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Non-trivial periodic solutions for a class of damped vibration problems

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Abstract. Employing a very recent local minimum theorem for differentiable functionals, we prove the existence of at least one non-trivial periodic solution for a class of damped vibration problems.

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1. Introduction

Consider the following damped vibration problem

$$\begin{cases} -\ddot{u}(t) - q(t)\dot{u}(t) + A(t)u(t) = \lambda \nabla F(t, u(t)) & a.e.\ \ t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases} \tag{1}$$

where $T > 0$, $q \in L^1(0, T; \mathbb{R})$, $Q(t) = \int_0^t q(s)ds$ for all $t \in [0, T]$, $Q(T) = 0$, $A : [0, T] \to \mathbb{R}^{N\times N}$ is a continuous map from the interval $[0, T]$ to the set of N-order symmetric matrices, $\lambda > 0$, $\mu \geq 0$ and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable with respect to t, for all $u \in \mathbb{R}^N$, continuously differentiable in u, for almost every $t \in [0, T]$, satisfies the following standard summability condition

$$\sup_{|\xi|\leq a} \max\{|F(\cdot, \xi)|,\ |\nabla F(\cdot, \xi)|\} \in L^1([0, T]) \tag{2}$$

for any $a > 0$, and $F(t, 0, \ldots, 0) = 0$ for all $t \in [0, T]$.

Assume that ∇F is continuous in $[0, T] \times \mathbb{R}^N$, then the condition (2) is satisfied.

Inspired by the monographs [19, 20], the existence and multiplicity of periodic solutions for Hamiltonian systems, as a special case of dynamical systems which are very important in the study of fluid mechanics, gas dynamics, nuclear physics and relativistic mechanics, have been investigated in many papers (see [1, 4, 5, 11, 12, 14, 15, 22, 24] and the references therein). For example, [11] Cordaro established a multiplicity result to an eigenvalue problem related to second-order Hamiltonian systems, and proved the existence of an open interval of positive eigenvalues in which the problem admits three distinct periodic solutions. In [14] Faraci studied the multiplicity of solutions of a second order nonautonomous system. In [4] Bonanno and Livrea ensured the existence of infinitely many periodic solutions for a class of second-order Hamiltonian systems under an appropriate oscillating behavior of the nonlinear term. Moreover, they obtained the multiplicity of periodic solutions for the system with a coercive potential and also in the noncoercive case.

Moreover, very recently, some researchers have paid attention to the existence and multiplicity of solutions for damped vibration problems by using the variational method, for instance, see [8, 9, 10, 17, 23, 25, 26, 27] and references therein. For example, the authors in [26] based on variational methods and critical point theory studied the existence of one solution and multiple solutions for damped vibration problems. Wu and Chen in [25] based on variational principle presented three existence theorems for periodic solutions of a class of damped vibration problems. In [27] the authors by a symmetric mountain pass theorem and a generalized mountain pass theorem, an existence result and a multiplicity result of homoclinic solutions of damped vibration problems. Chen in [8, 9] studied a class of non-periodic damped vibration systems with subquadratic terms and with asymptotically quadratic terms, respectively, and obtained infinitely many nontrivial homoclinic orbits by a variant fountain theorem developed recently by Zou [29]. In [17] using variational methods and critical point theory the existence of three distinct weak solutions for a class of perturbed damped vibration problems with nonlinear terms depending on two real parameters was investigated.

In the present paper, motivated by [26], using a very recent local minimum theorem for differentiable functionals due to Bonanno [2, Theorem 5.1](Theorem 2.1), we establish the existence of at least one non-trivial solution for the problem (1) for any fixed positive parameter λ belonging to an exact interval.

2. Preliminaries

Our main tool is a local minimum theorem, very recently obtained by Bonanno [2, Theorem 5.1] that we here recall in its equivalent formulation [2, Proposition 2.1 and Remark 2.1]) (see also [21] for the related result).

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\vartheta(r_1, r_2) = \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Theorem 2.1. [2, Theorem 5.1] *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\vartheta(r_1, r_2) < \rho(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in]\frac{1}{\rho(r_1, r_2)}, \frac{1}{\vartheta(r_1, r_2)}[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ $\forall u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

For more details on Theorem 2.1, we refer the reader to [3, 6, 7, 13, 16, 18], where it has already been applied to nonlinear second-order differential problems.

We assume that the matrix A satisfies the following conditions:

$(A1)$ $A(t) = (a_{kl}(t))$, $k = 1, \dots, N$, $l = 1, \dots, N$, is a symmetric matrix with $a_{kl} \in L^\infty[0,T]$ for any $t \in [0,T]$,

$(A2)$ there exists $\delta > 0$ such that $(A(t)\xi, \xi) \geq \delta|\xi|^2$ for any $\xi \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$.

Let us recall some basic concepts. Denote

$$E = \{u : [0,T] \to \mathbb{R}^N |\ u \text{ is absolutely continuous},\ \ u(0) = u(T),\ \ \dot{u} \in L^2([0,T], \mathbb{R}^N)\}$$

with the inner product

$$\prec u, v \succ_E = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (u(t), v(t))]dt.$$

The corresponding norm is defined by

$$\|u\|_E = \Big(\int_0^T (|\dot{u}(t)|^2 + |u(t)|^2)dt\Big)^{\frac{1}{2}}\ \forall\ u \in E.$$

For every $u, v \in E$, we define

$$\prec u, v \succ = \int_0^T [e^{Q(T)}(\dot{u}(t), \dot{v}(t)) + e^{Q(T)}(A(t)u(t), v(t))]dt,$$

and we observe that, by the assumptions $(A1)$ and $(A2)$, it defines an inner product in E. Then E is a separable and reflexive Banach space with the norm

$$\|u\| = \prec u, u \succ^{\frac{1}{2}} \quad , \quad \forall u \in E.$$

Obviously, E is an uniformly convex Banach space.

Clearly, the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_E$ (see [15]).

Since $(E, \|\cdot\|)$ is compactly embedded in $C([0,T], \mathbb{R}^N)$ (see [19]), there exists a positive constant c such that

$$\|u\|_\infty \leq c \parallel u \parallel, \tag{3}$$

where $\|u\|_\infty = \max_{t\in[0,T]} \mid u(t) \mid$.

We mean by a (weak) solution of the problem (1), any function $u \in E$ such that

$$\int_0^T e^{Q(t)}(\dot{u}(t), \dot{v}(t))dt + \int_0^T e^{Q(t)}(A(t)u(t), v(t))dt - \lambda \int_0^T e^{Q(t)}(\nabla F(t, u(t)), v(t))dt = 0$$

for every $v \in E$.

3. Main result

Given a non-negative constant θ and a nonzero point $x_0 \in \mathbb{R}^N$ such that

$$\theta \neq c|x_0|\Big((\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt\Big)^{\frac{1}{2}},$$

put

$$a(\theta, x_0) := \frac{\int_0^T e^{Q(t)} \sup_{|\xi|\leq\theta} F(t,\xi)dt - \int_0^T e^{Q(t)} F(t, x_0)dt}{\theta^2 - c^2|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}.$$

We formulate our main result as follows.

Theorem 3.1. *Suppose that Assumptions (A1) and (A2) hold. Assume that there exist a non-negative constant θ_1, a positive constant θ_2 and a nonzero point $x_0 \in \mathbb{R}^N$ with*

$$\frac{\theta_1}{c(\delta \int_0^T e^{Q(t)}dt)^{\frac{1}{2}}} < |x_0| < \frac{\theta_2}{c((\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)}dt)^{\frac{1}{2}}}$$

such that

(a$_1$) $\quad a(\theta_2, x_0) < a(\theta_1, x_0)$.

Then, for any $\lambda \in \left] \frac{1}{2c^2}\frac{1}{a(\theta_1, x_0)}, \ \frac{1}{2c^2}\frac{1}{a(\theta_2, x_0)} \right[$ *the problem (1) admits at least one non-trivial solution $u_0 \in E$ such that* $\frac{\theta_1}{c} < ||u_0|| < \frac{\theta_2}{c}$.

Proof. In order to apply Theorem 2.1 to our problem, we take $X = E$ and we introduce the functionals $\Phi, \ \Psi : X \to \mathbb{R}$ defined as follows

$$\Phi(u) = \frac{1}{2}\|u\|^2$$

and

$$\Psi(u) = \int_0^T e^{Q(t)}(F(t, u(t))dt$$

for every $u \in X$. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^T e^{Q(t)}(\nabla F(t, u(t)), v(t))dt,$$

for every $v \in X$. $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this end, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$, then u_n converges uniformly to u on $[0, T]$ as $n \to \infty$; see [19]. Since F is continuously differentiable in u for almost every $t \in [0, T]$, ∇F is continuous in $\mathbb{R}^{\mathbb{N}}$ for every $t \in [0, T]$, so

$$\nabla F(t, u_n) \to \nabla F(t, u) \text{ as } n \to \infty.$$

Hence, $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [28]. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^T e^{Q(t)}(\dot{u}(t), \dot{v}(t))dt + \int_0^T e^{Q(t)}(A(t)u(t), v(t))dt,$$

for every $v \in X$. Since Φ' is uniformly monotone on X, coercive and hemicontinuous in X, applying [28, Theorem 26. A] it admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Now, put $r_1 := \frac{1}{2}(\frac{\theta_1}{c})^2$, $r_2 := \frac{1}{2}(\frac{\theta_2}{c})^2$ and $w(t) := x_0$ for all $t \in [0, T]$. We clearly observe that $w \in X$ and

$$\frac{1}{2}|x_0|^2\delta \int_0^T e^{Q(t)}dt \le \Phi(w) \le \frac{1}{2}|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)}dt.$$

This together with the condition

$$\frac{\theta_1}{c(\delta \int_0^T e^{Q(t)}dt)^{\frac{1}{2}}} < |x_0| < \frac{\theta_2}{c((\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)}dt)^{\frac{1}{2}}}$$

yields

$$r_1 < \Phi(w) < r_2.$$

Bearing (3) in mind, we see that

$$\begin{aligned}\Phi^{-1}(]-\infty, r_2[) &= \{u \in X;\ \Phi(u) < r_2\} \\ &= \left\{u \in X;\ \frac{||u||^2}{2} < r_2\right\} \\ &\subseteq \{u \in X;\ |u(t)| \le \theta_2 \text{ for each } t \in [0,T]\},\end{aligned}$$

and it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \int_0^T e^{Q(t)}F(t,u(t))dt \le \int_0^T e^{Q(t)} \sup_{|\xi|\le\theta_2} F(t,\xi)dt.$$

Therefore, one has

$$\begin{aligned}\beta(r_1,r_2) &\le \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ &\le \frac{\int_0^T e^{Q(t)\sup_{|\xi|\le\theta_2}} F(t,\xi)dt - \Psi(w)}{r_2 - \Phi(w)} \\ &\le \frac{\int_0^T e^{Q(t)} \sup_{|\xi|\le\theta_2} F(t,\xi)dt - \int_0^T e^{Q(t)} F(t,x_0)dt}{\frac{1}{2}(\frac{\theta_2}{c})^2 - \frac{1}{2}|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt} \\ &= \frac{2c^2(\int_0^T e^{Q(t)} \sup_{|\xi|\le\theta_2} F(t,\xi)dt - \int_0^T e^{Q(t)} F(t,x_0)dt)}{\theta_2^2 - c^2|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt} \\ &= 2c^2 a(\theta_2, x_0).\end{aligned}$$

On the other hand, arguing as before, one has

$$\begin{aligned}\rho(r_1,r_2) &\ge \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(w) - r_1} \\ &\ge \frac{\Psi(w) - \int_0^T e^{Q(t)} \sup_{|\xi|\le\theta_1} F(t,\xi)dt}{\Phi(w) - r_1} \\ &\ge \frac{\int_0^T e^{Q(t)} F(t,x_0)dt - \int_0^T e^{Q(t)} \sup_{|\xi|\le\theta_1} F(t,\xi)dt}{\frac{1}{2}|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt - \frac{1}{2}(\frac{\theta_1}{c})^2} \\ &= \frac{2c^2(\int_0^T e^{Q(t)} F(t,x_0)dt - \int_0^T e^{Q(t)} \sup_{|\xi|\le\theta_1} F(t,\xi)dt)}{c^2|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt - \theta_1^2} \\ &= 2c^2 a(\theta_1, x_0).\end{aligned}$$

Hence, from Assumption (a_2), one has $\beta(r_1,r_2) < \rho(r_1,r_2)$. Therefore, employing Theorem 2.1, for each $\lambda \in \left]\frac{1}{2c^2}\frac{1}{a(\theta_1,x_0)}, \frac{1}{2c^2}\frac{1}{a(\theta_2,x_0)}\right[$, the functional $\Phi - \lambda\Psi$ admits at least one critical point $u_0 \in X$ such that $r_1 < \Phi(u) < r_2$, that is $\frac{\theta_1}{c} < ||u_0|| < \frac{\theta_2}{c}$. Since the solutions of the problem (1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$ (see [26, Theorem 2.2]), we have the conclusion. $\square$

Now, we point out an immediate consequence of Theorem 3.1.

Theorem 3.2. *Suppose that Assumptions (A1) and (A2) hold. Assume that there exist a positive constant θ and a nonzero point $x_0 \in \mathbb{R}^N$ with*

$$|x_0| < \frac{\theta}{c((\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)} dt)^{\frac{1}{2}}}$$

such that

(a$_2$) $$\frac{\int_0^T e^{Q(t)} \sup_{|\xi| \le \theta} F(t,\xi) dt}{\theta^2} < \frac{1}{c^2|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)} dt} \int_0^T e^{Q(t)} F(t,x_0) dt;$$

(a$_3$) $F(t,0) = 0$ *for a.e.* $t \in [0,T]$.

Then, for every $\lambda \in \left] \frac{|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)} dt}{2 \int_0^T e^{Q(t)} F(t,x_0) dt}, \frac{\theta^2}{2c^2 \int_0^T e^{Q(t)} \sup_{|\xi| \le \theta} F(t,\xi) dt} \right[$ *the problem (1) admits at least one non-trivial solution $u_0 \in E$ such that $\|u_0\|_\infty < \theta$.*

Proof. The conclusion follows from Theorem 3.1, by taking $\theta_1 = 0$ and $\theta_2 = \theta$. Indeed, owing to Assumptions (a$_2$) and (a$_3$), one has

$$\begin{aligned} a(\theta, x_0) &= \frac{\int_0^T e^{Q(t)} \sup_{|\xi| \le \theta} F(t,\xi) dt - \int_0^T e^{Q(t)} F(t,x_0) dt}{\theta^2 - c^2|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)} dt} \\ &< \frac{\left(\frac{\theta^2}{c^2|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)} dt} - 1\right) \int_0^T e^{Q(t)} F(t,x_0) dt}{\theta^2 - c^2|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)} dt} \\ &= \frac{\int_0^T e^{Q(t)} F(t,x_0) dt}{c^2|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)} dt} = a(0, x_0) \end{aligned}$$

Hence, taking (3) into account, the conclusion follows from Theorem 3.1. $\square$

A consequence of Theorem 3.2 is the following existence result.

Theorem 3.3. *Suppose that Assumptions (A1) and (A2) hold. Let $b \in L^1([0,T])$ such that $b(t) \ge 0$ a.e. $t \in [0,T]$ and $b \not\equiv 0$, $G \in C^1(\mathbb{R}^N, \mathbb{R})$ such that $G(0,\ldots,0) = 0$ and*

$$\lim_{x \to 0^+} \frac{\max\limits_{|\xi| \le x} G(\xi)}{|x|^2} = +\infty. \tag{4}$$

Then, for each $\lambda \in (0, \ \lambda^*)$, where $\lambda^* := \dfrac{1}{2c^2 \int_0^T e^{Q(t)} b(t) dt} \sup_{\theta > 0} \frac{\theta^2}{\max_{|\xi| \le \theta} G(\xi)}$ *the problem*

$$\begin{cases} -\ddot{u}(t) - q(t)\dot{u}(t) + A(t)u(t) = \lambda \nabla b(t) G(u(t)) & a.e. \ t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases} \tag{5}$$

admits at least a nontrivial periodic solution.

Proof. For fixed $\lambda \in (0, \ \lambda^*)$, there exists a positive constant θ such that

$$\lambda < \frac{1}{2c^2 \int_0^T e^{Q(t)} b(t) dt} \frac{\theta^2}{\max_{|\xi| \le \theta} G(\xi)}.$$

Moreover, using (4) we can choose point x_0 satisfying $|x_0| < \frac{\theta}{c((\sum_{i,j=1}^{N}\|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt)^{\frac{1}{2}}}$ such that

$$\frac{(\sum_{i,j=1}^{N}\|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}{2\lambda\int_0^T e^{Q(t)}b(t)dt} < \frac{G(x_0)}{|x_0|^2}.$$

Hence, Theorem 3.2 leads to the conclusion. □

Now, we present the following example to illustrate the result.

Example 3.1. Let $N = 2$ and $T = 1$. Let $A : [0,1] \to \mathbb{R}^{2\times 2}$ be the identity matrix and let $G(\xi_1,\xi_2) = e^{-(\xi_1^2+\xi_2^2)}(\xi_1+\xi_2+\xi_1^2+\xi_2^2)$ for all $(\xi_1,\xi_2) \in \mathbb{R}^2$, $b(t) = 2t$ for all $t \in [0,1]$ and $q(t) = 2t-1$ for all $t \in [0,1]$. It is clear that

$$\lim_{x\to 0^+}\frac{\max\limits_{|(\xi_1,\xi_2)|\le x} G(\xi_1,\xi_2)}{x^2} = +\infty.$$

Hence, using Theorem 3.3, for each

$$\lambda \in \left(0,\ \frac{e}{c^2(e-1)}\sup_{\theta>0}\frac{\theta^2}{\max_{|(\xi_1,\xi_2)|\le\theta} e^{-(\xi_1^2+\xi_2^2)}(\xi_1+\xi_2+\xi_1^2+\xi_2^2)}\right),$$

the problem (5), in this case, admits at least one nontrivial periodic solution.

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