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Radical-Depended Graph of a Commutative Ring

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Abstract. Let R be a commutative ring with identity and $\sqrt{I}$ be the radical of an ideal I of R. We introduce the radical-depended graph $\mathcal{G}_I(R)$ whose vertex set is $\{x \in R \setminus \sqrt{I} \mid xy \in I$ for some $y \in R \setminus \sqrt{I}\}$ and distinct vertices x and y are adjacent if and only if $xy \in I$. In this paper, several properties of $\mathcal{G}_I(R)$ are investigated and some results on the parameters of this graph are given. It follows that if I is a quasi primary ideal, then $\mathcal{G}_I(R) = \emptyset$. It is shown that if I is a 2-absorbing ideal of R which is not quasi primary, then $\mathcal{G}_I(R)$ is the complete bipartite graph $K_{1,1}$ or $K_{m,n}$ for some $m, n \geq 2$. Moreover, it is proved that $\mathcal{G}_I(R)$ is a connected graph with diameter at most 3, and if it has a cycle, then its girth is at most 4. Also, it is shown that if R is a Noetherian ring, then the clique number of $\mathcal{G}_I(R)$ is equal to $|\operatorname{Min}(R/I)|$ for every ideal I of R.

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1. Introduction

The zero-divisor graph of a commutative ring was introduced by I. Beck in [5] and further studied by D. D. Anderson and M. Naseer in [3]. However, they let all the elements of R be vertices of the graph, and they were mainly interested in colorings. We adopt the approach used by D. F. Anderson and P. S. Livingston in [2] and consider only nonzero zero-divisors as vertices of the graph. Let R be a commutative ring with nonzero identity, I a proper ideal of R, and $Z(R)$ the set of zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the graph with vertices $Z(R)^* = Z(R)\setminus 0$, and distinct vertices x and y are adjacent if and only if $xy = 0$. In [12], Redmond introduced an ideal-based zero-divisor graph of R as a generalization of $\Gamma(R)$. Let I be an ideal of R. The ideal-based zero-divisor graph of R is the graph $\Gamma_I(R)$ with vertices $\{x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. Therefore, if $I = 0$, then $\Gamma_I(R) = \Gamma(R)$, and I is a prime ideal if and only if $\Gamma_I(R) = \emptyset$.

In this paper, we study the radical-depended subgraph $\mathcal{G}_I(R)$ of R that is a subgraph of $\Gamma_{\sqrt{I}}(R)$ with the vertices $\{x \in R \setminus \sqrt{I} \mid xy \in I$ for some $y \in R \setminus \sqrt{I}\}$ and distinct vertices x and y are adjacent if and only if $xy \in I$. Therefore, I is a quasi primary ideal (i.e., $\sqrt{I}$ is a prime ideal [9]) if and only if $\mathcal{G}_I(R) = \emptyset$, and if I is a radical ideal, then $\mathcal{G}_I(R) = \Gamma_{\sqrt{I}}(R)$.

Let us recall some notions and notations from graph theory that will be used later. A graph is said to be connected if for each pair of distinct vertices x and y, there is a

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finite sequence of distinct vertices $x = x_1, \dots, x_n = y$ such that each pair $\{x_i, x_{i+1}\}$ is an edge. Such a sequence is said to be a path and the distance, $d(x, y)$, between connected vertices x and y is the length of a shortest path connecting them. The diameter of a connected graph G, denoted $\mathrm{diam}(G)$, is the supremum of the distances between vertices (and let $d(x, y) = \infty$ if no such path exists). A cycle in a graph G is a path that begins and ends at the same vertex. The girth of G, written $\mathrm{gr}(G)$, is the length of the shortest cycle in G (and $\mathrm{gr}(G) = \infty$ if G has no cycles). A vertex x of a connected graph G is a cut-point of G if $G \setminus \{x\}$ is not connected. The connectivity of a graph G, denoted by $\kappa(G)$, is defined to be the minimum number of vertices which is necessary to remove from G in order to produce a disconnected graph. A complete graph is a graph where all vertices are adjacent. The complete graph on n vertices is denoted by K_n. For a graph G, a complete subgraph of G is called a clique. The clique number, $\omega(G)$, is the greatest integer $n \geq 1$ such that $K_n \subseteq G$, and $\omega(G) = \infty$ if $K_n \subseteq G$ for all $n \geq 1$. The complete bipartite graph, denoted $K_{m,n}$, is the graph whose vertex set is the disjoint union of two sets, V_1 and V_2, satisfying $|V_1| = m$, $|V_2| = n$, and whose edge set is precisely $\{\{v_1, v_2\} \mid v_1 \in V_1 \text{ and } v_2 \in V_2\}$.

Here is a brief summary of the paper. It is shown that $\mathcal{G}_I(R)$ is a connected graph with $\mathrm{diam}(\mathcal{G}_I(R)) \leq 3$ (Theorem 2.3), and if it has a cycle, then $\mathrm{gr}(G) \leq 4$ (Theorem 2.11). This graph has no cut-points (Theorem 2.4), and we provide bounds on $\kappa(\mathcal{G}_I(R))$ (Theorem 2.5).

A proper ideal I is called n-absorbing if $x_1 \cdots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I (see [1, 4, 11]). It is shown that if I is a 2-absorbing ideal of R, then $\mathcal{G}_I(R) = \emptyset$ or $\mathcal{G}_I(R) \cong K_{1,1}$ or $\mathcal{G}_I(R) \cong K_{m,n}$ for some $m, n \geq 2$ (Theorem 2.8). Thus, in this case, $\mathrm{diam}(\mathcal{G}_I(R)) \in \{0, 1, 2\}$ (Corollary 2.9) and $\mathrm{gr}(G) \in \{0, 4, \infty\}$ (Remark 2.2).

It is proved that if $I = q_1 \cap \cdots \cap q_m$ is a minimal primary decomposition of an ideal I of R with $n(\leq m)$ isolated prime ideals, then $\omega(\mathcal{G}_I(R)) \leq n$. In particular, if $m = n$, then $\omega(\mathcal{G}_I(R)) = n$ (Theorem 2.13). Thus, if R is a Noetherian ring, then for every ideal I of R, $\omega(\mathcal{G}_I(R)) = |\mathrm{Min}(R/I)|$, where $\mathrm{Min}(R/I)$ is the set of all minimal prime ideals of R/I (Corollary 2.14). It is also obtained that if I is an n-absorbing ideal of R, then $\omega(\mathcal{G}_I(R)) = |Min(R/I)| \leq n$ (Corollary 2.15).

2. On Radical-Depended Graph

Lemma 2.1. *Let R be a ring. If I is a quasi primary ideal, then $\mathcal{G}_I(R) = \Gamma_{\sqrt{I}}(R) = \Gamma(R/\sqrt{I}) = \emptyset$. In particular, this equality holds when I is an ideal of a zero-dimensional ring R.*

For a graph G, the vertices set and the edges set of G are denoted by $V(G)$ and $E(G)$ respectively. In the following example, we see that $\Gamma_{\sqrt{I}}(R)$ and its subgraph $\mathcal{G}_I(R)$ may or may not isomorphic graphs.

Example 2.1. (1) Let $R = \mathbb{Z}_{24}$ and $I =< 12 >$. Then $\mathcal{G}_I(R) \ncong \Gamma_{\sqrt{I}}(R)$, since the vertices $2, 10, 14, 22$ of $\Gamma_{\sqrt{I}}(R)$ are not vertices of $\mathcal{G}_I(R)$.
(2) Let $R = \mathbb{Z}$ and $I = 12\mathbb{Z}$. Then $V(\Gamma_{\sqrt{I}}(R)) = \{6k+2, 6k+3, 6k+4 \mid k \in \mathbb{Z}\}$, $E(\Gamma_{\sqrt{I}}(R)) = \big\{\{6k+2, 6k'+3\} \mid k, k' \in \mathbb{Z}\big\} \cup \big\{\{6k+3, 6k'+4\} \mid k, k' \in \mathbb{Z}\big\}$, $V(\mathcal{G}_I(R)) = \{12k+4, 6k+3, 12k+8 \mid k \in \mathbb{Z}\}$ and $E(\mathcal{G}_I(R)) = \big\{\{12k+4, 6k'+3\} \mid k, k' \in \mathbb{Z}\big\} \cup \big\{\{6k+3, 12k'+8\} \mid k, k' \in \mathbb{Z}\big\}$. It is easy to check that $\varphi : \Gamma_{\sqrt{I}}(R) \to \mathcal{G}_I(R)$

defined by $\varphi(6k+2) = 12k+4$, $\varphi(6k+3) = 6k+3$ and $\varphi(6k+4) = 12k+8$ is a graph isomorphism.

Let S be a nonempty set of vertices of a graph G. The induced subgraph generated by S, denoted by $< S >$, is the subgraph H of G with vertex set S where vertices are adjacent in H precisely when adjacent in G.

Remark 2.1. Let R be a ring, I be an ideal of R and $adj(x) = \{y + I \in \Gamma(R/I) \mid xy \in I\}$. Let $< \Lambda >$ be the induced subgraph of $\Gamma(R/I)$ generated by

$$\Lambda = \{x + I \in \Gamma(R/I) \mid x \notin \sqrt{I} \text{ and } adj(x) \nsubseteq \sqrt{I}/I\}.$$

$< \Lambda >$ is also a subgraph of $\Gamma_{\sqrt{I}/I}(R/I)$. In Example 2.1, $2+I$ and $3+I$ are adjacent in $\Gamma_{\sqrt{I}/I}(R/I)$, but they are not adjacent in $< \Lambda >$. Hence $< \Lambda >$ may be a proper subgraph of $\Gamma_{\sqrt{I}/I}(R/I)$. It is easy to see that $x+I$ and $y+I$ are adjacent in $< \Lambda >$ if and only if x and y are adjacent in $\mathcal{G}_I(R)$. Moreover, if $x+I$ and $y+I$ are adjacent in $< \Lambda >$, then $x+i$ and $y+j$ are adjacent in $< \Lambda >$ for all $i, j \in I$.
Now, we use $< \Lambda >$ to construct $\mathcal{G}_I(R)$. Let $\{x_\alpha\}_{\alpha \in \Delta}$ be the vertex set of $< \Lambda >$. Define a graph G_i with vertices $\{x_\alpha + i \mid \alpha \in \Delta\}$ and $x_\alpha + i$ and $x_\beta + i$ are adjacent in G_i if and only if $x_\alpha + I$ and $x_\beta + I$ are adjacent in $< \Lambda >$. Thus the union of G_i's is the vertex set of $\mathcal{G}_I(R)$ and edge set of $\mathcal{G}_I(R)$ is (1) all edges of G_i's, (2) for distinct $\alpha, \beta \in \Delta$ and for any $i, j \in I$, $x_\alpha + i$ and $x_\beta + j$ are adjacent in $\mathcal{G}_I(R)$ if and only if $x_\alpha + I$ and $x_\beta + I$ are adjacent in $< \Lambda >$. Indeed, the relationship between the subgraph $< \Lambda >$ of $\Gamma(R/I)$ and the subgraph $\mathcal{G}_I(R)$ of $\Gamma_I(R)$ is similar to that between $\Gamma(R/I)$ and $\Gamma_I(R)$ which has been expressed in [12, p. 4429]. This subgraph will be used in Theorem 2.2 to characterize $\mathcal{G}_I(R)$.

The degree of a vertex v in a graph G is the number of edges of G incident with v. For any nonnegative integer r, the graph G is called r-regular if the degree of each vertex is equal to r. A subgraph H of G is called a spanning subgraph when $V(G) = V(H)$. A 1-regular spanning subgraph H of G is called a 1-factor or a perfect matching of G. A graph G is 1-factorable if the edges of G are partitioned into 1-factors of G. Every r-regular bipartite graph is 1-factorable (cf. [6, p. 192]). If the edges of G are partitioned into subgraphs $H_1, \ldots, H_n$, then we write $G \cong H_1 \oplus \cdots \oplus H_n$, and if $H_i \cong H_j$ for all $1 \le i, j \le n$, then we write $G \cong nH$, where $H \cong H_i$. Using these notions, it has been shown that in [10, Theorem 2.1], $\Gamma_I(R) \cong |I|^2\Gamma(R/I)$ if I is a radical ideal of R. Now, by a similar method, we give a characterization for $\Gamma_I(R)$ when $\sqrt{I}$ is finite, and a characterization for $\mathcal{G}_I(R)$ when I is finite.

Theorem 2.2. *Let R be a ring and I an ideal of R.*

(1) *If $\sqrt{I}$ is finite, then $\Gamma_I(R) \cong |I|^2\Gamma(R/I) \oplus |X| \cdot K_{|I|}$ where $X = \{x + I \in \Gamma(R/I) \mid x^2 \in I\}$. In particular, if I is a 2-absorbing ideal of R, then $X = \Gamma(R/I) \cap (\sqrt{I}/I)$.*

(2) *If I is finite, then $\mathcal{G}_I(R) \cong |I|^2 < \Lambda >$.*

Proof. (1) Let e be the edge of $\Gamma(R/I)$ between the vertices a and b. Since every element of the form $a+i$ is adjacent to every element of the form $b+j$, for all $i, j \in I$, it is easy to see that there exists a subgraph of $\Gamma_I(R)$, denoted by $H^{(e)}$, which is isomorphic to the complete bipartite graph $K_{|I|,|I|}$. On the other hand, by [6, p. 192], we have $K_{|I|,|I|} \cong M_1^{(e)} \oplus \cdots \oplus M_{|I|}^{(e)}$, where each of $M_i^{(e)}$ is a perfect matching of $K_{|I|,|I|}$. Now consider $L_i := \oplus_{e \in E(\Gamma(R/I))} M_i^{(e)}$ which is a subgraph of $\Gamma_I(R)$. On

the other hand, for all distinct $i, j \in I$, $a+i$ is adjacent to $a+j$ if and only if $a^2 \in I$. Thus there exists a subgraph of $\Gamma_I(R)$, denoted by N_a, which is isomorphic to the complete graph $K_{|I|}$. Hence $\Gamma_I(R) \cong L_1 \oplus \cdots \oplus L_{|I|} \oplus |X| \cdot K_{|I|}$. Now the assertion follows from the fact that each L_i is partitioned into $|I|$ edge-disjoint subgraph where each of them is isomorphic to $\Gamma(R/I)$.
The "in particlar" statement follows from the fact that if I is a 2-absorbing ideal of R, then $\sqrt{I} = \{x \in R \mid x^2 \in I\}$ [4, Theorem 2.1].
(2) The proof is similar to the proof of (1) by considering $< \Lambda >$ instead of $\Gamma(R/I)$. Note that $X = \{x + I \in \Lambda \mid x^2 \in I\} = \emptyset$. □

The following theorem presents a result analogous to the case for $\Gamma(R)$ found in [2, Theorem 2.3] and for $\Gamma_I(R)$ found in [12, Theorem 2.4].

Theorem 2.3. *Let R be a ring and I be an ideal of R. Then $\mathcal{G}_I(R)$ is a connected graph and* $\operatorname{diam}(\mathcal{G}_I(R)) \leq 3$.

Proof. Let I be an ideal of a ring R, and x and y be distinct vertices of $\mathcal{G}_I(R)$. If $xy \in I$, then $x{-}y$ is a path in $\mathcal{G}_I(R)$. Let $xy \notin I$. Then there exist $a, b \in R \setminus (\sqrt{I} \cup \{x, y\})$ such that $ax \in I$ and $by \in I$. If $a = b$, then $x{-}a{-}y$ is a path in $\mathcal{G}_I(R)$. If $a \neq b$ and $ab \in \sqrt{I}$, i.e. $a^n b^n \in I$ for some positive integer n, then we have a path $x{-}a^n{-}b^n{-}y$ (for when $a^n \neq b^n$) or a path $x{-}a^n{-}y$ in $\mathcal{G}_I(R)$. If $a \neq b$ and $ab \notin \sqrt{I}$, then $x{-}ab{-}y$ is a path in $\mathcal{G}_I(R)$. □

Theorem 2.4. *If I is a nonzero proper ideal of R, then $\mathcal{G}_I(R)$ has no cut-points.*

Proof. Assume that the vertex x of $\mathcal{G}_I(R)$ is a cut-point. Then there exist vertices u, w such that x lies on every path from u to w. By Theorem 2.3, a shortest path from u to w in $\mathcal{G}_I(R)$ is of the form $u - x - w$ or $u - x - y - w$ for some $y \in \mathcal{G}_I(R)$. In each of these paths, we can replace x by $x + i$ for each $0 \neq i \in I$, since every vertex adjacent to x is adjacent to $x+i$ and distinct from $x+i$ ($x \notin \sqrt{I}$), a contradiction. □

The next result should be compared with [12, Theorem 3.3].

Theorem 2.5. *Let R be a ring, I be a nonzero proper ideal of R which is not quasi primary and Λ be as in Remark* 2.1. *Then $|I| - 1 \leq \kappa(\mathcal{G}_I(R)) \leq |I|\kappa(< \Lambda >)$. In particular, $\kappa(\mathcal{G}_I(R)) = \infty$ if I is infinite.*

Proof. First we show that $\kappa(\mathcal{G}_I(R)) \leq |I|\kappa(< \Lambda >)$. Suppose that $< \Lambda >$ is disconnected by removing the vertices $a_1 + I, \ldots, a_k + I$. Define H to be the graph obtained form $\mathcal{G}_I(R)$ by removing the set $\{a_\alpha + i \mid 1 \leq \alpha \leq k, i \in I\}$, which has $k \cdot |I|$ elements.

By way of contradiction assume that H is connected. Suppose that $b + I$ is not connected to $c + I$ after $a_1 + I, \ldots, a_k + I$ are removed from $< \Lambda >$. Then b and c are vertices of H. Suppose that $b - x_1 - \cdots - x_t - c$ is a path in H. If $x_i + I = x_{i+1} + I$ for some $1 \leq i \leq t$, then $x_i^2 \in I$, which is a contradiction ($x_i \notin \sqrt{I}$). Thus $b + I - x_1 + I - \cdots - x_t + I - c + I$ is a path in $< \Lambda >$ after removing $a_1 + I, \ldots, a_k + I$. This is a contradiction. Hence H is disconnected and we have $\kappa(\mathcal{G}_I(R)) \leq |I|\kappa(< \Lambda >)$.

Now we show that $|I| - 1 \leq \kappa(\mathcal{G}_I(R))$. If I is finite, set $t = |I| - 1$; otherwise let t be any positive integer. Let $a_1, \ldots, a_t$ be any collection of vertices of $\mathcal{G}_I(R)$. Define the graph $H = \mathcal{G}_I(R) - \{a_1, \ldots, a_t\}$.
Let x, y be two vertices of H. We show that there is a path between x and y in H.

By Theorem 2.3, $\operatorname{diam}(\mathcal{G}_I(R)) \leq 3$. Therefore, we have three cases:
Case 1: $d(x, y) = 1$; so we are done.
Case 2: $d(x, y) = 2$. Let $x - v - y$ be a shortest path from x to y in $\mathcal{G}_I(R)$. If $v \neq a_\alpha$ for any $1 \leq \alpha \leq t$, then this is also a path in H. Assume that $v = a_\alpha$ for some $1 \leq \alpha \leq t$. Since the set $\{a_\alpha + i \mid i \in I\}$ has $|I|$ element, we can choose $u \in \{a_\alpha + i \mid i \in I\}$ such that $u \neq a_\beta$ for any $1 \leq \beta \leq t$. Since $xv \in I$ and $vy \in I$, $xu \in I$ and $uy \in I$. Hence $x - u - y$ is a path in H.
Case 3: $d(x, y) = 3$. Let $x - u - v - y$ be a shortest path frome x to y in $\mathcal{G}_I(R)$. Since $u, v \in R \setminus \sqrt{I}$, $u + I \neq v + I$. Thus, since $|I| > t$, we can choose $a \in \{u + i \mid i \in I\}$ and $b \in \{v + i \mid i \in I\}$ such that $a, b \notin \{a_1, \ldots, a_t\}$. Now $xu \in I, uv \in I$ and $vy \in I$ implies that $xa \in I, ab \in I$ and $by \in I$. Hence $x - a - b - y$ is a path from x to y in H.
Hence in all cases H is connected. □

Corollary 2.6. *Let R be a ring, I be a finite ideal of R and Λ be as in Remark* 2.1. *Then* $|I| - 1 \leq \kappa(|I|^2 < \Lambda >) \leq |I|\kappa(< \Lambda >)$.

Proof. It follows from Theorem 2.2 (2) and Theorem 2.5. □

Theorem 2.7. ([4, Theorem 2.4]) *Let I be a 2-absorbing ideal of R. Then one of the following statements must hold:*
(1) $\sqrt{I} = p$ *is a prime ideal of R such that* $p^2 \subseteq I$.
(2) $\sqrt{I} = p_1 \cap p_2$, $p_1p_2 \subseteq I$, *and* $(\sqrt{I})^2 \subseteq I$, *where p_1, p_2 are the only distinct prime ideals of R that are minimal over I.*

Theorem 2.8. *Let R be a ring and I be a 2-absorbing ideal of R. Then* $\mathcal{G}_I(R) = \emptyset$ *or* $\mathcal{G}_I(R) \cong K_{1,1}$ *or* $\mathcal{G}_I(R) \cong K_{m,n}$ *for some* $m, n \geq 2$.

Proof. Let I be a 2-absorbing ideal of R such that $\mathcal{G}_I(R) \neq \emptyset$. Then $\sqrt{I}$ is not a prime ideal and so, by Theorem 2.7, $\sqrt{I} = p_1 \cap p_2$ and $p_1p_2 \subseteq I$ where p_1, p_2 are the only distinct prime ideals of R which are minimal over I. Now for $x, y \in R \setminus \sqrt{I}$ with $xy \in I$, we have $xy \in p_1$ and $xy \in p_2$. Since p_1 and p_2 are prime, we have $x \in p_1$ or $y \in p_1$ and $x \in p_2$ or $y \in p_2$ and $x, y \notin p_1 \cap p_2$. Without loss of generality, we may assume that $x \in p_1 \setminus p_2$ and $y \in p_2 \setminus p_1$. Since $p_1p_2 \subseteq I$, $\mathcal{G}_I(R)$ is a complete bipartite graph with parts $p_1 \setminus p_2$ and $p_2 \setminus p_1$. Let $|p_1 \setminus p_2| = m$ and $|p_2 \setminus p_1| = n$. If $m = 1$ and $n \geq 2$, or $n = 1$ and $m \geq 2$, the $\mathcal{G}_I(R)$ is a star graph, a contradiction to Theorem 2.4. Thus $\mathcal{G}_I(R) \cong K_{1,1}$ or $\mathcal{G}_I(R) \cong K_{m,n}$ for some $m, n \geq 2$. □

Corollary 2.9. *Let R be a ring and I a 2-absorbing ideal of R. Then* $\operatorname{diam}(\mathcal{G}_I(R)) \leq 2$.

Corollary 2.10. *Let R be a ring and I a 2-absorbing ideal of R. If I is not a radical ideal, then* $\mathcal{G}_I(R) = \emptyset$ *or* $\mathcal{G}_I(R) \cong K_{m,n}$ *for some* $m, n \geq 2$.

Proof. Let $\mathcal{G}_I(R) \neq \emptyset$. Thus $\sqrt{I} = p_1 \cap p_2$, $p_1p_2 \subseteq I$ where p_1, p_2 are the only distinct prime ideals of R which are minimal over I and $p_1p_2 \subseteq I$. By the proof of Theorem 2.8, it suffices to show that $|p_1 \setminus p_2| \neq 1$ and $|p_2 \setminus p_1| \neq 1$. Otherwise, if for instance $|p_1 \setminus p_2| = 1$, then $p_1 = \{x\} \cup \sqrt{I}$ for some $x \in R$. Thus for any $r \in R \setminus p_2$, we have $rx \in p_1 \setminus \sqrt{I}$. Hence $rx = x$ and so $(1 - r)x = 0 \in p_2$. Therefore $1 - r \in p_2$. This means that for any $r \in R$ either $r + p_2 = p_2$ or $r + p_2 = 1 + p_2$. Thus R/p_2 is a field and hence p_2 is a maximal ideal of R. It implies that p_1 and p_2 are comaximal and so by Theorem 2.7, $\sqrt{I} = p_1 \cap p_2 = p_1 p_2 \subseteq I$, which is a contradiction. $\square$

A graph theoretical result says that if a grph G contains a cycle, then $\mathrm{gr}(G) \leq 2\,\mathrm{diam}(G) + 1$ [8, Proposition 1.3.2]. By using this fact and Theorem 2.3, we have $\mathrm{gr}(\mathcal{G}_I(R)) \leq 7$. In [7, Theorem 1.6], it has been shown that $\mathrm{gr}(\Gamma(R)) \leq 4$. By combining this result and the fact that $\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma(R/I)$ [12, Corollary 2.7], we conclude that if $\Gamma_I(R)$ has a cycle, then $\mathrm{gr}(\Gamma_I(R)) \leq 4$. This can be compared with the following result.

Theorem 2.11. *Let I be an ideal of a ring R. If $\mathcal{G}_I(R)$ contains a cycle, then $\mathrm{gr}(\mathcal{G}_I(R)) \leq 4$.*

Proof. Let $x_0 - x_1 - \cdots - x_n - x_0$ with $n \geq 4$ be a cycle in $\mathcal{G}_I(R)$. Then we have two cases:
Case 1: Let $x_i x_j \notin I$ for any $j > i + 1$ such that either $0 \leq i < j \leq n - 1$ or $1 \leq i < j \leq n$. Then
(a) If $x_1 x_{n-1} = x_0$ or x_n, then $x_0^2 \in I$ or $x_n^2 \in I$, which is a contradiction.
(b) Let $x_1 x_{n-1} \neq x_i \, (i = 0, n)$. If $x_1 x_{n-1} \notin \sqrt{I}$, then $x_0 - (x_1 x_{n-1}) - x_n - x_0$ is a cycle of length 3 in $\mathcal{G}_I(R)$. Now assume that $x_1 x_{n-1} \in \sqrt{I}$. Thus there exists a positive integer t such that $x_1^t x_{n-1}^t \in I$. If $x_1^t \neq x_0$ and $x_{n-1}^t \neq x_n$, then $x_0 - x_1^t - x_{n-1}^t - x_n - x_0$ is a cycle of length 4 in $\mathcal{G}_I(R)$. Otherwise, $x_0 - x_{n-1}^t - x_n - x_0$ or $x_0 - x_1^t - x_n - x_0$ is a cycle of length 3 in $\mathcal{G}_I(R)$.
Case 2: Let $x_i x_j \in I$ for some i, j with the conditions of Case 1. Then we can replace the path $x_i - x_{i+1} - \cdots - x_j$ by the path $x_i - x_j$ in the cycle $x_0 - x_1 - \cdots - x_n - x_0$, to obtain a shorter cycle and use Case 1. $\square$

Remark 2.2. Let R be a ring and I be an ideal of R. If I is a 2-absorbing ideal, then by Theorem 2.8, $\mathcal{G}_I(R) = \emptyset$ or $\mathcal{G}_I(R) \cong K_{1,1}$ or $\mathcal{G}_I(R) \cong K_{m,n}$ for some $m, n \geq 2$ and so $\mathrm{gr}(\mathcal{G}_I(R)) = 0$ or $\mathrm{gr}(\mathcal{G}_I(R)) = \infty$ or $\mathrm{gr}(\mathcal{G}_I(R)) = 4$. If I is a 2-absorbing ideal which is not radical, then $\mathrm{gr}(\mathcal{G}_I(R)) = 0$ or $\mathrm{gr}(\mathcal{G}_I(R)) = 4$ by Corollary 2.10. In particular, if I is a 2-absorbing ideal which is neither radical nor quasi primary, then $\mathrm{gr}(\mathcal{G}_I(R)) = 4$.

Recall that the number of graph vertices of the largest complete subgraph of a graph G, denoted by $\omega(G)$, is the clique number of G.

Lemma 2.12. *Let $I \subseteq J$ be two ideals of R such that $\sqrt{I} = \sqrt{J}$. Then $\mathcal{G}_I(R)$ is a subgraph of $\mathcal{G}_J(R)$. In particular $\omega(\mathcal{G}_I(R)) \leq \omega(\mathcal{G}_J(R))$.*

Proof. Let x and y be two adjacent vertices in $\mathcal{G}_I(R)$. Then $x, y \in R \setminus \sqrt{I} = R \setminus \sqrt{J}$ and $xy \in I \subseteq J$. Hence x and y are adjacent vertices in $\mathcal{G}_J(R)$.
The "in particular" statement is clear, since every clique in $\mathcal{G}_I(R)$ can be extended to a clique in $\mathcal{G}_J(R)$. $\square$

Theorem 2.13. *Let R be a ring and $I = q_1 \cap \cdots \cap q_m$ be a minimal primary decomposition of the ideal I of R with $n (\leq m)$ isolated prime ideals. Then $\omega(\mathcal{G}_I(R)) \leq n$. In particular if $m = n$, then $\omega(\mathcal{G}_I(R)) = n$.*

Proof. Let $\sqrt{q_i} = p_i \ (1 \leq i \leq m)$, where p_i's are prime ideals of R. Without loss of generality we suppose that $p_1, \ldots, p_n$ are isolated prime ideals of I. Set $J = q_1 \cap \cdots \cap q_n$. Then $I \subseteq J$ and $\sqrt{I} = \sqrt{J}$. By Lemma 2.12, $\omega(\mathcal{G}_I(R)) \leq \omega(\mathcal{G}_J(R))$.

Now we show that $\omega(\mathcal{G}_J(R)) = n$. This also proves the last part of theorem. Since $p_1, \ldots, p_n$ are isolated prime ideals of I, $\sqrt{J} = p_1 \cap \cdots \cap p_n$ and for all $1 \leq j \leq n$, $\sqrt{J} \neq \hat{p}_j$ where $\hat{p}_j = \cap\{p_i \mid 1 \leq i \leq n, i \neq j\}$.

Consider $x_j \in \hat{p}_j \setminus p_j$ for all $1 \leq j \leq n$. Then $x_i x_j \in \sqrt{J}$ for all $1 \leq i \neq j \leq n$, so there exists a positive integer t_{ij} such that $(x_i x_j)^{t_{ij}} \in J$. Therefore $\{x_1^t, \ldots, x_n^t\}$ is a clique in $\mathcal{G}_J(R)$ for $t = \max\{t_{ij} \mid 1 \leq i, j \leq n, i \neq j\}$. Hence, $\omega(\mathcal{G}_J(R)) \geq n$. Now we must show that $\omega(\mathcal{G}_J(R)) \leq n$. In fact, by induction on n, we show that if J is an ideal of R such that $\sqrt{J} = \cap\{p_i \mid 1 \leq i \leq n\}$ and for each $1 \leq j \leq n$, $\sqrt{J} \neq \hat{p}_j$, then $\omega(\mathcal{G}_J(R)) \leq n$. For $n = 2$, by Theorem 2.8, $\mathcal{G}_J(R)$ is a bipartite graph, hence $\omega(\mathcal{G}_J(R)) = 2$. Suppose $n > 2$ and the result is true for any integer less than n. Let $\sqrt{J} = \cap\{p_i \mid 1 \leq i \leq n\}$ and for each $1 \leq j \leq n$, $\sqrt{J} \neq \hat{p}_j$. Let $\{x_1, \ldots, x_k\}$ be a clique in $\mathcal{G}_J(R)$. Hence, $x_1 x_j \in J \subseteq p_1$ for any $2 \leq j \leq k$. Without loss of generality, suppose that $x_1 \notin p_1$. Therefore, $x_2, \ldots, x_k \in p_1$, so $x_2, \ldots, x_k \notin \hat{p}_1$. Let K be an ideal of R such that $\sqrt{K} = \hat{p}_1$. Then by induction hypothesis $\omega(\mathcal{G}_K(R)) \leq n-1$. Since $\{x_1, \ldots, x_k\}$ is a clique in $\mathcal{G}_J(R)$, for all $2 \leq i \neq j \leq n$, $x_i x_j \in J \subseteq \sqrt{J} \subseteq \sqrt{K}$. Then there exists a positive integer t_{ij} such that $(x_i x_j)^{t_{ij}} \in K$. Therefore $\{x_2^t, \ldots, x_k^t\}$ is a clique in $\mathcal{G}_K(R)$ for $t = \max\{t_{ij} \mid 2 \leq i, j \leq k, i \neq j\}$. Thus $k - 1 \leq n - 1$, and hence $\omega(\mathcal{G}_J(R)) \leq n$.

□

Corollary 2.14. *Let R be a Noetherian ring and $\mathrm{Min}(R)$ be the set of minimal prime ideals of R. Then $\omega(\mathcal{G}_I(R)) = |\mathrm{Min}(R/I)|$ for each ideal I of R.*

Proof. Since R is Noetherian, $|\mathrm{Min}(R/I)|$ is finite for each ideal I of R. Therefore, $\sqrt{I}$ is a finite intersection of minimal prime ideals of I. Now, the result follows from the proof of Theorem 2.13. □

If I is a 2-absorbing ideal, then by Theorem 2.8, $\omega(\mathcal{G}_I(R)) = 0$ or $\omega(\mathcal{G}_I(R)) = 2$. We can generalize this result as follows.

Corollary 2.15. *Let R be a ring and I be an n-absorbing ideal of R . Then $\omega(\mathcal{G}_I(R)) = |Min(R/I)| \leq n$.*

Proof. It follows from combining [1, Theorem 2.5] and the proof of Theorem 2.13. □

Theorem 2.16. *Let R be a ring, I be an ideal of R and Λ be as in Remark* 2.1. *Then $\omega(\mathcal{G}_I(R)) = \omega(< \Lambda >)$.*

Proof. Since $\mathcal{G}_I(R)$ contains copies of $< \Lambda >$, $\omega(< \Lambda >) \leq \omega(\mathcal{G}_I(R))$. It is enough to consider the case where $\omega(< \Lambda >) = n < \infty$. Assume that $H =< \{a_1, a_2, \ldots, a_{n+1}\} >$ is a complete subgraph of $\mathcal{G}_I(R)$ and H^* is the subgraph of $< \Lambda >$ on the vertices $a_1 + I, a_2 + I, \ldots, a_{n+1} + I$. Note that vertices x and y are adjacent in $\mathcal{G}_I(R)$ if and only if $x + I$ and $y + I$ are adjacent in $< \Lambda >$. Thus H^* is a complement subgraph of $< \Lambda >$. Hence $a_j + I = a_k + I$ for some $1 \leq j \neq k \leq n + 1$. Therefore, $a_j a_k \in I$ implies that $a_j^2 \in I$, which is a contradiction. □

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