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Homoclinic solutions for a class of damped vibration systems

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Abstract. In this paper, we establish a new existence result on homoclinic solutions for a non periodic damped vibration system

$$\ddot{x}(t) + q(t)\dot{x}(t) + V'(t, x(t)) = 0,$$

where q is a continuously differentiable function and V $\in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$, $V(t,x) = -K(t,x)+W(t,x)$. This homoclinic solution is obtained as a limit of solutions of a certain sequence of nil-boundary value problems which are obtained by the minimax methods.

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1. Introduction

Consider the following damped vibration system

$$(DV) \qquad \ddot{x}(t) + q(t)\dot{x}(t) + V'(t, x(t)) = 0,$$

where $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $V'(t,x) = \frac{\partial V}{\partial x}(t,x)$ and $q \in C^1(\mathbb{R}, \mathbb{R})$.
As usual, we say that a solution x of (DV) is homoclinic (to 0) if $x \in C^2(\mathbb{R}, \mathbb{R}^N)$, $x \neq 0$, $x(t) \to 0$ and $\dot{x}(t) \to 0$ as $t \to \pm\infty$.

It is well known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, its perturbed system probably produce chaotic phenomena. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic orbits of (DV) emanating from 0.

When $q(t) = 0$ for all $t \in \mathbb{R}$, (DV) is just the following second order Hamiltonian system

$$(HS) \qquad \ddot{x}(t) + \nabla V(t, x(t)) = 0.$$

In last decades, the existence and multiplicity of homoclinic orbits for systems (HS) have been extensively investigated by many authors see [1],[3]-[18],[20]-[25] and references therein.
In 2008, Wu and Zhou [26] studied the existence of solutions of the following damped vibration problems

$$\begin{cases} \ddot{x}(t) + q(t)\dot{x}(t) = Ax(t) + W'(t, x(t)) = 0 \ \text{ for } \ t \in [0,T] \\ x(0) = x(T) = \dot{x}(0) - e^{Q(T)}\dot{x}(T) = 0 \end{cases} \tag{1}$$

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using variational method.

In 2011, Zhang and Yan [27] studied the existence of homoclinic solutions of a special case of (DV)

$$\ddot{x}(t) + c\dot{x}(t) - L(t)x(t) + W'(t, x(t)) = 0 \text{ for } t \in \mathbb{R}\,,\ x \in \mathbb{R}^{\mathrm{N}}. \tag{2}$$

They introduce the concept of fast homoclinic solutions and establish some criteria to guarantee the existence of fast homoclinic solutions for (2) under the case where $W(t, x)$ is subquadratic at infinity and $c \geq 0$.

In 2013, Peng Chen and X.H.Tang [2] studied the existence and multiplicity of the following damped vibration problems

$$\ddot{x}(t) + q(t)\dot{x}(t) - L(t)x(t) + W'(t, x(t)) = 0 \text{ for } t \in \mathbb{R}\,, x \in \mathbb{R}^{\mathrm{N}}, \tag{3}$$

where $L(t)$ and $W(t, x)$ are neither autonomous nor periodic in t.

In the recent paper [11], F.Khelifi and M.Timoumi proved that system (DV) possesses at least one non trivial even homoclinic solution under some suitable assumptions where V is of the type $V(t, x) = -K(t, x) + W(t, x)$.

As far as the case $q(t) \neq 0$, is concerned, to our best knowledge, there is few research about the existence of such solutions for (DV) when V is of the type $V = -K + W$.

In the present paper, we shall study the existence of homoclinic solutions for (DV) when $q(t) \neq 0$ and assuming that $V(t, x)$ is not periodic in t and $W(t, x)$ satisfies a kind of new superquadratic condition which is different from the corresponding condition in the known results. We obtain the existence of homoclinic solution as the limit of solutions of a certain sequence of boundary-value problem which are obtained by the minimax methods.

Our result is presented as follows:

Theorem 1.1. *Let $M_1 = \sup\{K(t, x),\ t \in \mathbb{R},\ |x| \leq 1\} < \infty$ holds. Moreover, assume that the following conditions hold:*

(K_1) *$K(t, 0) = 0$, and there exist constants $a > 0$ and $\beta \in]1, 2]$ such that $K(t, x) \geq a|x|^2, K(t, x) \leq K'(t, x).x \leq \beta K(t, x) \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N$.*

(K_2) *For all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, $K'(t, x) \to 0$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$,*

(W_1) *$W(t, 0) = 0$ and $W'(t, x) = o(|x|)$ as $|x| \to 0, uniformly\ in\ t$, and there exist, $M_0 > 0$ such that*

$$\frac{|W'(t, x)|}{|x|} \leq M_0,$$

for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.

(W_2) *$W(t, x) - w(t)|x|^2 = o(|x|^2)$ as $|x| \to \infty$ uniformly in t, where $w \in L^\infty(\mathbb{R}, \mathbb{R})$ with $w_\infty = \inf_{t\in\mathbb{R}} w(t) > \frac{2M_1M_\infty}{m_0}$,*
where $M_\infty = \sup\limits_{t\in\mathbb{R}} e^{Q(t)}$ and $m_0 = \inf\limits_{t\in\mathbb{R}} e^{Q(t)}$.

(W_3) *$\overline{W}(t, x) = \frac{1}{2}W'(t, x)x - W(t, x) \to \infty$ as $|x| \to \infty$, and*

$$\inf\{\frac{\overline{W}(t, x)}{|x|^2} : t \in \mathbb{R};\ with\ c \leq |x| < d\} > 0,$$

for any $c, d > 0$.

(q) *q and Q are bounded functions and $q \in C^1(\mathbb{R}, \mathbb{R})$, where $Q(t) = \int_0^t q(s)ds$.*

Then the system (DV) has at least one nontrivial homoclinic solution $x \in H^1_Q(\mathbb{R}, \mathbb{R}^N)$ *such that* $\dot{x}(t) \to 0$ *as* $|t| \to +\infty$.

Remark 1.1. Theorem 1.1 treats the asymptotically quadratic case on W. Consider the functions

$$q(t) = \sin(t), \quad Q(t) = 1 - \cos(t),$$

$$K(t,x) = (1 + e^{-|t|})|x|^2, \quad W(t,x) = d(t)|x|^2 \left(1 - \frac{1}{\ln(e + |x|)}\right),$$

where $d \in L^\infty(\mathbb{R}, \mathbb{R})$ and $\inf_{t\in\mathbb{R}} d(t) > 4 + 32\pi^2$.

A straightforward computation shows that W and K satisfy the assumptions of Theorem 1.1, but W does not satisfy the global Ambrosetti-Rabinowitz condition, and K can not be written in the form $\frac{1}{2}(L(t)x, x)$ and does not satisfy the corresponding results in [1, 4, 10, 13, 19, 22, 24, 25]. Hence, Theorem 1.1 also extends the results in [7, 21].

2. Proof of the main results

By the idea of [12], we approximate a homoclinic solution of (DV) by a solution of the following problem:

$$\begin{cases} \ddot{x}(t) + q(t)\dot{x}(t) - K'(t, x(t)) + W'(t, x(t)) = 0 \; for \; t \; \in]-T, T[\\ x(-T) = x(T) = 0 \end{cases} \tag{4}$$

where T is a positive constant.

For $1 \le s < \infty$, let $L^s_Q(-T, T; \mathbb{R}^N)$ be the Banach space of measurable functions x defined on $[-T, T]$ with values in $\mathbb{R}^N$ satisfying $\int_{-T}^{T} e^{Q(t)}|x(t)|^s dt < \infty$, with the norm

$$\|x\|_{L^s_Q} = \left(\int_{-T}^{T} e^{Q(t)}|x(t)|^s dt\right)^{\frac{1}{s}}.$$

The space $L^2_Q(-T, T; \mathbb{R}^N)$ provided with the inner product

$$< x, y >= \int_{-T}^{T} e^{Q(t)} x(t).y(t) dt, \; x, y \in L^2_Q(-T, T; \mathbb{R}^N)$$

is a Hilbert space. Let E be the space defined by

$$E = \left\{x \in L^2_Q([-T, T], \mathbb{R}^N), \dot{x} \in L^2_Q(-[T, T], \mathbb{R}^N), x(-T) = x(T) = 0\right\}$$

The space E provided with the inner product

$$\langle x, y \rangle = \int_{-T}^{T} e^{Q(t)}[x(t).y(t) + \dot{x}(t).\dot{y}(t)]dt,$$

and the associated norm

$$\|x\| = \left(\int_{-T}^{T} e^{Q(t)}(|x(t)|^2 + |\dot{x}(t)|^2)dt\right)^{\frac{1}{2}}$$

is a Hilbert space.

Consider the functional $I : E \to \mathbb{R}$ defined by

$$\Phi(x) = \int_{-T}^{T} e^{Q(t)} \left[\frac{1}{2}|\dot{x}(t)|^2 + K(t, x(t)) - W(t, x(t)) \right] dt. \tag{5}$$

It is easy to check that $\Phi \in C^1(E, \mathbb{R})$ and for all $x, y \in E$, we have

$$\Phi'(x)y = \int_{-T}^{T} e^{Q(t)} \left[(\dot{x}(t).\dot{y}(t) + K'(t, x(t)).y(t) - W'(t, x(t)).y(t) \right] dt. \tag{6}$$

Moreover, the critical points of Φ in $H_0^1([-T, T])$ are the classical solutions of (DV) in $[-T, T]$ satisfying $x(T) = x(-T) = 0$. We will obtain a critical point of Φ by using an improved version of the Mountain Pass Theorem. For completeness, we give this theorem.

Recall that a sequence (x_j) is a C-sequence for the functional φ if $\varphi(x_j)$ is bounded and $(1 + \|x_j\|)\varphi'(x_j) \to 0$. A functional φ satisfies the C-condition if and only if any C-condition for φ contains a convergent subsequence.

Lemma 2.1. [19] *Let H be a real Banach space and $I \in C^1(H, \mathbb{R})$ satisfying the C-condition. If I satisfies the following conditions:*

(i) $I(0) = 0$,

(ii) there exist constants ρ, $\alpha > 0$ such that $I_{|\partial B_\rho(0)} \geq \alpha$,

(iii) there exists $e \in H \backslash \overline{B}_\rho(0)$ such that $I(e) \leq 0$.

Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where $B_\rho(0)$ is the open ball in H centered in 0, with radius ρ, $\partial B_\rho(0)$ as its boundary and

$$\Gamma = \{g \in C([0, 1], H) : g(0) = 0, g(1) = e\}.$$

Proof. As shown in [4], a deformation lemma can be proved with the C-condition replacing the usual (PS)-condition, and it turns out that the Mountain Pass Theorem in [19] holds true under the C-condition. □

By Sobolev's embedding theorem, $H^1(\mathbb{R}; \mathbb{R}^N)$ is continuously embedded into $L^\infty(\mathbb{R}, \mathbb{R}^N)$. Thus there exists $C > 0$ such that

$$\|x\|_{L^\infty(\mathbb{R},\mathbb{R}^N)} \leq C\|x\|_{H^1}, \quad \forall\, x \in H^1(\mathbb{R}; \mathbb{R}^N).$$

Since $x \in E$ can be regarded as belonging to $H^1(\mathbb{R}; \mathbb{R}^N)$ if one extends it by zero in $\mathbb{R}\backslash[-T, T]$, then we have

$$\|x\|_{L^\infty([-T,T],\mathbb{R}^N)} \leq C\|x\|_{H^1}, \quad \forall\, x \in H_0^1([-T, T], \mathbb{R}^N),$$

where C is independent of $T > 0$.

It follows from the above inequality that

$$\|x\|_{L^\infty([-T,T],\mathbb{R}^N)} \leq C\|x\|_{H^1} \leq \gamma_2\|x\|, \;\; \forall x \in E, \tag{7}$$

where $\gamma_2 = \frac{C}{\sqrt{m_0}}$.

Note that the inequality (7) holds true with constant $C = \sqrt{2}$ if $T > \frac{1}{2}$ (see [10]). Subsequently, we may assume this condition is fulfilled.

Lemma 2.2. *Assume that* (K_1) *holds, then*

$$K(t,x) \le K(t, \frac{x}{|x|})|x|^{\beta},\ \forall t \in \mathbb{R},\ |x| \ge 1. \tag{8}$$

Proof. To prove this lemma it suffices to show that for every $x \in \mathbb{R}^N$ and t $\in [-T,T]$ the function $(0,+\infty) \to \mathbb{R},\ s \mapsto K(t, s^{-1}x)s^{\beta}$ is nondecreasing; which is an immediate consequence of (K_1). The proof of Lemma 2.2 is complete. □

Lemma 2.3. *Under the assumptions of Theorem 1.1, the problem (4) possesses a nontrivial solution.*

Proof. We show that the functional Φ satisfies the (C)-condition. Let

$$\Phi(y_j) \textit{ be bounded and } (1 + \|y_j\|)\Phi'(y_j) \to 0.$$

Observe that for j large, it follows from (K_1) that there exists a constant M such that

$$\begin{aligned} M &\ge \Phi(y_j) - \frac{1}{2}\Phi'(y_j)y_j \\ &= \int_{-T}^{T} e^{Q(t)}(\frac{1}{2}W'(t,y_j).y_j - W(t,y_j))dt + \int_{-T}^{T} e^{Q(t)}(K(t,y_j) - \frac{1}{2}K'(t,y_j).y_j)dt \\ &\ge \int_{-T}^{T} e^{Q(t)}\overline{W}(t,y_j(t))dt. \end{aligned} \tag{9}$$

By negation, if (y_j) is not bounded, passing to a subsequence if necessary we may assume that $\|y_j\| \to +\infty$ as $j \to +\infty$. Set $z_j = \frac{y_j}{\|y_j\|}$, then $\|z_j\| = 1$ and by (7) one has

$$\|z_j\|_{\infty} \le \gamma_2. \tag{10}$$

Note that

$$\begin{aligned} \Phi'(y_j)y_j &= \int_{-T}^{T} e^{Q(t)}|\dot{y}_j(t)|^2 dt + \int_{-T}^{T} e^{Q(t)}K'(t,y_j(t))y_j(t)dt \\ &- \int_{-T}^{T} e^{Q(t)}W'(t,y_j(t))y_j(t)dt \\ &\ge \overline{a}\|y_j\|^2 - \int_{-T}^{T} e^{Q(t)}W'(t,y_j(t)).y_j(t)dt \\ &\ge \|y_j\|^2\left(\overline{a} - \int_{-T}^{T} e^{Q(t)}\frac{W'(t,y_j(t)).y_j(t)}{\|y_j\|^2}dt\right), \end{aligned} \tag{11}$$

where $\overline{a} = \min\{1,a\} > 0$. Thus implies that

$$\lim_{j\to+\infty} \int_{-T}^{T} e^{Q(t)}\frac{W'(t,y_j(t)).y_j(t)}{\|y_j\|^2}dt \ge \overline{a}. \tag{12}$$

Set for $s \ge 0$

$$h(s) := \inf\left\{\overline{W}(t,x)|\ t \in [-T,T] \textit{ and } x \in \mathbb{R}^N \textit{ with } |x| \ge s\right\}. \tag{13}$$

By (W_3), one has

$$h(s) \to +\infty \textit{ as } s \to +\infty.$$

For $0 \leq l < m$, let

$$\Omega_j(l,m) = \{t \in [-T,T] \mid l \leq |y_j(t)| < m\},$$

and

$$C_l^m = \inf\left\{\frac{\overline{W}(t,x)}{|x|^2}, t \in [-T,T] \; and \; l \leq |x| < m\right\}. \tag{14}$$

Then by (W_3), $C_l^m > 0$. One has

$$\overline{W}(t,y_j(t)) \geq C_l^m |y_j(t)|^2, \; for \; all \; t \in \; \Omega_j(l,m). \tag{15}$$

It follows from (9) that

$$\begin{aligned} M &\geq \int_{-T}^{T} e^{Q(t)}\overline{W}(t,y_j)dt \\ &= \int_{\Omega_j(0,l)} e^{Q(t)}\overline{W}(t,y_j)dt + \int_{\Omega_j(l,m)} e^{Q(t)}\overline{W}(t,y_j)dt \\ &+ \int_{\Omega_j(m,\infty)} e^{Q(t)}\overline{W}(t,y_j(t))dt \\ &\geq \int_{\Omega_j(0,l)} e^{Q(t)}\overline{W}(t,y_j)dt + C_l^m \int_{\Omega_j(l,m)} e^{Q(t)}|y_j|^2 dt \\ &+ m_0 h(m)|\Omega_j(m,\infty)|, \end{aligned} \tag{16}$$

which implies that

$$|\Omega_j(m,\infty)| \leq \frac{M}{m_0 h(m)} \to 0 \;\; as \; m \to +\infty \; uniformly \; in \; j, \tag{17}$$

and for any fixed $0 < l < m$

$$\int_{\Omega_j(l,m)} e^{Q(t)}|z_j|^2 dt = \frac{1}{\|y_j\|^2}\int_{\Omega_j(l,m)} e^{Q(t)}|y_j|^2 dt \leq \frac{M}{C_l^m \|y_j\|^2} \to 0 \tag{18}$$

as $j \to +\infty$. Moreover, by (7) and (17), we have

$$\begin{aligned} \int_{\Omega_j(m,\infty)} e^{Q(t)}|z_j|^2 dt &\leq \|z_j\|^2_{L^\infty([-T,T])}|\Omega_j(m,\infty)|M_\infty \\ &\leq \gamma_2^2 M_\infty |\Omega_j(m,\infty)| \to 0, \end{aligned} \tag{19}$$

as $m \to +\infty$ uniformly in j. Let $0 < \varepsilon < \frac{\overline{a}}{3}$, by (W_1) there exist $l_\varepsilon > 0$ such that

$$|W'(t,x)| \leq \frac{\varepsilon}{\gamma_2^2}|x| \;\; for \; all \; |x| \leq l_\varepsilon. \tag{20}$$

Consequently,

$$\int_{\Omega_j(0,l_\varepsilon)} e^{Q(t)}\frac{|W'(t,y_j)||z_j|^2}{|y_j|}dt \leq \frac{\varepsilon}{\gamma_2^2}\int_{\Omega_j(0,l_\varepsilon)} |z_j|^2 dt \leq \varepsilon. \tag{21}$$

By (17), we can take m_ε large such that

$$\int_{\Omega_j(m_\varepsilon,\infty)} e^{Q(t)}|z_j|^2 dt \leq \frac{\varepsilon}{M_0}. \tag{22}$$

Hence, by (W_1) we obtain

$$\begin{aligned}\int_{\Omega_j(m_\varepsilon,\infty)} e^{Q(t)}\frac{|W'(t,y_j)||z_j|^2}{|y_j|}dt &\leq M_0\int_{\Omega_j(m_\varepsilon,\infty)} e^{Q(t)}|z_j|^2dt \\ &\leq \varepsilon. \end{aligned}\tag{23}$$

By (18) there is j_0 such that

$$\begin{aligned}\int_{\Omega_j(l_\varepsilon,m_\varepsilon)} e^{Q(t)}\frac{|W'(t,y_j)||z_j|^2}{|y_j|}dt &\leq M_0\int_{\Omega_j(l_\varepsilon,m_\varepsilon)} e^{Q(t)}|z_j|^2dt \\ &\leq \varepsilon, \end{aligned}\tag{24}$$

for all $j \geq j_0$. Therefore, combining (21)-(24) we have

$$\begin{aligned}\int_{-T}^{T} e^{Q(t)}\frac{W'(t,y_j).y_j}{\|y_j\|^2}dt &\leq \int_{[-T,T]\setminus\{t\in[-T,T]/|y_j(t)|=0\}} e^{Q(t)}\frac{|W'(t,y_j)||z_j|^2}{|y_j|}dt \\ &\leq 3\varepsilon \\ &< \overline{a}, \end{aligned}\tag{25}$$

which contradicts (12). Hence, (y_j) is bounded in E_T. In a similar way to Proposition B.35 in [[15]], we can prove that (y_j) has a convergent sub sequence. Hence Φ satisfies the C-condition.

Now, let us show that Φ satisfies assumption (ii) of Lemma 2.1. By (W_1) and (W_2), given $0 < \varepsilon < \frac{\overline{a}}{2}$, there exists some $C_\varepsilon > 0$ such that

$$|W(t,x)| \leq \varepsilon|x|^2 + C_\varepsilon|x|^p \tag{26}$$

for all $x \in \mathbb{R}^N$ and $t \in [-T,T]$, where $p > 2$. It follows from (K_1), (H_6), (7) and (26) that

$$\begin{aligned}\Phi(x) &= \int_{-T}^{T} e^{Q(t)}\left[\frac{1}{2}|\dot{x}(t)|^2 + K(t,x(t)) - W(t,x(t))\right]dt \\ &\geq \left(\frac{\overline{a}}{2}-\varepsilon\right)\|x\|^2 - 2\gamma_2^pTC_\varepsilon\|x\|^p. \end{aligned}\tag{27}$$

Hence, there exist $\alpha > 0$ and $\rho > 0$ such that $\Phi(x) \geq \alpha$ for all $x \in E_T$ with $\|x\| = \rho$.

We show that Φ satisfies assumption(iii) of Lemma 2.1 . By (W_2), there exists $B > 0$ such that

$$W(t,x) \geq w_\infty|x|^2 - B,\ \forall t \in [-T,T],\ x \in \mathbb{R}^N. \tag{28}$$

Let

$$e(t) = \xi|\sin(\omega t)|e_1,\ t \in [-T,T],$$

where $\omega = \frac{2\pi}{T}$, $e_1 = (1,0,...,0)$ and $\xi \in \mathbb{R}\setminus\{0\}$. Clearly, $e \in E_T$.

By (18), (28), and (8) we have

$$\begin{aligned}\Phi(e) =&\frac{1}{2}\int_{-T}^{T} e^{Q(t)}|\dot{e}(t)|^2dt + \int_{-T}^{T} e^{Q(t)}K(t,e(t))dt - \int_{-T}^{T} e^{Q(t)}W(t,e(t))dt \\ =&\frac{1}{2}\xi^2\omega^2\int_{-T}^{T} e^{Q(t)}|\cos(\omega t)|^2dt + \int_{\{t\in[-T,T];|e(t)|\leq 1\}} e^{Q(t)}K(t,e(t)|)dt \\ &+ \int_{\{t\in[-T,T];|e(t)|\geq 1\}} e^{Q(t)}K(t,e(t))dt - \int_{-T}^{T} e^{Q(t)}W(t,e(t))dt \\ \leq&\frac{M_\infty}{2}T^2\omega^2\int_{-T}^{T}|\cos(\omega t)|^2dt + M_\infty M_1\int_{\{t\in[-T,T];|e(t)|\geq 1\}}|e(t)|^\beta dt + 2TM_1M_\infty \\ &- m_0w_\infty\xi^2\int_{-T}^{T}|\sin(\omega t)|^2dt + 2TBM_\infty \\ \leq&\xi^2(\frac{\omega^2}{2}M_\infty + M_\infty M_1 - w_\infty m_0)T + 2TM_\infty(M1 + B). \end{aligned} \tag{29}$$

Since $w_\infty > \frac{2M_1M_\infty}{m_O}$ and $T > \sqrt{\frac{2}{M_1}}\pi$, then $\frac{\omega^2}{2}M_\infty + M_\infty M_1 - w_\infty m_0 < 0$. So $\Phi(e) \to -\infty$ as $\xi \to \infty$. So, we can choose large enough $\xi \in \mathbb{R}$ such that $\|e\| > \rho$ and $\Phi(e) < 0$.

Clearly $\Phi(0) = 0$; then, by application of Lemma 2.1 there exists a critical point $x_T \in E_T$ of Φ such that $\Phi(x_T) \geq \alpha$ for all $T > \sqrt{\frac{2}{M_1}}\pi$. □

Lemma 2.4. *x_T is bounded uniformly in $T > \sqrt{\frac{2}{M_1}}\pi$.*

Proof. Define the set of paths

$$\Gamma_T = \{g \in C([0,1], E_T) : g(0) = 0, g(1) = e\},$$

then there exists a solution x_T of system (11) at which

$$\inf_{g\in\Gamma}\max_{s\in[0,1]}\Phi(g(s)) \equiv D_T$$

is achieved. Let $\hat{T} > T$. Since any function in E_T can be regarded as belonging to $E_{\hat{T}}$ if one extends it by zero in $[-\hat{T},\hat{T}]\diagdown[-T,T]$, then $\Gamma_T \subset \Gamma_{\hat{T}}$ and

$$D_{\hat{T}} \leq D_{\frac{1}{2}} \quad \text{uniformly} \;\; in\ T > \sqrt{\frac{2}{M_1}}\pi. \tag{30}$$

Notice that $\Phi'(x_T) = 0$, and together with (30), one has

$$\Phi(x_T) \leq D_{\frac{1}{2}}, \;\; (1 + \|x_T\|)\|\Phi'(x_T)\| = 0. \tag{31}$$

The rest of the proof is similar to the that in Lemma 2.3. Hence there exists a constant $M_2 > 0$, independent of T such that

$$\|x_T\| \leq M_2, \;\; \forall T > \sqrt{\frac{2}{M_1}}\pi. \tag{32}$$

The proof is complete. □

Take a sequence $T_n \to \infty$, and consider the problem (4) on the interval $[-T_n, T_n]$. By Lemma 2.3 , there exists a nontrivial solution $x_n = x_{T_n}$ of problem (4).

Lemma 2.5. *Let (x_n) be the sequence given above. Then there exists a subsequence (x_{n_j}) convergent to x_0 in $C^1_{loc}(\mathbb{R},\mathbb{R}^N)$.*

Proof. First we prove that the sequences $\|x_n\|_{L^\infty_{T_n}}$, $\|\dot{x}_n\|_{L^\infty_{T_n}}$, and$\|\ddot{x}_n\|_{L^\infty_{T_n}}$ are bounded. From (7) and (32), for n large enough, one has

$$\|x_n\|_{L^\infty_{T_n}} \leq \gamma_2 M_2 = M_3. \tag{33}$$

Suppose that $\dot{x}_n(t) = (\dot{x}_{n_1}(t), \dot{x}_{n_2}(t), ..., \dot{x}_{n_N}(t))$ for each $t \in \mathbb{R}$. By the Mean Value theorem, there exists $t_{n_i} \in [t-1,t]$, for all $t \in \mathbb{R}$, such that $\dot{x}_{n_i}(t_{n_i}) = \int_{t-1}^{t} \dot{x}_{n_i}(s)ds = x_{n_i}(t) - x_{n_i}(t-1)$ for any $i \in \{1,2,.....,N\}$. As (x_n) satisfies

$$\ddot{x}_n(t) + q(t)\dot{x}_n(t) + V'(t,x_n(t)) = 0, \tag{34}$$

we obtain

$$\begin{aligned}\dot{x}_n(t) - \dot{x}_n(t_{n_i}) =& q(t_{n_i})x_n(t_{n_i}) - q(t)x_n(t) + \int_{t_{n_i}}^{t} \dot{q}(s)x_n(s)ds \\ & - \int_{t_{n_i}}^{t} V'(s,x_n(s))ds. \end{aligned} \tag{35}$$

It follows from (35) that there exists $M_4 > 0$ such that

$$|\dot{x}_{n_i}(t)| < M_4, , \quad \forall\, i \in \{1,2.....,N\}, \forall t \in \mathbb{R}.$$

From the above inequality we deduce that there exists $M_5 > 0$ such that

$$\|\dot{x}_n\|_{L^\infty_{T_n}} < M_5. \tag{36}$$

Moreover, using (34), we deduce that there exists $M_6 > 0$ such that

$$\|\ddot{x}_n\|_{L^\infty_{T_n}} < M_6. \tag{37}$$

Second, we show that the sequences (x_n) and $(\dot{x}_n)$ are equicontinuous. In deed, for any $n \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$, by (36)

$$\begin{aligned}|x_n(t_1) - x_n(t_2)| &= |\int_{t_2}^{t_1} \dot{x}_n(s)|ds \\ &\leq \int_{t_2}^{t_1} |\dot{x}_n|(s)ds \\ &\leq M_5|t_1 - t_2|. \end{aligned} \tag{38}$$

Similarly, by (37), one gets

$$|\dot{x}_n(t_1) - \dot{x}_n(t_2)| \leq M_6|t_1 - t_2|. \tag{39}$$

By using the Arzela-Ascoli Theorem, we obtain the existence of a sub sequence (x_{n_j}) and a function x_0 such that

$$x_{n_j} \to x_0 \quad as\ j \to \infty \quad in \quad C^1_{loc}(\mathbb{R},\mathbb{R}^N). \tag{40}$$

The proof is complete. □

Lemma 2.6. *Let $x_0 : \mathbb{R} \to \mathbb{R}^N$ be the function given by (40). Then x_0 is the homoclinic solution of (DV).*

Proof. First we show that x_0 is a solution of (DV). Let (x_{n_j}) be the sequence given by Lemma 2.4, then

$$\ddot{x}_{n_j}(t) + q(t)\dot{x}_{n_j}(t) + V'(t, x_{n_j}(t)) = 0, \tag{41}$$

for every $j \in \mathbb{N}$ and $t \in [-T_{n_j}, T_{n_j}]$. Take $b, c \in \mathbb{R}$ with $b < c$. There exists $j_0 \in \mathbb{R}$ such that for all $j > j_0$; we get $[b, c] \subset [-T_{n_j}, T_{n_j}]$ and

$$\ddot{x}_{n_j}(t) = -q(t)\dot{x}_{n_j}(t) - V'(t, x_{n_j}(t)), \ \ \forall\, t \in [b, c]. \tag{42}$$

Integrating (42) from b to $t \in [b, c]$, we have

$$\begin{aligned} \dot{x}_{n_j}(t) - \dot{x}_{n_j}(b) &= q(b)x_{n_j}(b) - q(t)x_{n_j}(t) + \int_b^t \dot{q}(s)x_n(s)ds \\ &- \int_b^t V'(s, x_{n_j}(s))ds, \ \ \forall\, t \in [b, c]. \end{aligned} \tag{43}$$

Since $x_{n_j} \to x_0$ uniformly on $[b, c]$ and $\dot{x}_{n_j} \to \dot{x}_0$ uniformly on $[b, c]$ as $j \to \infty$, then, from (43), we obtain

$$\begin{aligned} \dot{x}_0(t) - \dot{x}_0(b) &= q(b)x_0(b) - q(t)x_0(t) + \int_b^t \dot{q}(s)x_0(s)ds \\ &- \int_b^t V'(s, x_0(s))ds, \ \ \forall\, t \in [b, c]. \end{aligned} \tag{44}$$

Because of the arbitrariness of b and c, we conclude that x_0 satisfies (DV).

Second, we prove that $x_0(t) \to 0$ as $|t| \to +\infty$. By the argument of Lemma 2.5, for each $i \in \mathbb{N}$ there is $n_i \in \mathbb{N}$ such that for all $n \geq n_i$ we have

$$\int_{-T_i}^{T_i} e^{Q(t)}(|x_n(t)|^2 + |\dot{x}_n(t)|^2)dt \leq \|x_n\|^2 \leq M_2^2. \tag{45}$$

Letting $n \to +\infty$, we obtain

$$\int_{-T_i}^{T_i} e^{Q(t)}(|x_0(t)|^2 + |\dot{x_0}(t)|^2)dt \leq M_2^2. \tag{46}$$

As $i \to +\infty$, we have

$$\int_{-\infty}^{+\infty} e^{Q(t)}(|x_0(t)|^2 + |\dot{x_0}(t)|^2)dt \leq M_2^2. \tag{47}$$

Hence, we get

$$\int_{|t|\geq r} e^{Q(t)}(|x_0(t)|^2 + |\dot{x_0}(t)|^2)dt \to 0 \; as \; r \to +\infty. \tag{48}$$

By Corollary 2.2 in [10], we have

$$|x_0(t)|^2 \leq \int_{t-1}^{t+1} (|x_0(s)|^2 + |\dot{x_0}(s)|^2)ds \tag{49}$$

for every $t \in \mathbb{R}$. By (48) and (49) we conclude that

$$x_0(t) \to 0 \, as \, |t| \to \infty. \tag{50}$$

We have to show that $\dot{x}_0(t) \to 0$ *as* $|t| \to \infty$. By Corollary 2.2 in [10] we have

$$|\dot{x}_0(t)|^2 \leq \int_{t-1}^{t+1} (|x_0(s)|^2 + |\dot{x_0}(s)|^2)ds + \int_{t-1}^{t+1} |\ddot{x_0}(s)|^2 ds, \tag{51}$$

for every $t \in \mathbb{R}$. Since $x_0 \in H^1_Q(\mathbb{R}, \mathbb{R}^N) \subset H^1(\mathbb{R}, \mathbb{R}^N)$, we get

$$\int_{t-1}^{t+1} (|x_0(s)|^2 + |\dot{x_0}(s)|^2)ds \to 0 \ as \ |t| \to \infty. \tag{52}$$

Hence, it suffices to prove that

$$\int_{t-1}^{t+1} |\ddot{x_0}(s)|^2 ds \to 0 \ as \ |t| \to \infty. \tag{53}$$

By (DV), we have

$$\begin{aligned}
\int_{t-1}^{t+1} |\ddot{x}(s)|^2 ds &= \int_{t-1}^{t+1} |q(s)\dot{x}(s) + V'(s, x(s))|^2 ds \\
&\leq \|q\|^2_\infty \int_{t-1}^{t+1} |\dot{x}(s)|^2 ds + \int_{t-1}^{t+1} |V'(s, x(s))|^2 ds \\
&+ 2\|q\|_\infty \left(\int_{t-1}^{t+1} |\dot{x}(s)|^2 ds\right)^{\frac{1}{2}} \left(\int_{t-1}^{t+1} |V'(s, x(s))|^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\int_{t-1}^{t+1} |\dot{x_0}(s)|^2 ds \to 0$ as $|t| \to \infty$, $x_0(t) \to 0$ as $|t| \to \infty$ and $V'(t, x_0) \to 0$ as $|x_0| \to 0$ uniformly in $t \in \mathbb{R}$, then (53) follows.

Let us show that x_0 is nontrivial. Consider the function Ψ defined by $\Psi(0) = 0$ and for $s > 0$

$$\Psi(s) = \max_{t \in \mathbb{R}, 0 < |x| \leq s} \frac{W'(t, x).x}{|x|^2}. \tag{54}$$

Then Ψ is a continuous, non-decreasing function and $\Psi(s) \geq 0$ for $s \geq 0$. The definition of Ψ implies that

$$\int_{-T_n}^{T_n} W'(t, x_n(t)).x_n(t)dt \leq \Psi(\|x_n\|_{L^\infty([-T_n, T_n], \mathbb{R}^N)})\|x_n\|^2, \tag{55}$$

for every $n \in \mathbb{N}$. Since $\Phi'(x_n).x_n = 0$, we have

$$\begin{aligned}
\int_{-T_n}^{T_n} W'(t, x_n(t)).x_n(t)dt &= \int_{-T_n}^{T_n} |\dot{x}_n(t)|^2 dt - \int_{-T_n}^{T_n} (A\dot{x}_n(t).x_n(t))dt \\
&+ \int_{-T_n}^{T_n} K'(t, x_n(t)).x_n(t)dt.
\end{aligned} \tag{56}$$

From (55), (56) and (K_1) we obtain

$$\begin{aligned}\Psi(\|x_n\|_{L^\infty([-T_n,T_n],\mathbb{R}^N)})\|x_n\|^2 &\geq \int_{-T_n}^{T_n} e^{Q(t)}|\dot{x}_n(t)|^2dt + \int_{-T_n}^{T_n} e^{Q(t)}K'(t,x_n(t)).x_n(t)dt \\ &\geq \int_{-T_n}^{T_n} e^{Q(t)}|\dot{x}_n(t)|^2dt + a\int_{-T_n}^{T_n} e^{Q(t)}|x_n(t)|^2dt \\ &\geq \min\{1,a\}\|x_n\|^2.\end{aligned}$$

Since $\|x_n\| > 0$, it follows that

$$\Psi(\|x_n\|_{L^\infty([-T_n,T_n],\mathbb{R}^N)}) \quad \geq \quad \min\{1,a\} > 0. \tag{57}$$

If $\|x_n\|_{L^\infty([-T_n,T_n],\mathbb{R}^N)} \to 0$ as $n \to \infty$, we would have
$\Psi(0) \geq \min\{1,a\} > 0$, a contradiction. Passing to a subsequence of (x_n) if necessary, there is a constant $M_7 > 0$ such that

$$\|x_n\|_{L^\infty([-T_n,T_n],\mathbb{R}^N)} \geq M_7 \tag{58}$$

for every $n \in \mathbb{N}$. Now, suppose $x_0 \equiv 0$ and let x_n be the function defined in Lemma 2.5 , extended by 0 in $\mathbb{R} \setminus [-T_n, T_n]$. For $R > 0$ we have

$$\begin{aligned}\|x_n\|^2 &= \int_{-T_n}^{T_n} e^{Q(t)}(|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt \\ &= \int_{\mathbb{R}} e^{Q(t)}(|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt \\ &= \int_{-R}^{R} e^{Q(t)}(|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt + \int_{\mathbb{R}\setminus[-R,R]} e^{Q(t)}(|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt \\ &\to 0 \; as \; n \to \infty.\end{aligned}$$

which is in contradiction with (33). Hence x_0 is nontrivial.

The proof of Theorem is complete. □

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