s f(A_j) + \lambda_{n+1}^s f(A_{n+1}) \\ &= \sum_{j=1}^{n+1} \lambda_j^s f(A_j), \end{aligned}$$

then, (ii) is proved.
It is obvious that (ii) implies (i). □

Theorem 3.6. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert space, I be an interval in $[0, \infty)$ and f be a real valued continuous function on I. Let A and A_j be positive operators on $\mathcal{H}$ with spectra contained in I $(j = 1, 2, ..., n)$. If f is an operator s-convex function on I then the following conditions hold:*

(i) For every $A \in B(\mathcal{H})^+$ and isometry $C \in B(\mathcal{K}, \mathcal{H})$, i.e., $C^*C = 1_{\mathcal{K}}$, we have

$$f(C^*AC) \leq 2^{1-s} C^* f(A) C. \tag{12}$$

(ii) For every $A_j \in B(\mathcal{H})^+$ and $C_j \in B(\mathcal{K}, \mathcal{H})$ with $\sum_{j=1}^n C_j^* C_j = 1_{\mathcal{K}}$ $(j = 1, ..., n)$, we have

$$f(\sum_{j=1}^n C_j^* A_j C_j) \leq 2^{1-s} \sum_{j=1}^n C_j^* f(A_j) C_j. \tag{13}$$

Proof. (i) Let $X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{K})^+$ for some positive operator $B \in B(\mathcal{K})^+$ with spectra contained in I and

$$U = \begin{pmatrix} C & D \\ 0 & -C^* \end{pmatrix}, V = \begin{pmatrix} C & -D \\ 0 & C^* \end{pmatrix} \in B(\mathcal{K} \oplus \mathcal{H}, \mathcal{H} \oplus \mathcal{K}),$$

where $D = \sqrt{1_{\mathcal{H}} - CC^*}$. Since $C^*D = \sqrt{1_{\mathcal{K}} - C^*C}C^* = 0 \in B(\mathcal{H}, \mathcal{K})$ and $DC = C\sqrt{1_{\mathcal{K}} - C^*C} = 0 \in B(\mathcal{K}, \mathcal{H})$, it follows that both U and V are unitary operators on $\mathcal{H} \oplus \mathcal{K}$. Then

$$U^*XU = \begin{pmatrix} C^*AC & C^*AD \\ DAC & DAD + CBC^* \end{pmatrix}$$

and

$$V^*XV = \begin{pmatrix} C^*AC & -C^*AD \\ -DAC & DAD + CBC^* \end{pmatrix}.$$

So, we have

$$\begin{pmatrix} C^*AC & 0 \\ 0 & D^*AD + CBC^* \end{pmatrix} = \frac{U^*XU + V^*XV}{2}.$$

Hence, it follows from the operator s-convexity of f and Lemmas 2.2, 2.3 that

$$\begin{aligned} \begin{pmatrix} f(C^*AC) & 0 \\ 0 & f(D^*AD + CBC^*) \end{pmatrix} &= f\begin{pmatrix} C^*AC & 0 \\ 0 & D^*AD + CBC^* \end{pmatrix} \\ &= f\left(\frac{U^*XU + V^*XV}{2}\right) \\ &\leq \frac{f(U^*XU) + f(V^*XV)}{2^s} \\ &= 2^{1-s}\frac{U^*f(X)U + V^*f(X)V}{2} \\ &= 2^{1-s}\begin{pmatrix} C^*f(A)C & 0 \\ 0 & D^*f(A)D + Cf(B)C^* \end{pmatrix}. \end{aligned}$$

Thus we have $f(C^*AC) \leq 2^{1-s}C^*f(A)C$ by seeing the (1,1)-components.
(ii) Let

$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix}, C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}.$$

Then $A \in B(\mathcal{H} \oplus ... \oplus \mathcal{H})$ is a positive operator with spectra contained in I and $C \in B(\mathcal{K}, \mathcal{H} \oplus ... \oplus \mathcal{H})$ is an isometry, i.e., $C^*C = 1_{\mathcal{K}}$. Hence it follows from (i) that

$$f(\sum_{j=1}^{n} C_j^* A_j C_j) = f(C^*AC) \leq 2^{1-s}C^*f(A)C = 2^{1-s}\sum_{j=1}^{n} C_j^* f(A_j)C_j$$

□

Corollary 3.7. *Let $\mathcal{H}$ be a Hilbert space, I be an interval in $[0,\infty)$ that contains 0 and f be a real valued continuous function on I. If f is an operator s-convex function on I with $f(0)=0$ then for each natural number n, the inequality*

$$f(\sum_{j=1}^{n} C_j^* A_j C_j) \leq 2^{1-s} \sum_{j=1}^{n} C_j^* f(A_j) C_j$$

holds for every n-tuple $(A_1,...,A_n)$ of bounded positive operators on $\mathcal{H}$ with spectra contained in I and every n-tuple $(C_1,...,C_n)$ of operators on $\mathcal{H}$ with $\sum_{j=1}^{n} C_j^* C_j \leq 1_{\mathcal{H}}$.

Proof. Define $C_{n+1} := (1_{\mathcal{H}} - \sum_{j=1}^{n} C_j^* C_j)^{\frac{1}{2}}$. Then $C_{n+1} \in B(\mathcal{H})$ and $\sum_{j=1}^{n+1} C_j^* C_j = 1_{\mathcal{H}}$. Put $A_{n+1}=0$, by Theorem 3.6, we obtain

$$\begin{aligned} f\left(\sum_{j=1}^{n} C_j^* A_j C\right) &= f\left(\sum_{j=1}^{n+1} C_j^* A_j C_j\right) \\ &\leq 2^{1-s} \sum_{j=1}^{n+1} C_j^* f(A_j) C_j = 2^{1-s} \sum_{j=1}^{n} C_j^* f(A_j) C_j. \end{aligned}$$

□

Corollary 3.8. *Let $\mathcal{H}$ be a Hilbert space and f an operator s-convex function on $[0,\infty)$ with $f(0)=0$ and $A,B \in B(\mathcal{H})$ with spectra contained in $(0,\infty)$. If $A \leq B$, then*

$$A^{-1} f(A) \leq 2^{1-s} B^{-1} f(B)$$

Proof. We define $C := B^{-1/2} A^{1/2}$. Then $CC^* = B^{-1/2} A B^{-1/2} \leq 1_{\mathcal{H}}$, so $C^* C \leq 1_{\mathcal{H}}$. Now by Corollary 3.7 we have

$$f(A) = f(C^* B C) \leq 2^{1-s} C^* f(B) C = 2^{1-s} A^{1/2} B^{-1/2} f(B) B^{-1/2} A^{1/2}.$$

Using the commutativity of A and $f(A)$ and the commutativity of B and $f(B)$, we have the desired result. □

Remark 3.1. Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be an operator s-convex function. Let $A_j \in B(\mathcal{H})^+$ be positive operators on $\mathcal{H}$ with spectra contained in I, and $\lambda_j \in \mathbb{R}^+$ $(j = 1,2,...)$ with $\sum_{j=1}^{n} \lambda_j = 1$. Let

$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix}, C = \begin{pmatrix} \sqrt{\lambda_1} \\ \sqrt{\lambda_2} \\ \vdots \\ \sqrt{\lambda_n} \end{pmatrix}.$$

It is easy to show that $A \in B(\mathcal{H} \oplus ... \oplus \mathcal{H})$ is a positive operator with spectra contained in I and C is an isometry in $B(\mathcal{H}, \mathcal{H} \oplus ... \oplus \mathcal{H})$. Using Theorem 3.6, we have

$$f(\sum_{j=1}^{n} \lambda_j A_j) \leq 2^{1-s} \sum_{j=1}^{n} \lambda_j f(A_j). \tag{14}$$

We define

$$\gamma_f(\lambda, A) := \sum_{j=1}^{n} \lambda_j^s f(A_j) - 2^{1-s} \sum_{j=1}^{n} \lambda_j f(A_j) \tag{15}$$
$$= \sum_{j=1}^{n} \lambda_j \left(\lambda_j^{s-1} - 2^{1-s}\right) f(A_j).$$

where $\lambda = (\lambda_1, ..., \lambda_n)$ and $A = (A_1, ..., A_n)$. Note that if $\lambda_j \leq \frac{1}{2}$ for $j = 1, 2, ..., n$, then $\gamma_f(\lambda, A) \geq 0$ (since $f(A_j), \lambda_j \geq 0$ for $j = 1, 2, ..., n$). It shows that if $\lambda_j \leq \frac{1}{2}$ for $j = 1, 2, ..., n$, then the inequality in (14) is a refinement of (11). But it is not true generally. For example, let $s \in (0, 1]$, $f(t) = t^s$, $A_1 = 0$ and $\lambda > \frac{1}{2}$. By Example 3.1, we know that f is an operator s-convex function on $[0, \infty)$. Using positivity of f, we have

$$(1-\lambda)^s f(A_1) + \lambda^s f(A_2) = \lambda^s f(A_2) < 2^{1-s} \lambda f(A_2) = 2^{1-s} \left((1-\lambda) f(A_1) + \lambda f(A_2)\right),$$

where A_2 is an arbitrary operator in $B(\mathcal{H})^+$. So, in this case, (14) is not refinement of (11).

4. Davis-Choi-Jensen's type inequality for operator s-convex functions

In this section, we introduce an inequality of Davis-Choi-Jensen's type and making use of it, we give another version of Jensen's inequality for operator s-convex functions. Now, we recall the definition of normalized positive linear maps which will be used throughout this section.

Definition 4.1. A linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is called positive if $A \in B(\mathcal{H})^+$ implies $\Phi(A) \in B(\mathcal{K})^+$.
A linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is called normalized if $\Phi(1_{\mathcal{H}}) = 1_{\mathcal{K}}$.
We denote $P\left[B(\mathcal{H}), B(\mathcal{K})\right]$ as the set of all positive linear maps $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ and $P_N\left[B(\mathcal{H}), B(\mathcal{K})\right]$ as the set of all normalized positive linear maps $\Phi \in P\left[B(\mathcal{H}), B(\mathcal{K})\right]$.

Theorem 4.1. *Let Φ be a normalized positive linear map in $P_N[B(\mathcal{H}), B(\mathcal{K})]$, and f is an operator s-convex function on an interval I. Then*

$$f\left(\Phi(A)\right) \leq 2^{1-s} \Phi\left(f(A)\right) \tag{16}$$

for every positive operator A on $\mathcal{H}$ with spectra contained in I.

Proof. Note that a C^*-algebra $C^*(A)$ generated by A and $1_{\mathcal{H}}$ is a commutative C^*-algebra, since A is positive. By Stinespring decomposition theorem, Φ restricted to $C^*(A)$ admits a decomposition $\Phi(X) = C^* \phi(X) C$ for all $X \in C^*(A)$, where ϕ is a representation of $C^*(A) \subset B(\mathcal{H})$, and C is an isometry in $B(\mathcal{K}, \mathcal{H})$. Hence it follows from Theorem 3.6 that

$$f(\Phi(A)) = f(C^* \phi(A) C) \leq 2^{1-s} C^* f(\phi(A)) C = 2^{1-s} C^* \phi(f(A)) C = 2^{1-s} \Phi(f(A))$$

□

Using Proposition 3.5, Theorem 4.1 and Remark 3.1 , we obtain the following theorem.

Theorem 4.2. *Let I be an interval in $[0,\infty)$ and $A_1,...,A_n \in B(\mathcal{H})$ be positive operators on $\mathcal{H}$, with spectra contained in I. Let $\Phi_j \in P_N\left[B(\mathcal{H}),B(\mathcal{K})\right]$ be normalized positive maps and $\lambda_1,...,\lambda_n \in \mathbb{R}^+$ be any finite number of positive real numbers with $\sum_{j=1}^n \lambda_j = 1$. If $f,g \in C(I)$, $f \le g$ and f is operator s-convex on I, then the following inequalities hold:*

$$f\left(\sum_{j=1}^n \lambda_j \Phi_j(A_j)\right) \le 2^{1-s}\sum_{j=1}^n \lambda_j^s \Phi_j\left(g(A_j)\right), \tag{17}$$

$$f\left(\sum_{j=1}^n \lambda_j \Phi_j(A_j)\right) \le 4^{1-s}\sum_{j=1}^n \lambda_j \Phi_j\left(g(A_j)\right) \tag{18}$$

Proof. Using (11) and (16), we have

$$f\left(\sum_{j=1}^n \lambda_j \Phi_j(A_j)\right) \le \sum_{j=1}^n \lambda_j^s f\left(\Phi_j(A_j)\right) \le 2^{1-s}\sum_{j=1}^n \lambda_j^s \Phi_j\left(f(A_j)\right). \tag{19}$$

On the other hand, by the spectral theorem and the positivity of Φ_j, it follows that

$$\Phi_j\left(f(A_j)\right) \le \Phi_j\left(g(A_j)\right), j = 1,2,...n. \tag{20}$$

Now, by (19) and (20), we have (17).

Similarly, by (14) and (16), we have

$$f\left(\sum_{j=1}^n \lambda_j \Phi_j(A_j)\right) \le 2^{1-s}\sum_{j=1}^n \lambda_j f\left(\Phi_j(A_j)\right) \le 4^{1-s}\sum_{j=1}^n \lambda_j \Phi_j\left(f(A_j)\right). \tag{21}$$

Now, by (20) and (21), we have the desired result. □

We note that, there is an analogue of Remark 3.1, for the inequalities in (17) and (18). We omit the details.

Here we present converses of inequality in (18). Notice that we don't assume the operator s-convexity of f.
Let $f : [m,M] \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a continuous s-convex function on $[m,M]$. For convenience, we define:

$$\mu_f := \frac{f(M)-f(m)}{M-m}\ ,\ \nu_f := \frac{Mf(m)-mf(M)}{M-m} \tag{22}$$

We remark that a straight line $l(t) = \mu_f t + \nu_f$ is a line thought two points $(m,f(m))$ and $(M,f(M))$. By (13) we have

$$f(\lambda m + (1-\lambda)M) \le 2^{1-s}\left(\lambda f(m) + (1-\lambda)f(M)\right),$$

for every $\lambda \in [0,1]$. Therefore we have $f(t) \le 2^{1-s}l(t)$ for every $t \in [m,M]$, since for every $t \in [m,M]$ there exist a unique $\lambda \in [0,1]$ such that $t = \lambda m + (1-\lambda)M$, so $l(t) = l(\lambda m + (1-\lambda)M) = \lambda f(m) + (1-\lambda)f(M)$.

Theorem 4.3. *Let $A_j \in B(\mathcal{H})^+$ be positive operators with spectra contained in $[m,M]$, $\Phi_j \in P_N\left[B(\mathcal{H}),B(\mathcal{K})\right]$ normalized positive linear maps (j=1,...,n). Let $\lambda_1,...,\lambda_n \in \mathbb{R}^+$ be any finite number of positive real numbers such that $\sum_{j=1}^n \lambda_j = 1$. Let $f,g \in C([m,M])$ and $F(u,v)$ be a real valued continuous function defined on $U \times V$,* *where $f[m,M]\subset U, g[m,M]\subset V$. If $F(u,v)$ is operator monotone on a first variable u and f is s-convex on $[m,M]$, then*

$$F\left[\sum_{j=1}^{n}\lambda_j\Phi_j\left(f(A_j)\right), g\left(\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)\right)\right] \leq \left\{\max_{m\leq t\leq M} F\left[2^{1-s}(\mu_f t+\nu_f), g(t)\right]\right\}1_{\mathcal{K}} \quad (23)$$

Proof. From $f(t)\leq 2^{1-s}l(t) = 2^{1-s}(\mu_f t+\nu_f)$ for every $t\in[m,M]$, it follows that $f(A_j)\leq 2^{1-s}(\mu_f A_j+\nu_f 1_{\mathcal{H}})$ for all $j=1,...,n$. Since Φ_j is normalized positive linear map, we have

$$\Phi_j\left(f(A_j)\right)\leq \Phi_j\left(2^{1-s}\left(\mu_f A_j+\nu_f 1_{\mathcal{H}}\right)\right) = 2^{1-s}\left(\mu_f\Phi_j(A_j)+\nu_f 1_{\mathcal{K}}\right),$$

for $j=1,...,n$. Further, multiplying them with λ_j, summing of all $j=1,...,n$, and using $\sum_{j=1}^{n}\lambda_j=1$, we have

$$\sum_{j=1}^{n}\lambda_j\Phi_j\left(f(A_j)\right)\leq 2^{1-s}\left(\mu_f\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)+\nu_f 1_{\mathcal{K}}\right). \quad (24)$$

On the other hand, since $m1_{\mathcal{H}}\leq A_j\leq M1_{\mathcal{H}}$, we have $m1_{\mathcal{K}}\leq\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)\leq M1_{\mathcal{K}}$. i.e. $\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)$ is a positive operator on $\mathcal{K}$ with spectra contained in $[m,M]$. Using Operator monotonicity of $F(.,v)$, we have

$$\begin{aligned} F&\left[\sum_{j=1}^{n}\lambda_j\Phi_j\left(f(A_j)\right), g\left(\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)\right)\right] \\ &\leq F\left[2^{1-s}\left(\mu_f\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)+\nu_f 1_{\mathcal{K}}\right), g\left(\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)\right)\right] \\ &\leq \left\{\max_{m\leq t\leq M} F\left[2^{1-s}(\mu_f t+\nu_f), g(t)\right]\right\}1_{\mathcal{K}}. \quad (25)\end{aligned}$$

Therefore, desired result is proved. □

In the following theorems we consider complementary problems to Jensen's type inequality (18) in Theorem 4.2.

Theorem 4.4. *Let $A_j,\Phi_j,\lambda_j, j=1,...,n$ be as in Theorem 4.3 and $f,g\in C([m,M])$. If f is s-convex function on $[m,M]$, then for every real number $\alpha\in\mathbb{R}$, we have*

$$\sum_{j=1}^{n}\lambda_j\Phi_j\left(f(A_j)\right)\leq \alpha g\left(\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)\right)+\beta 1_{\mathcal{K}}, \quad (26)$$

where

$$\beta=\max_{m\leq t\leq M}\left\{2^{1-s}(\mu_f t+\nu_f)-\alpha g(t)\right\}.$$

Proof. It is easy to show that $m1_{\mathcal{K}} \leq \sum_{j=1}^{n} \lambda_j \Phi_j(A_j) \leq M1_{\mathcal{K}}$. Hence $g\left(\sum_{j=1}^{n} \lambda_j \Phi_j(A_j)\right)$ is well defined.

Let $F(u,v) = u - \alpha v$. Then F is operator monotone on u and hence it follows from Theorem 4.3 that

$$\sum_{j=1}^{n} \lambda_j \Phi_j\left(f(A_j)\right) - \alpha g\left(\sum_{j=1}^{n} \lambda_j \Phi_j(A_j)\right) \leq \max_{m \leq t \leq M} F\left(2^{1-s}(\mu_f t + \nu_f), \alpha g(t)\right) 1_{\mathcal{K}}$$
$$= \max_{m \leq t \leq M} \left\{2^{1-s}(\mu_f t + \nu_f) - \alpha g(t)\right\} 1_{\mathcal{K}},$$

which gives the desired inequality (27). □

If we let $g = f$ in Theorem 4.4, then we obtain the following corollary.

Corollary 4.5. *Let $A_j, \Phi_j, \lambda_j, j = 1, ..., n$ be as in Theorem 4.3 and $f \in C([m,M])$. If f is s-convex function on $[m,M]$, then for every real number $\alpha \in \mathbb{R}$ we have*

$$\sum_{j=1}^{n} \lambda_j \Phi_j\left(f(A_j)\right) \leq \alpha f\left(\sum_{j=1}^{n} \lambda_j \Phi_j(A_j)\right) + \beta 1_{\mathcal{K}}, \tag{27}$$

where

$$\beta = \max_{m \leq t \leq M} \left\{2^{1-s}(\mu_f t + \nu_f) - \alpha f(t)\right\}.$$

Theorem 4.6. *Let $A_j, \Phi_j, \lambda_j, j = 1, ..., n$ be as in Theorem 4.3 and $f, g \in C([m,M])$ and suppose that either of the following conditions holds:*
(i) $g(t) > 0$ for all $t \in [m,M]$,
(ii) $g(t) < 0$ for all $t \in [m,M]$.
If f is s-convex function on $[m,M]$, then

$$\sum_{j=1}^{n} \lambda_j \Phi_j\left(f(A_j)\right) \leq \alpha_0 g\left(\sum_{j=1}^{n} \lambda_j \Phi_j(A_j)\right), \tag{28}$$

where $\alpha_0 = \max_{m \leq t \leq M} \left\{\frac{2^{1-s}}{g(t)}(\mu_f t + \nu_f)\right\}$ in the case (i),
or $\alpha_0 = \min_{m \leq t \leq M} \left\{\frac{2^{1-s}}{g(t)}(\mu_f t + \nu_f)\right\}$ in the case (ii).

Proof. Suppose that (i) holds. let $F(u,v) := v^{-1/2} u v^{-1/2}$. By Theorem 4.3 we have

$$\sum_{j=1}^{n} \lambda_j \Phi_j\left(f(A_j)\right) \leq \max_{m \leq t \leq M} \left\{\frac{2^{1-s}}{g(t)}(\mu_f t + \nu_f)\right\} g\left(\sum_{j=1}^{n} \lambda_j \Phi_j(A_j)\right),$$

then the proof in the case (i) is complete.

Next, Suppose that (ii) holds. Let $g_1(t) = -g(t) > 0$ for all $t \in [m, M]$. Then we have

$$\begin{aligned}
\sum_{j=1}^{n} \lambda_j \Phi_j\left(f(A_j)\right) &\leq \max_{m\leq t\leq M}\left\{\frac{2^{1-s}}{g_1(t)}(\mu_f t+\nu_f)\right\} g_1\left(\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)\right)\\
&= -\max_{m\leq t\leq M}\left\{\frac{2^{1-s}}{-g(t)}(\mu_f t+\nu_f)\right\} g\left(\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)\right)\\
&= \min_{m\leq t\leq M}\left\{\frac{2^{1-s}}{g(t)}(\mu_f t+\nu_f)\right\} g\left(\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)\right).
\end{aligned}$$

Then we have the desired results. □

If we let $g = f$ in Theorem 4.6, then we obtain the following corollary.

Corollary 4.7. *Let $A_j, \Phi_j, \lambda_j, j = 1, ..., n$ be as in Theorem 4.6 and $f \in C([m, M])$. Suppose that either of the following conditions holds:*
(i) $f(t) > 0$ for all $t \in [m, M]$,
(ii) $f(t) < 0$ for all $t \in [m, M]$.
If f is s-convex function on $[m, M]$, then

$$\sum_{j=1}^{n} \lambda_j \Phi_j\left(f(A_j)\right) \leq \alpha_0 f\left(\sum_{j=1}^{n}\lambda_j\Phi_j(A_j)\right), \tag{29}$$

where $\alpha_0 = \max_{m\leq t\leq M}\left\{\frac{2^{1-s}}{f(t)}(\mu_f t+\nu_f)\right\}$ in the case (i),
or $\alpha_0 = \min_{m\leq t\leq M}\left\{\frac{2^{1-s}}{f(t)}(\mu_f t+\nu_f)\right\}$ in the case (ii).

5. Some integral inequalities of Jensen's type for operator s-convex functions

If (X, Σ, μ) is a measure space and B is a Banach space, a map $\varphi : X \to B$ is called simple if there exist $b_1, ..., b_n \in B$ and $E_1, ..., E_n \in \Sigma$ which satisfy that $E_i \cap E_j = \emptyset$ for $i \neq j$, such that

$$\varphi(x) = \sum_{i=1}^{n} b_i \chi_{E_i}(x), \ x \in X$$

where $\chi_{E_i}(x) = 1$ if $x \in E_i$ and $\chi_{E_i}(x) = 0$ if $x \notin E_i$. A map $\varphi : X \to B$ is called μ-measurable if there exists a sequence of simple maps φ_n from X to B with

$$\lim_{n\to\infty} \|\varphi_n(x) - \varphi(x)\| = 0$$

μ-almost everywhere.

A map $\varphi : X \to B$ is called weakly μ-measurable if for each $\Psi \in B^*$ the function $\Psi(\varphi)$ is μ-measurable, where B^* is the dual space of B. By Pettiss measurability theorem, a μ-measurable map from a measure space to a Banach space is weakly μ-measurable [8].

Let $\mathcal{H}$ be a Hilbert space and $B = B(\mathcal{H})$. If $\varphi : X \to B(\mathcal{H})$ is measurable, then it is easily seen that for every pair of elements h_1, h_2 in $\mathcal{H}$, the function $\varphi_{h_1,h_2} : X \to \mathbb{C}$

defined by $\varphi_{h_1,h_2}(x) = \langle \varphi(x)h_1, h_2\rangle$ is also measurable. If $\mathcal{H}$ is finite dimensional, then clearly the converse is also true. This explains why in [9] the notion of measurability is given in this latter form.
For $A, B \in B_h(\mathcal{H})$ the subsets $(A, +\infty)$ and $[A, +\infty)$ of $B(\mathcal{H})$ are defined as follows:

$$(A, +\infty) = \{C \in B_h(\mathcal{H}) | \sigma(C - A) \subseteq (0, +\infty)\},$$
$$[A, +\infty) = \{C \in B_h(\mathcal{H}) | \sigma(C - A) \subseteq [0, +\infty)\}.$$

The subsets $(-\infty, B)$ and $(-\infty, B]$ are defined similarly. Using these subsets, we may define other intervals. For example (A, B) is defined as $(-\infty, B) \cap (A, +\infty)$.

Note that the subsets $(-\infty, B]$ and $[A, +\infty)$ are closed subsets of $B(\mathcal{H})$, while, unlike its apparent form, an interval in the form (A, B) is not necessarily open.

The following proposition asserts that for a measurable function $\varphi : X \to B_h(\mathcal{H})$ the sequence of measurable simple functions which converges a.e. on X to φ maybe chosen to satisfy some more properties.

Proposition 5.1. *Let (X, Σ) be a measurable space and suppose that $\varphi : X \to B_h(\mathcal{H})$ is a measurable function. If there is an open interval $(\alpha, \beta) \subseteq \mathbb{R}$ with $\sigma(\varphi(x)) \subseteq (\alpha, \beta)$ for all $x \in X$, then there is a sequence of measurable simple functions $\varphi_n : X \to B_h(\mathcal{H})$ which converges a.e. on X to φ, moreover satisfies*

$$\forall n \in \mathbb{N},\ \forall x \in X,\ \sigma(\varphi_n(x)) \subseteq (\alpha, \beta)$$

One can see the proof of this proposition in [2].

Let (X, Σ, μ) be a measure space, and let B be a Banach space. A μ-measurable map $\varphi : X \to B$ is said to be Bochner integrable if there exists a sequence of simple maps $\{\varphi_n\}$ from X to B such that

$$\lim_{n\to\infty} \int_{\Omega} \|\varphi(x) - \varphi_n(x)\| d\mu = 0. \tag{30}$$

In this case, for any $E \in \Sigma$, the Bochner integral of φ over E is defined by

$$\int_E \varphi(x) d\mu = \lim_{n\to\infty} \int_E \varphi_n(x) d\mu,$$

in the sense of strong convergence in B, where $\int_E \varphi_n(x)d\mu$ is defined in an obvious way [8]. By [8, Chapter II, Theorem 2], a μ-measurable function $\varphi : X \to B$ is Bochner integrable if and only if $\int_X \|\varphi\| d\mu < \infty$. Hence in the case where (X, Σ, μ) is a finite measure space, if a measurable function $\varphi : X \to B$ is bounded, then it is integrable. We can see that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying (30) may be chosen so that it converges everywhere on X to φ and $\|\varphi_n(x)\| \le \|\varphi(x)\|$ for all $n \in \mathbb{N}$ and $x \in X$.

Since in the next section we will consider the special case $B = B(\mathcal{H})$, the following lemma will turn out to be useful (see [2, Lemma 3.4]).

Lemma 5.2. *(i) If the measurable function $\varphi : X \to B(\mathcal{H})$ is integrable then $\int_X \varphi d\mu \in B(\mathcal{H})$ satisfies*

$$\forall h \in \mathcal{H},\ \left\langle \left(\int_X \varphi d\mu \right) h, h \right\rangle = \int_X \langle \varphi h, h\rangle d\mu,$$

where $\langle \varphi h, h\rangle : X \to \mathbb{C}$ *is given by* $\langle \varphi h, h\rangle(x) = \langle \varphi(x)h, h\rangle$.
(ii) If $\varphi, \psi : X \to B_h(\mathcal{H})$ *are integrable and if* $\varphi(x) \leq \psi(x)$ *for all* $x \in X$, *then*

$$\int_X \varphi d\mu \leq \int_X \psi d\mu.$$

Theorem 5.3. *Let* (X, Σ, μ) *be a probability measure space,* $\mathcal{H}$ *be a Hilbert space,* I *be an open interval in* $[0, \infty)$ *and* $f : I \to \mathbb{R}$ *be an operator s-convex function on* I. *Suppose* $\varphi : X \to B(\mathcal{H})^+$ *is a measurable function for which there exist* α *and* β *in* $\mathbb{R}$ *such that*

$$\forall x \in X,\ \sigma(\varphi(x)) \subseteq [\alpha, \beta] \subset I.$$

Then $f \circ \varphi$ *is Bochner integrable and*

$$f\left(\int_X \varphi d\mu\right) \leq 2^{1-s} \int_X f \circ \varphi d\mu. \tag{31}$$

Proof. First suppose φ is a measurable simple function in the form $\sum_{j=1}^n a_j \chi_{E_j}$ with $a_j \in B(\mathcal{H})^+$ for all $j = 1..., n$. Clearly, $f \circ \varphi = \sum_{j=1}^n f(a_j)\chi_{E_j}$ is also a measurable simple function and hence Bochner integrable. Since $\sigma(a_j) \subset [\alpha, \beta]$ for every $j = 1, ..., n$ and $\sum_{j=1}^n \mu(E_j) = 1$, we have also

$$\sigma\left(\int_X \varphi d\mu\right) = \sigma\left(\sum_{j=1}^n \mu(E_j)a_j\right) \subset [\alpha, \beta].$$

Hence $f\left(\int_X \varphi d\mu\right)$ is well defined and

$$f\left(\int_X \varphi d\mu\right) = f\left(\sum_{j=1}^n \mu(E_j)a_j\right) \leq 2^{1-s}\sum_{j=1}^n \mu(E_j)f(a_j) = 2^{1-s}\int_X f \circ \varphi d\mu.$$

Now for the general case, using the assumption, there are α' and β' in $\mathbb{R}$ with

$$\forall x \in X,\ \sigma(\varphi(x)) \subset [\alpha, \beta] \subset (\alpha', \beta') \subset [\alpha', \beta'] \subset I.$$

Then there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of measurable simple functions which converges everywhere on X to φ. and which satisfies

$$\forall n \in \mathbb{N},\ \forall x \in X,\ \sigma(\varphi_n(x)) \subset (\alpha', \beta').$$

Hence $(\varphi_n)_{n\in\mathbb{N}}$ is uniformly bounded. By the dominated convergence theorem, $\int_X \|\varphi - \varphi_n\| d\mu \to 0$, whence $\int_X \varphi_n d\mu \to \int_X \varphi d\mu$. Note also, since $\alpha 1_{\mathcal{H}} \leq \varphi(x) \leq \beta 1_{\mathcal{H}}$ for all $x \in X$, $\alpha 1_{\mathcal{H}} \leq \int_X \varphi d\mu \leq \beta 1_{\mathcal{H}}$. Hence $\sigma\left(\int_X \varphi d\mu\right) \subset [\alpha, \beta] \subset I$. Similarly, $\sigma\left(\int_X \varphi_n d\mu\right) \subset [\alpha', \beta'] \subset I$. Thus, by Lemma 2.3, we have

$$f\left(\int_X \varphi_n d\mu\right) \to f\left(\int_X \varphi d\mu\right). \tag{32}$$

Moreover, the same lemma implies that the sequence $(f \circ \varphi_n)_{n\in\mathbb{N}}$ of measurable simple functions converges pointwise on X to $f \circ \varphi$. Hence $f \circ \varphi : X \to B(\mathcal{H})^+$ is also measurable. On the other hand,

$$\forall x \in X,\ \|f \circ \varphi(x)\| = \|f\|_{\infty, \sigma(\varphi(x))} \leq \|f\|_{\infty, [\alpha, \beta]}.$$

Again by the dominated convergence theorem,

$$\int_X f \circ \varphi_n d\mu \to \int_X f \circ \varphi d\mu. \tag{33}$$

Finally, by the first part of the proof, for each $n \in \mathbb{N}$ we have

$$f\left(\int_X \varphi_n d\mu\right) \leq 2^{1-s} \int_X f \circ \varphi_n d\mu.$$

By using (32) and (33) we have the desired result in (31). □

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