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Decomposition of A-ideals in MV-modules

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Abstract. In this paper, by considering the notion of MV-modules, we present definition of radical of an ideal in MV-algebras by prime ideals that in last was defined by maximal ideals. Also, we define the notions of primary and P-primary A-ideals in MV-modules. Then we show that under conditions, if an A-ideal has a primary decomposition, then it has a reduced primary decomposition. Finally, we characterize proper A-ideals that have a reduced primary decomposition.

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1. Introduction

MV-algebras were defined by C.C. Chang [2, 3] as algebras corresponding to the Lukasiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: CN-algebras, Wajsberg algebras, bounded commutative BCK-algebras and bricks. It is discovered that MV-algebras are naturally related to the Murray-von Neumann order of projections in operator algebras on Hilbert spaces and that they play an interesting role as invariants of approximately finite-dimensional C^*-algebras. They are also naturally related to Ulam's searching games with lies. MV-algebras admit a natural lattice reduct and hence a natural order structure. Many important properties can be derived from the fact, established by Chang that nontrivial MV-algebras are subdirect products of MV-chains, that is, totally ordered MV-algebras. To prove this fundamental result, Chang introduced the notion of prime ideal in an MV-algebra. A *product MV-algebra* (or PMV-algebra, for short) is an MV-algebra which has an associative binary operation "$\cdot$". It satisfies an extra property which will be explained in preliminaries. During the last years, PMV-algebras were considered and their equivalence with a certain class of l-rings with strong unit was proved. It seems quite natural to introduce modules over such algebras, generalizing the divisible MV-algebras and the MV-algebras obtained from Riesz spaces and to prove natural equivalence theorems. Hence, the notion of MV-modules was introduced as an action of a PMV-algebra over an MV-algebra by A. Di Nola [6]. In 2014, F. Forouzesh, E. Eslami and A. Borumand Saeid defined prime A-ideals and radical of A-ideals by maximal A-ideals in MV-modules [8, 9]. Since MV-modules are in their infancy, stating and opening of any subject in this field can be useful. Since the notion of A-ideal in MV-modules is important, for completion of study of ideals in MV-modules,

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in this paper, we present definitions of primary decomposition and reduced primary decomposition of an A-ideal by prime A-ideals (no maximal A-ideals). The simplification of an A-ideal helps us for better studying it. Hence, the decomposition of an A-ideal can be useful and important.

2. Preliminaries

Definition 2.1. [4] An *MV-algebra* is a structure $M = (M, \oplus, ', 0)$ of type $(2, 1, 0)$ such that:
$(MV1)$ $(M, \oplus, 0)$ is an abelian monoid,
$(MV2)$ $(a')' = a$,
$(MV3)$ $0' \oplus a = 0'$,
$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$,
If we define the constant $1 = 0'$ and operations $\odot$ and $\ominus$ by $a \odot b = (a' \oplus b')'$, $a \ominus b = a \odot b'$, then
$(MV5)$ $(a \oplus b) = (a' \odot b')'$,
$(MV6)$ $x \oplus 1 = 1$,
$(MV7)$ $(a \ominus b) \oplus b = (b \ominus a) \oplus a$,
$(MV8)$ $a \oplus a' = 1$,
for every $a, b \in A$. It is clear that $(M, \odot, 1)$ is an abelian monoid. Now, if we define auxiliary operations $\vee$ and $\wedge$ on M by $a \vee b = (a \odot b') \oplus b$ and $a \wedge b = a \odot (a' \oplus b)$, for every $a, b \in M$, then $(M, \vee, \wedge, 0)$ is a *bounded distributive lattice*. An MV-algebra M is a *Boolean* algebra if and only if the operation " $\oplus$ " is idempotent, i.e., $x \oplus x = x$, for every $x \in X$. In MV-algebra M, the following conditions are equivalent: (i) $a' \oplus b = 1$, (ii) $a \odot b' = 0$, (iii) $b = a \oplus (b \ominus a)$, (iv) $\exists c \in A$ such that $a \oplus c = b$, for every $a, b, c \in M$. For any two elements a, b of MV-algebra M, $a \leq b$ if and only if a, b satisfy in the above equivalent conditions $(i) - (iv)$. An ideal of MV-algebra M is a subset I of M, satisfying the following condition: $(I1)$ $0 \in I$, $(I2)$ $x \leq y$ and $y \in I$ implies that $x \in I$, $(I3)$ $x \oplus y \in I$, for every $x, y \in I$. A proper ideal P of M is a prime ideal if and only if $x \ominus y \in P$ or $y \ominus x \in P$, for every $x, y \in M$. Equivalently, P is prime if and only if $x \wedge y \in P$ implies $x \in P$ or $y \in P$, for $x, y \in M$. A proper ideal I of M is a maximal ideal of M if and only if no proper ideal of M strictly contains I. In MV-algebra M, the *distance function* $d : M \times M \to M$ is defined by $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ which satisfies (i) $d(x, y) = 0$ if and only if $x = y$, (ii) $d(x, y) = d(y, x)$, (iii) $d(x, z) \leq d(x, y) \oplus d(y, z)$, (iv) $d(x, y) = d(x', y')$, (v) $d(x \oplus z, y \oplus t) \leq d(x, y) \oplus d(z, t)$, for every $x, y, z, t \in M$. Let I be an ideal of MV-algebra M. Then we denote $x \sim y$ $(x \equiv_I y)$ if and only if $d(x, y) \in I$, for every $x, y \in M$. So $\sim$ is a congruence relation on M. Denote the equivalence class containing x by $\frac{x}{I}$ and $\frac{M}{I} = \{\frac{x}{I} : x \in M\}$. Then $(\frac{M}{I}, \oplus, ', \frac{0}{I})$ is an MV-algebra, where $(\frac{x}{I})' = \frac{x'}{I}$ and $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$, for all $x, y \in M$.(See [4])

Lemma 2.1. [4] *In every MV-algebra A, the natural order " $\leq$ " has the following properties:*
(i) $x \leq y$ if and only if $y' \leq x'$,
(ii) if $x \leq y$, then $x \oplus z \leq y \oplus z$, for every $z \in A$.

Proposition 2.2. [4] *Every proper ideal of an MV-algebra is an intersection of prime ideals.*

Proposition 2.3. [4] *Let M be an MV-algebra and $z \in M$. Then the principal ideal generated by z is denoted by $\prec z \succ$ and $\prec z \succ = \{x \in M : nz = \underbrace{z \oplus \cdots \oplus z}_{n\ times} \geq x,\ for\ some\ n \geq 0\}$ is an ideal of M.*

Proposition 2.4. [4] *Let I be an ideal of A. Then*

$$\prec I \cup \{z\} \succ = \{x \in A : x \leq nz \oplus a,\ for\ some\ n \in \mathbb{N}\ and\ a \in I\}$$

is an ideal of A.

Definition 2.2. [6, 7] (i) An l-group is an algebra $(G, +, -, 0, \vee, \wedge)$, where the following properties hold:
(a) $(G, +, -, 0)$ is a group,
(b) $(G, \vee, \wedge)$ is a lattice,
(c) $x \leq y$ implies that $x + a \leq y + a$, for any $x, y, a, b \in G$.
A strong unit $u > 0$ is a positive element with property that for any $g \in G$ there exits $n \in \omega$ such that $g \leq nu$. The Abelian l-groups with strong unit will be simply called lu-groups.
The category whose objects are MV-algebras and whose homomorphisms are MV-homomorphisms is denoted by *MV*. The category whose objects are pairs (G, u), where G is an Abelian l-group and u is a strong unit of G and whose homomorphisms are l-group homomorphisms is denoted by *Ug*. The functor that establishes the categorial equivalence between *MV* and *Ug* is

$$\Gamma : Ug \longrightarrow MV,$$

where $\Gamma(G, u) = [0, u]_G$, for every lu-group (G, u) and $\Gamma(h) = h|_{[0,u]}$, for every lu-group homomorphism h. The above results allows us to consider an MV-algebra, when necessary, as an interval in the positive cone of an l-group. Thus, many definitions and properties can be transferred from l-groups to MV-algebras. For example, the group addition becomes a partial operation when it is restricted to an interval, so we define a *partial addition* on an MV-algebra M as follows:
$x + y$ is defined if and only if $x \leq y'$ and in this case, $x + y = x \oplus y$, for every $x, y \in M$. Moreover, if $z + x \leq z + y$, then $x \leq y$.
(ii) A *product MV-algebra* (or PMV-algebra, for short) is a structure $A = (A, \oplus, ., ', 0)$, where $(A, \oplus, ', 0)$ is an MV-algebra and "." is a binary associative operation on A such that the following property is satisfied: if $x + y$ is defined, then $x.z + y.z$ and $z.x + z.y$ are defined and $(x + y).z = x.z + y.z$, $z.(x + y) = z.x + z.y$, for every $x, y, z \in A$, where "$+$" is the partial addition on A. A unity for the product is an element $e \in A$ such that $e.x = x.e = x$, for every $x \in A$. If A has a unity for product, then $e = 1$.

Lemma 2.5. [5] *Let A be a PMV-algebra. Then $a \leq b$ implies that $a.c \leq b.c$ and $c.a \leq c.b$, for any $a, b, c \in A$. If A has unity for product, then $a.b \leq a \wedge b$, for any $a, b \in A$.*

Lemma 2.6. [5] *A finite PMV-algebra A has unity for product if and only if A is a Boolean algebra and in this case $a.b = a \wedge b$, for any $a, b \in A$.*

Definition 2.3. [6] Let $A = (A, \oplus, ., ', 0)$ be a PMV-algebra, $M = (M, \oplus, ', 0)$ be an MV-algebra and the operation $\Phi : A \times M \longrightarrow M$ be defined by $\Phi(a, m) = am$, which satisfies the following axioms:

$(AM1)$ If $x+y$ is defined in M, then $ax+ay$ is defined in M and $a(x+y)=ax+ay$,
$(AM2)$ If $a+b$ is defined in A, then $ax+bx$ is defined in M and $(a+b)x=ax+bx$,
$(AM3)$ $(a.b)x=a(bx)$, for every $a,b\in A$ and $x,y\in M$.
Then M is called a (left) *MV-module* over A or briefly an A-module. We say M is a *unitary* MV-module if A has a unity for the product, that is
$(AM4)$ $1_Ax=x$, for every $x\in M$.

Lemma 2.7. [6] *Let A be a PMV-algebra and M be an A-module. Then*
(a) $0x=0$,
(b) $a0=0$,
(c) $ax'\leq(ax)'$,
(d) $a'x\leq(ax)'$,
(e) $(ax)'=a'x+(1x)'$,
(f) $x\leq y$ *implies that* $ax\leq ay$,
(g) $a\leq b$ *implies that* $ax\leq bx$,
(h) $a(x\oplus y)\leq ax\oplus ay$,
(i) $d(ax,ay)\leq ad(x,y)$,
(j) *if* $x\equiv_I y$, *then* $ax\equiv_I ay$, *where* I *is an ideal of* A,
(k) *if* M *is a unitary* MV*-module, then* $(ax)'=a'x+x'$, *for every* $a,b\in A$ *and* $x,y\in M$.

Lemma 2.8. [8] *Let A be a PMV-algebra and M be an A-module. Then $(a\oplus b)x\leq ax\oplus bx$, for every $a,b\in A$ and $x\in M$.*

Definition 2.4. [6] Let A be a PMV-algebra and M be an A-module. Then an ideal $N\subseteq M$ is called an A-*ideal* of M if $(I4)$ $ax\in N$, for every $a\in A$ and $x\in N$.

Note: From now on, in this paper, we let A is a PMV-algebra, M be an MV-algebra, $\mathcal{PI}(M)$ be the set of all prime ideals of M and $\mathcal{PI}_J(M)$ be the set of all prime ideals of M that contain $J\in\mathcal{I}(M)$.

3. Primary ideals in MV-algebras

In this section, we present definition of radical of an ideal in MV-algebras by prime ideals that in [9] was defined by maximal ideals. Also, we introduce the notion of primary ideals in MV-algebras and we get some results that we use in the section 4.

Definition 3.1. Let $I\in\mathcal{I}(M)$. Then the intersection of all prime ideals of M, including I, is called *radical* of I and it is denoted by $rad_M(I)$ or briefly $rad(I)$. If there is not any prime ideal of M including I, then we let $rad(I)=M$.

Example 3.1. (i) Let $M=\{0,1,2\}$ and operation $\oplus$ be defined by

$\oplus$	0	1	2
0	0	1	2
1	1	1	2
2	0	2	2

If $0'=2,\ 1'=1$ and $2'=0$, then $(M,\oplus,',0,1)$ is an MV-algebra. It is easy to show that $I=\{0,1\}$ is only prime ideal of M and so $rad(\{0\})=\{0,1\}$ and $rad(I)=I$.
(ii) Let $M_2(\mathbb{R})$ be the ring of square matrixes of order 2 with real elements and let 0

be the matrix with all elements 0. If we define the order relation on components

$$A=(a_{ij})_{i,j=1,2}\geq 0 \ if \ and \ only \ if \ a_{ij}\geq 0 \ for \ any \ i,j,$$

then $M_2(\mathbb{R})$ is an l-ring. If $v=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$, then $(M_2(\mathbb{R}),v)$ is an lu-ring and so $M=\Gamma(M_2(\mathbb{R}),v)$ is an MV-algebra. It is easy to see that $I(M)=\{\{0\},M\}$ and $\{0\}$ is not a prime ideal of M. Then $rad(\{0\})=M$.

Lemma 3.1. *In M, the following conditions are equivalent:*
(a) $a=a\ominus(b\ominus a)$,
(b) $a\ominus b=(a\ominus b)\ominus b$,
(c) $(a\ominus c)\ominus(b\ominus c)=(a\ominus b)\ominus c$,
(d) $a\wedge a'=0$,
(e) $a\vee a'=1$,
(f) $a=a\ominus a'$,
(g) $a'=a'\ominus a$,
(h $b'\wedge a=a\ominus b$*'*
(i) $b\wedge a=a\ominus b'$
(j) $a\wedge(b\ominus c)=(a\wedge b)\ominus c$, *for every* $a,b,c\in M$.

Proof. The proof is routine. □

Definition 3.2. M is called an implicative MV-algebra if $x\ominus(y\ominus x)=x$, for every $x,y\in M$.

Example 3.2. Let $M_1=\{0,1,2,3\}$, $M_2=\{0,1\}$, and operations $\oplus_1$ and $\oplus_2$ be defined by

$\oplus_1$	0	1	2	3
0	0	1	2	3
1	0	1	3	3
2	2	3	2	3
3	3	3	3	3

$\oplus_2$	0	1
0	0	1
1	1	1

If $0'=3,\ 1'=2,\ 2'=1$ and $3'=0$, then $(M_1,\oplus_1,',0,1)$ is an implicative MV-algebra. Also, if $0'=1$ and $1'=0$, then $(M_2,\oplus_2,',0,1)$ is an implicative MV-algebra.

Definition 3.3. Let $\emptyset\neq S\subseteq M$. We say that S is $\wedge$-closed, if $a\wedge b\in S$, for all $a,b\in S$.

Example 3.3. In Example 3.2, consider $S=\{0,1,2\}\subseteq M_1$ and $T=\{1,2\}\subseteq M_1$. It is easy to see that S is $\wedge$-closed and T is not $\wedge$-closed.

Lemma 3.2. *Let $I\in\mathcal{I}(M)$, $S\subseteq M$ be $\wedge$-closed and $S\cap I=\emptyset$. Then there exists a maximal ideal P of M such that $P\supseteq I$ and $P\cap S=\emptyset$. Furthermore, P is a prime ideal of M.*

Proof. The existence of an ideal P easily follows from $Zorn's$ Lemma. Let there exist $x,y\in M$ such that $x\wedge y\in P$, $x\notin P$ and $y\notin P$. Then P is properly contained in both $\prec P\cup\{x\}\succ=P_1$ and $\prec P\cup\{y\}\succ=P_2$. By maximality of P, $P_1\cap S\neq\emptyset$ and $P_2\cap S\neq\emptyset$. Let $s_i\in P_i\cap S$, $i=1,2$. Then $s_1\wedge s_2\leq s_i$, i=1,2 implies $s_1\wedge s_2\in P_1\cap P_2=P$. On the other hand, $s_1\wedge s_2\in S$, which is a contradiction. Therefore, P is a prime ideal of M. □

Theorem 3.3. *Let M be implicative and $I \in \mathcal{I}(M)$. Then*

$$rad(I) = \{x \in M : \forall P \in \mathcal{PI}_I(M), \exists c \in M \setminus P \textit{ such that } c \wedge x \in I\}.$$

Proof. Let

$$T = \{x \in M : \forall P \in \mathcal{PI}_I(M), \exists c \in M \setminus P \textit{ such that } c \wedge x \in I\}$$

and $x \in rad(I)$. Then $x \in P$, for every $P \in \mathcal{PI}_I(M)$. If $x \in I$, then by considering $c = 1$, we have $x \in T$. Now, let $x \notin I$. If $x \notin T$, then there exists $P_1 \in \mathcal{PI}_I(M)$ such that $c \wedge x \notin I$, for every $c \in M \setminus P_1$. Let $S = \{(c \wedge x) \ominus y : y \in I \textit{ and } c \in M \setminus P_1\}$. First, we show that S is $\wedge$-closed. Let $(c_1 \wedge x) \ominus y_1, (c_2 \wedge x) \ominus y_2 \in S$, where $c_1, c_2 \in M \setminus P_1$ and $y_1, y_2 \in I$. By Lemma 3.1 (j) and (i),

$$\begin{aligned} ((c_1 \wedge x) \ominus y_1) \wedge ((c_2 \wedge x) \ominus y_2) &= ((c_1 \wedge x) \ominus y_1) \wedge (c_2 \wedge x)) \ominus y_2 \\ &= ((c_2 \wedge x) \wedge ((c_1 \wedge x) \ominus y_1)) \ominus y_2, \\ &= (((c_2 \wedge x) \wedge (c_1 \wedge x)) \ominus y_1) \ominus y_2 \\ &= y_2' \wedge (((c_1 \wedge c_2) \wedge x) \ominus y_1), \\ &= (y_2' \wedge ((c_1 \wedge c_2) \wedge x)) \ominus y_1 \\ &= ((y_2' \wedge c_1 \wedge c_2) \wedge x) \ominus y_1. \end{aligned}$$

Now, we show that $y_2' \wedge c_1 \wedge c_2 \in M \setminus P_1$. Let $y_2' \wedge c_1 \wedge c_2 \in P_1$. Since $c_1 \wedge c_2 \notin P_1$, $y_2' \in P_1$ and so $1 \in P_1$. Since $x \leq 1 \in P_1$, we get $x \in P_1$, for every $x \in M$ and so $P_1 = M$, which is a contradiction. Hence, $y_2' \wedge c_1 \wedge c_2 \in M \setminus P_1$ and so $((y_2' \wedge c_1 \wedge c_2) \wedge x) \ominus y_1 \in S$. It means that$((c_1 \wedge x) \ominus y_1) \wedge ((c_2 \wedge x) \ominus y_2) \in S$ and so S is $\wedge$-closed. Now, we prove that $S \cap I = \emptyset$. If $S \cap I \neq \emptyset$, then there exist $c' \in M \setminus P_1$ and $y' \in I$ such that $(c' \wedge x) \ominus y' \in I$. It results that $c' \wedge x \in I$. But, by definition of S, $c \wedge x \notin I$, for every $c \in M \setminus P_1$, which is a contradiction. Then $S \cap I = \emptyset$ and so by Lemma 3.2, there exists $P_2 \in \mathcal{PI}_I(M)$ such that $P_2 \cap S = \emptyset$. Since $(c \wedge x) \ominus x = 0 \in P$ and $x \in P$, $c \wedge x \in P$, for every $c \in M \setminus P$ and for every $P \in \mathcal{PI}_I(M)$. Then $(c \wedge x) \in P_2$. On the other hand, $c \wedge x = (c \wedge x) \ominus 0 \in S$. Hence, $c \wedge x \in P_2 \cap S$, which is a contradiction. It implies that $x \in T$. Therefore, $rad(I) \subseteq T$.
Now, let $x \in T$. Hence, for every $P \in \mathcal{PI}_I(M)$ there exists $c \in M \setminus P$ such that $c \wedge x \in I \subseteq P$. Since $c \notin P$, we get $x \in P$, for every $P \in \mathcal{PI}_I(M)$. It means that $x \in rad(I)$ and so $T \subseteq rad(I)$. Therefore, $T = rad(I)$. □

Proposition 3.4. *Let M be implicative and $I \in \mathcal{I}(M)$. If for every $P \in \mathcal{PI}(M)$, $P \cap I \neq \{0\}$ implies that $I \subseteq P$, then*

$$rad(I) = \{x \in X : \ \forall\ P \in \mathcal{PI}(M) \textit{ with } P \cap I \neq \{0\},\ \exists c \in M \setminus P \textit{ such that } c \wedge x \in I\}.$$

Proof. By Theorem 3.3, the proof is clear. □

Theorem 3.5. *Let M be an MV-algebra and $I, J, I_1, \cdots, I_n$ be ideals of M. Then*
(i) $I \subseteq rad(I)$,
(ii) $I \subseteq J$ implies $rad(I) \subseteq rad(J)$,
(iii) $rad(I) \cup rad(J) \subseteq rad(I \cup J)$.
Moreover, if M is implicative and $P \cap I_k \neq \{0\}$ implies that $I_k \subseteq P$, for every $P \in \mathcal{PI}(M)$ and $1 \leq k \leq n$, then
(iv) $rad(rad(I)) = rad(I)$,
(v) $rad(\bigcap_{k=1}^{n} I_k) = \bigcap_{k=1}^{n} rad(I_k)$.

Proof. The proofs of (i), (ii) and (iii) are easy.
(iv) By (i), $rad(I) \subseteq rad(rad(I))$. Now, let $x \in rad(rad(I))$ and $P \in \mathcal{PI}(M)$ with $P \cap I \neq \{0\}$. Then by (i), $P \cap rad(I) \neq \{0\}$. Since $x \in rad(rad(I))$, by Proposition 3.4, there exists $c_1 \in M \setminus P$ such that $c_1 \wedge x \in rad(I)$. Since $c_1 \wedge x \in rad(I)$ and $P \cap I \neq \{0\}$, by Proposition 3.4, there exists $c_2 \in M \setminus P$ such that $(c_2 \wedge c_1) \wedge x = c_2 \wedge (c_1 \wedge x) \in I$. It is clear that $c = c_1 \wedge c_2 \in M \setminus P$. Similarly, for every $P \in \mathcal{PI}(M)$ with $P \cap I \neq \{0\}$ there is $c \in M \setminus P$ such that $c \wedge x \in I$. Hence, by Proposition 3.4, $x \in rad(I)$. Therefore, $rad(rad(I)) \subseteq rad(I)$.
(v) Let $x \in rad(\bigcap_{k=1}^{n} I_k)$ and $P \in \mathcal{PI}_{I_t}(M)$, for $1 \leq t \leq n$. Since $I_t \subseteq P$, we get $\bigcap_{k=1}^{n} I_k \subseteq I_t \subseteq P$. Since $x \in rad(\bigcap_{k=1}^{n} I_k)$, by Theorem 3.3, there exists $c \in M \setminus P$ such that $c \wedge x \in \bigcap_{k=1}^{n} I_k \subseteq I_t$ and so $c \wedge x \in I_t$. Hence, $x \in rad(I_t)$. Similarly, $x \in rad(I_k)$, for every $1 \leq k \leq n$ and so $x \in \bigcap_{k=1}^{n} rad(I_k)$. Hence, $rad(\bigcap_{k=1}^{n} I_k) \subseteq \bigcap_{k=1}^{n} rad(I_k)$.
Now, let $x \in \bigcap_{k=1}^{n} rad(I_k)$ and $P \in \mathcal{PI}(M)$ with $P \cap (\bigcap_{k=1}^{n} I_k) \neq \{0\}$. Then $P \cap I_k \neq \{0\}$, for every $1 \leq k \leq n$. Since $x \in rad(I_k)$, by Proposition 3.4, there is $c_k \in M \setminus P$ such that $c_k \wedge x \in I_k$, for every $1 \leq k \leq n$. Let $c = c_1 \wedge \cdots \wedge c_n$. It is clear that $c \notin P$. On the other hand, since $(c \wedge x) \leq (c_k \wedge x) \in I_k$, $c \wedge x \in I_k$, for every $1 \leq k \leq n$. Then $c \wedge x \in \bigcap_{k=1}^{n} I_k$. Therefore, by Proposition 3.4, $x \in rad(\bigcap_{k=1}^{n} I_k)$ and so $\bigcap_{k=1}^{n} rad(I_k) \subseteq rad(\bigcap_{k=1}^{n} I_k)$ □

Definition 3.4. Let Q be a proper ideal of M. Then Q is called a *primary* ideal of M if $a \wedge b \in Q$, then there exists $c \in M \setminus P$ such that $c \wedge b \in Q$ or $a \wedge c \in Q$, for every $P \in \mathcal{PI}_Q(M)$ and $a, b \in M$.

Example 3.4. In Example 3.2, $I = \{0, 1\}$ and $J = \{0, 2\}$ are primary ideals of M_1.

Proposition 3.6. *Let M be implicative and Q be an ideal of M. Then Q is a primary ideal of M if and only if $a \wedge b \in Q$ implies that $a \in rad(Q)$ or $b \in rad(Q)$, for any $a, b \in M$.*

Proof. $(\Rightarrow)$ Let Q be a primary ideal of M and $a \wedge b \in Q$, for $a, b \in M$. If $a \in Q$, then $a \in rad(Q)$. Let $a \notin Q$. Then there exits $c \in M \setminus P$ such that $c \wedge b \in Q$ or $a \wedge c \in Q$, for every $P \in \mathcal{PI}_Q(M)$. If $c \wedge b \in Q$, then $c \wedge b \in P$, for every $P \in \mathcal{PI}_Q(M)$. Since $c \notin P$, $b \in P$, for every $P \in \mathcal{PI}_Q(M)$. It results that $b \in \bigcap_{Q \subseteq P} P = rad(Q)$. Similarly, if $a \wedge c \in Q$, then $a \in rad(Q)$.
$(\Leftarrow)$ Let $Q \in \mathcal{I}(M)$. If $a \wedge b \in Q$, then $a \in rad(Q)$ or $b \in rad(Q)$, for $a, b \in M$ and so by Theorem 3.3, there exists $c \in M \setminus P$ such that $c \wedge b \in Q$ or $a \wedge c \in Q$, for every $P \in \mathcal{PI}_Q(M)$. It means that Q is a primary ideal of M. □

Theorem 3.7. *In an MV-algebra, every prime ideal is a primary ideal.*

Proof. Let M be an MV-algebra, Q be a prime ideal of M, $a \wedge b \in Q$ and $a \notin Q$, for $a, b \in M$. Then by considering $c = 1 \in M \setminus P$, for every $P \in \mathcal{PI}_Q(M)$, we have $c \wedge b = b \in Q$. Hence, P is a primary ideal of M. □

Example 3.5. In Example 3.1 (ii), $I = \{0\}$ is a primary ideal of M, but it is not a prime ideal of M.

Theorem 3.8. *Let M be implicative and $I \cap P \neq \{0\}$ implies that $I \subseteq P$, for every $I \in \mathcal{I}(M)$ and $P \in \mathcal{PI}(M)$. Then the radical of every primary ideal of M is a prime ideal of M.*

Proof. Let Q be a primary ideal of M. If $rad(Q) = M$, then $1 \in rad(Q)$. Hence, by Theorem 3.3, for every $P \in \mathcal{PI}_Q(M)$, there exists $c \in M \setminus P$ such that $c \wedge 1 = c \in Q \subseteq P$ and so $c \in P$, which is a contradiction. Now, let $a \wedge b \in rad(Q)$, for $a, b \in M$. Then there exists $c \in M \setminus P$ such that $(c \wedge a) \wedge b = c \wedge (a \wedge b) \in Q$, for every $P \in \mathcal{PI}_Q(M)$. If $a \notin rad(Q)$, then by Theorem 3.3, there is $P \in \mathcal{PI}_Q(M)$ such that $c \wedge a \notin Q$, for every $c \in M \setminus P$. Since Q is a primary ideal of M and $(c \wedge a) \wedge b \in Q$, there is $c' \in M \setminus P$ such that $c' \wedge b \in Q$, for every $P \in \mathcal{PI}_Q(M)$ and so $b \in rad(Q)$. Therefore, $rad(Q)$ is a prime ideal of M. □

4. Primary decomposition of A-ideals in MV-modules

In this section, we define the notions of primary and P-primary A-ideals of an MV-module. As a fundamental result, we introduce an MV-module that all its proper A-ideals have reduced primary decomposition.

Proposition 4.1. *Let M be an A-module and N be an A-ideal of M. Then $Q_N = \{x \in A : xM \subseteq N\}$ is an ideal of A.*

Proof. Let $x, y \in Q_N$, for $x, y \in A$. Then $xm,\ ym \in N$ and so $xm \oplus ym \in N$, for every $m \in M$. Since by Lemma 2.8, $(x \oplus y)m \leq xm \oplus ym \in N$, $(x \oplus y)m \in N$, for every $m \in M$. Hence, $x \oplus y \in Q_N$. Now, let $x \leq y$ and $y \in Q_N$, for $x, y \in A$. By Lemma 2.7 (g), $xm \leq ym \in N$ and so $xm \in N$, for every $m \in M$. Therefore, $x \in Q_N$ and so Q_N is an ideal of A. □

Definition 4.1. Let M be an A-module and N be a proper A-ideal of M. Then N is called a *primary* A-ideal of M, if for any $x \in A$ and $m \in M$, $xm \in N$ implies that $m \in N$ or $\exists c \in A \setminus P$ such that $(c \wedge x)M \subseteq N$, for every $P \in \mathcal{PI}_{Q_N}(A)$.

Example 4.1. Let $A = \{0, 1, 2, 3\}$ and the operations "$\oplus$" and "." on A are defined as follows:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Consider $0' = 3,\ 1' = 2,\ 2' = 1$ and $3' = 0$. Then it is easy to show that $(A, \oplus, ', ., 0)$ is a PMV-algebra and $(A, \oplus, ', 0)$ is an MV-algebra. Now, let the operation $\bullet$: $A \times A \longrightarrow A$ be defined by $a \bullet b = a.b$, for every $a, b \in A$. It is easy to show that A is an MV-module on A and $I = \{0, 1\}$, $J = \{0, 2\}$ are primary A-ideals of A.

Proposition 4.2. *Let M be a unitary A-module and N be a prime A-ideal of M. Then N is a primary A-ideal of M.*

Proof. Let $xm \in N$ and $m \notin N$, for $x \in A$ and $m \in M$. Then we consider $c = 1 \in A \setminus P$ and so $(c \wedge x)M = xM \subseteq N$, for every $P \in \mathcal{PI}_{Q_N}(A)$. □

Theorem 4.3. *Let M be a unitary A-module and N be a primary A-ideal of M. Then Q_N is a primary ideal of A.*

Proof. If $Q_N = A$, then $1 \in Q_N$ and so $M = N$, which is a contradiction. Let $a \wedge b \in Q_N$ and $a \notin Q_N$, for $a, b \in A$. Then by Lemmas 2.5 and 2.7 (g), $(b.a)m \leq (b \wedge a)m \in N$ and so $b(am) = (b.a)m \in N$, for every $m \in M$. Since $a \notin Q_N$, there exists $m' \in M$ such that $am' \notin N$. Moreover, since $b(am') \in N$ and $am' \notin N$, there exists $c \in A \setminus P$ such that $(c \wedge b)M \subseteq N$, for every $P \in \mathcal{PI}_{Q_N}(A)$. It results that $c \wedge b \in Q_N$. Therefore, Q_N is a primary ideal of A. □

Note. In Theorem 4.3, if A is implicative such that $I \cap P \neq \{0\}$ implies that $I \subseteq P$, then by Theorem 3.8, $rad(Q_N)$ is a prime ideal of A and

$$rad(Q_N) = \{x \in A : \forall P \in \mathcal{PI}_{Q_N}(A), \exists c \in A \setminus P \text{ such that } (c \wedge x)M \subseteq N\}.$$

Definition 4.2. Let M be an A-module and N be a proper A-ideal of M. Then N is called a *P-primary* A-ideal of M, if N is a primary A-ideal of M and $rad(Q_N) = P$.

Lemma 4.4. *Let A be implicative, M be an A-module, $N_1, \cdots, N_k$ be P'-primary A-ideal of M such that $Q_{\bigcap_{i=1}^k N_i} \neq 0$. If $P \cap I \neq \{0\}$ implies that $I \subseteq P$, for every ideal $I \in \mathcal{I}(A)$ and $P \in \mathcal{PI}(A)$, then $\bigcap_{i=1}^k N_i$ is a P'-primary A-ideal of M.*

Proof. It is clear that $\bigcap_{i=1}^k N_i \neq M$. Let $xm \in \bigcap_{i=1}^k N_i$ and $m \notin \bigcap_{i=1}^k N_i$, for $x \in A$ and $m \in M$. Then $xm \in N_i$, for every $1 \leq i \leq k$ and there exists $1 \leq j \leq k$ such that $m \notin N_j$. Since $xm \in N_j$ and $m \notin N_j$, there exists $c_j \in A \setminus P$ such that $(c_j \wedge x)M \subseteq N_j$, for every $P \in \mathcal{PI}_{Q_{N_j}}(A)$. It results that $x \in rad(Q_{N_j}) = P' = rad(Q_{N_i})$, for every $1 \leq i \leq k$. Hence, there exists $c_i \in A \setminus P$ such that $(c_i \wedge x)M \subseteq N$, for every $P \in \mathcal{PI}_{Q_{N_i}}(A)$. Now, we show that $rad(Q_{\bigcap_{i=1}^k N_i}) = P'$. For every $1 \leq i \leq k$,

$$x \in Q_{\bigcap_{i=1}^k N_i} \Leftrightarrow xM \subseteq \bigcap_{i=1}^{k} N_i \Leftrightarrow xM \subseteq N_i \Leftrightarrow x \in Q_{N_i} \Leftrightarrow x \in \bigcap_{i=1}^{k} Q_{N_i}.$$

Then $Q_{\bigcap_{i=1}^k N_i} = \bigcap_{i=1}^k Q_{N_i}$ and so by Theorem 3.5 (v),

$$rad(Q_{\bigcap_{i=1}^k N_i}) = rad(\bigcap_{i=1}^{k} Q_{N_i}) = \bigcap_{i=1}^{k} rad(Q_{N_i}) = \bigcap_{i=1}^{k} P' = P'.$$

Let $c = c_1 \wedge c_2 \cdots \wedge c_k$ and there exists $P \in \mathcal{PI}_{Q_{\bigcap_{i=1}^k N_i}}(A)$ such that $c \in P$. Then there is $1 \leq i \leq k$ such that $c_i \in P$. Since $\{0\} \neq Q_{\bigcap_{i=1}^k N_i} \subseteq Q_{N_i}$, we get $Q_{N_i} \cap P \neq \{0\}$ and so $Q_{N_i} \subseteq P$, for every $1 \leq i \leq k$. It results that $c_i \notin P$, for every $1 \leq i \leq k$, which is a contradiction. Hence, $c \in A \setminus P$, for every $P \in \mathcal{PI}_{Q_{N_{\bigcap_{i=1}^k N_i}}}(A)$. On the other hand, since $(c_i \wedge x).m \in N_i$,

$$(c \wedge x)m = (c_1 \wedge \cdots \wedge c_i \wedge x)m = (c_i \wedge x)m \in N_i$$

and so $(c \wedge x)m \in \bigcap_{i=1}^k N_i$, for every $m \in M$. Therefore, $\bigcap_{i=1}^k N_i$ is a P'-primary A-ideal of M. □

Definition 4.3. Let M be an A-module, N be a proper A-ideal of M and there exist proper A-ideals $A_1, A_2, \cdots, A_n$ of M such that A_i is a P_i-primary of M, for every $1 \leq i \leq n$ and $N = A_1 \cap A_2 \cap \cdots \cap A_n$. Then we say $A_1 \cap A_2 \cap \cdots \cap A_n$ is a *primary decomposition* of N and so N has a primary decomposition. Furthermore, this decomposition is *reduced* if

(i) $A_i \not\supseteq \bigcap_{i\neq j} A_j$,
(ii) $rad(Q_{A_i}) \neq rad(Q_{A_j})$, for every $1 \leq i, j \leq n$.

Example 4.2. (i) Let A be unital and finite. If we consider A as A-module, where $xy = x.y$, for every $x, y \in A$, then since $ax \leq 1x = x$, for every $a, x \in A$, any ideal of A is an A-ideal of A and by Lemma 2.6, every prime ideal of A is a prime A-ideal of A. Hence, by Proposition 2.2, every proper A-ideal of A has a primary decomposition.
(ii) In Example 4.1, $\{0, 2\} \cap \{0, 1\}$ is a primary decomposition of $\{0\}$. This decomposition is reduced, too.

Theorem 4.5. *Let A be implicative, M be an A-module, N be an A-ideal of M that has a primary decomposition and $I \cap P \neq \{0\}$ implies that $I \subseteq P$, for every $I \in \mathcal{I}(A)$ and $P \in \mathcal{PI}(A)$. Then N has a reduced primary decomposition.*

Proof. Let $N = A_1 \cap \cdots \cap A_n$, where A_i is a primary ideal of M, for every $1 \leq i \leq n$. If $A_j \supseteq \bigcap_{i=1}^{n} A_i$, where $i \neq j$, then we set $N = A_1 \cap \cdots \cap A_{j-1} \cap A_{j+1} \cap \cdots \cap A_n$, for every $1 \leq j \leq n$ and so by renumbering, $N = \bigcap_{i=1}^{k} A'_i$, where $k \leq n$ and $A'_j \not\supseteq \bigcap_{i=1}^{k} A'_i$, for every $1 \leq j \leq k$. Let $T = \{P_1, \cdots, P_m\}$, where $P_i \neq P_j$ and $m \leq k$, for every $1 \leq i, j \leq m$ and $rad(Q_{A'_i}) = P_i$, for some $1 \leq i \leq k$. Now, we resume

$$N = (A'_{i_1} \cap \cdots \cap A'_{i_t}) \cap (A'_{j_1} \cap \cdots \cap A'_{j_l}) \cap \cdots \cap (A'_{s_1} \cap \cdots \cap A'_{s_w}),$$

where by Lemma 4.4,

$$rad(Q_{\bigcap_{h=1}^{t} A'_{i_h}}) = \bigcap_{h=1}^{t} rad(Q_{A'_{i_h}}) = \bigcap_{h=1}^{t} p_1 = p_1, \cdots,$$
$$rad(Q_{\bigcap_{h=1}^{w} A'_{s_h}}) = \bigcap_{h=1}^{w} rad(Q_{A'_{s_h}}) = \bigcap_{h=1}^{w} p_m = p_m.$$

Therefore, I has a reduced primary decomposition. □

Definition 4.4. Let M be an A-module. Then
(i) M is called *Noetherian* if M satisfies the ascending chain condition (ACC): that is any chain $N_1 \subseteq N_2 \subseteq \cdots$ of A-ideal of M is stationary.
(ii) We say M satisfies the *maximum condition*, if every non-empty family of submodules of M has a maximum element.

Example 4.3. Every finite A-module is a Noetherian A-module.

Theorem 4.6. *Let M be an A-module. Then M is Noetherian if and only if M has maximum condition.*

Proof. The proof is routine. □

Definition 4.5. Let M be an A-module. Then M is called a *Boolean A-module* if $ax \oplus ay \leq a(x \oplus y)$, for every $a \in A$ and $x, y \in M$.

Example 4.4. If M is a Boolean-algebra, then every A-module M is a Boolean A-module. Since $x \leq x \oplus y$ and $y \leq x \oplus y$, by Lemma 2.7 (f), $ax \leq a(x \oplus y)$ and $ay \leq a(x \oplus y)$, for every $a \in A$ and $x, y \in M$ and so by Lemma 2.1 (ii), $ax \oplus ay \leq a(x \oplus y) \oplus ay$ and $a(x \oplus y) \oplus ay \leq a(x \oplus y) \oplus a(x \oplus y) = a(x \oplus y)$. Hence, $ax \oplus ay \leq a(x \oplus y)$, for every $a \in A$ and $x, y \in M$.

Theorem 4.7. *Let A be finite and M be a Boolean Noetherian A-module. Then every proper A-ideal of M has a reduced primary decomposition.*

Proof. Let

$$T = \{N : N \text{ is a proper } A\text{-ideal of M such that N has no any reduced primary decomposition}\}.$$

We show that $T = \emptyset$. Let $T \neq \emptyset$. Since M is Noetherian, by Theorem 4.6, T has a maximum element G. It is clear that G is not a primary A-ideal of M. So there exists $x \in A$ and $m \in M$ such that $xm \in G$, $m \notin G$ and for every $c \in A \setminus P$, $(c \wedge x)M \nsubseteq G$, where $P \in \mathcal{PI}_{Q_G}(A)$. We give an index $i \geq 1$ to every $c \in A \setminus P$. Let $B_i = \{m \in M : (c_1 \wedge c_2 \cdots \wedge c_i \wedge x)m \in G\}$, for every $i \geq 1$ and $m \in B_i$. Then

$$(c_1 \wedge c_2 \wedge \cdots \wedge c_i \wedge c_{i+1} \wedge x)m \leq (c_1 \wedge \cdots \wedge c_i \wedge x)m \in G$$

and so $(c_1 \wedge c_2 \wedge \cdots \wedge c_i \wedge c_{i+1} \wedge x)m \in G$. Hence, $m \in B_{i+1}$ and so $B_i \subseteq B_{i+1}$, for every $i \geq 1$. Since M is Noetherian, there exists $k \in \mathbb{N}$ such that $B_k = B_n$, for every $n \geq k$. We show that B_k is an A-ideal of M. Let $m_1, m_2 \in B_k$. Then $(c_1 \wedge \cdots \wedge c_k \wedge x)m_1, (c_1 \wedge \cdots \wedge c_k \wedge x)m_2 \in G$. By Lemma 2.7 (h),

$$(c_1 \wedge \cdots \wedge c_k \wedge x).(m_1 \oplus m_2) \leq (c_1 \wedge \cdots \wedge c_k \wedge x)m_1 \oplus (c_1 \wedge \cdots \wedge c_k \wedge x)m_2 \in G$$

and so $(c_1 \wedge \cdots \wedge c_k \wedge x).(m_1 \oplus m_2) \in G$. Hence, $m_1 \oplus m_2 \in B_K$. Now, let $m_1 \leq m_2 \in B_k$. Since $(c_1 \wedge \cdots \wedge c_k \wedge x)m_1 \leq (c_1 \wedge \cdots \wedge c_k \wedge x)m_2 \in G$, $(c_1 \wedge \cdots \wedge c_k \wedge x)m_1 \in G$ and so $m_1 \in B_k$. On the other hand,

$$\begin{aligned}(c_1 \wedge \cdots \wedge c_k \wedge x)(am) &= ((c_1 \wedge \cdots \wedge c_k \wedge x).a)m \\ &\leq (c_1 \wedge \cdots \wedge c_k \wedge x \wedge a)m \leq (c_1 \wedge \cdots \wedge c_k \wedge x)m \in G\end{aligned}$$

and so $am \in B_k$, for every $a \in A$ and $m \in B_k$. Hence, B_k is an A-ideal of M.
Let $D = \{(c_1 \wedge \cdots \wedge c_k \wedge x)m' \oplus g : m' \in M \textit{ and } g \in G\}$. We show that D is an A-ideal of M. Let $d_1, d_2 \in D$. It is easy to show that $d_1 \oplus d_2 \in D$. Let $d \in D$ and $a \in A$. So there exist $m' \in M$ and $g \in G$ such that

$$\begin{aligned} ad &= a((c_1 \wedge \cdots \wedge c_k \wedge x)m' \oplus g) \leq a((c_1 \wedge \cdots \wedge c_k \wedge x)m') \oplus ag \\ &= (a.(c_1 \wedge \cdots \wedge c_k \wedge x))m' \oplus ag \\ &\leq (a \wedge c_1 \wedge \cdots \wedge c_k \wedge x)m' \oplus ag \leq (c_1 \wedge \cdots \wedge c_k \wedge x)m' \oplus ag \in D \end{aligned}$$

Hence, D is an A-ideal of M. Now, we prove that $G = D \cap B_k$, $G \subsetneq D$ and $G \subsetneq B_k$. Let $g \in G$. Then $g = (c_1 \wedge \cdots \wedge c_k \wedge x)0 \oplus g \in D$. On the other hand, $(c_1 \wedge \cdots \wedge c_k \wedge x)g \in G$. So $g \in B_k$ and so $G \subseteq D \cap B_k$. Let $m \in D \cap B_k$. Since $m \in B_k$, $(c_1 \wedge \cdots \wedge c_k \wedge x)m \in G$ and since $m \in D$, there exist $m' \in M$ and $g \in G$ such that $m = (c_1 \wedge \cdots \wedge c_k \wedge x)m' \oplus g$. Since

$$\begin{aligned} &((c_1 \wedge \cdots \wedge c_k \wedge x).(c_1 \wedge \cdots \wedge c_k \wedge x))m' \oplus (c_1 \wedge \cdots \wedge c_k \wedge x)g \\ &\quad = (c_1 \wedge \cdots \wedge c_k \wedge x)((c_1 \wedge \cdots \wedge c_k \wedge x)m') \oplus (c_1 \wedge \cdots \wedge c_k \wedge x)g \\ &\quad = (c_1 \wedge \cdots \wedge c_k \wedge x)((c_1 \wedge \cdots \wedge c_k \wedge x)m' \oplus g) = (c_1 \wedge \cdots \wedge c_k \wedge x)m \in G, \end{aligned}$$

by Lemma 2.6,

$$\begin{aligned}(c_1 \wedge \cdots \wedge c_k \wedge x)m' &= ((c_1 \wedge \cdots \wedge c_k \wedge x) \wedge (c_1 \wedge \cdots \wedge c_k \wedge x))m' \\ &= ((c_1 \wedge \cdots \wedge c_k \wedge x).(c_1 \wedge \cdots \wedge c_k \wedge x))m' \in G\end{aligned}$$

and so $m \in G$. Hence, $D \cap B_k \subseteq G$. It is enough to show that $G \subsetneq D$ and $G \subsetneq B_k$. We have $(c \wedge x)M \nsubseteq G$, for every $c \in A \setminus P$, where $P \in \mathcal{PI}_{Q_G}(A)$. Then there exists $t \in M$ such that $(c \wedge x)t \notin G$. But if $c = c_1 \wedge \cdots \wedge c_k$, then $(c \wedge x)t = (c \wedge x)t + 0 \in D$ and so $G \subsetneq D$. On the other hand, there existed $m \in M$ and $x \in A$ such that $xm \in G$ and $m \notin G$, but $(c_1 \wedge \cdots \wedge c_k \wedge x)m = ((c_1 \wedge \cdots \wedge c_k).x)m = (c_1 \wedge \cdots \wedge c_k)(xm) \in G$. It means that $m \in B_k$ and so $G \subsetneq B_k$. By the maximality of G, D and B_k have primary decomposition. It results that G has primary decomposition, which is a contradiction. Therefore, $T = \emptyset$.

□

5. Conclusion

The equivalence between the category of lu-modules over (R, v) and the category of MV-modules over $\Gamma(R, v)$ was proved by Di Nola, where (R, v) is an lu-ring [6]. We studied ideals in MV-algebras and presented definition of radical of an ideal in MV-algebras by prime ideals that it was defined by maximal ideals in [9]. Also, we introduced the notion of primary ideals in MV-algebras. Then we studied A-ideals in MV-modules and defined the notions of primary and P-primary A-ideals of an MV-module in order to define primary decomposition of A-ideals. Also, we introduced MV-modules that their proper A-ideals have reduced primary decomposition. In fact, we opened new fields to anyone that is interested to studying and development of MV-modules.

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