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Uniqueness of entire functions whose difference polynomials sharing a polynomial of certain degree with finite weight

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Abstract. The purpose of the paper is to study the possible uniqueness relation of entire functions when the difference polynomial generated by them sharing a non zero polynomial of certain degree. The result obtained in the paper will improve and generalize a number of recent results in a compact and convenient way.

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1. Introduction, Definitions and Results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let $a \in \mathbb{C}$. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [6]). For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly outside a set of finite linear measure. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\longrightarrow\infty$, outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r,a)=S(r,f)$. The order of f is defined by

$$\sigma(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}.$$

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

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We say that a finite value z_0 is called a fixed point of f if $f(z_0) = z_0$ or z_0 is a zero of $f(z) - z$.

For the sake of simplicity we also use the notation

$$m^* := \begin{cases} 0, & \text{if } m = 0 \\ m, & \text{if } m \in \mathbb{N} \end{cases}$$

Let $f(z)$ be a transcendental meromorphic function, n be a positive integer. During the last few decades many authors investigated the value distributions of $f^n f^{'}$. Specially in 1959, W.K. Hayman (see [5], Corollary of Theorem 9) proved the following theorem.

Theorem A. [5] Let f be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.

The case $n = 2$ was settled by Mues [14] in 1979. Bergweiler and Eremenko [1] showed that $ff^{'} - 1$ has infinitely many zeros.

For an analog of the above results Laine and Yang investigated the value distribution of difference products of entire functions in the following manner.

Theorem B. [10] Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2$, $f^n(z)f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.

Afterwards, Liu and Yang improved Theorem B and obtained the next result.

Theorem C. [13] Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2$, $f^n(z)f(z+c) - p(z)$ has infinitely many zeros, where $p(z)$ is a non-zero polynomial.

Next we recall the uniqueness result corresponding to Theorem A, obtained by Yang and Hua [17] which may be considered a gateway to a new research in the direction of sharing values of differential polynomials.

Theorem D. [13] Let f and g be two non-constant entire functions, $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n f^{'}$ and $g^n g^{'}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1, c_2, c \in \mathbb{C}$ satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.

In 2001, Fang and Hong studied the uniqueness of differential polynomials of the form $f^n(f-1)f^{'}$ and proved the following uniqueness result.

Theorem E. [4] Let f and g be two transcendental entire functions, and let $n \geq 11$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f = g$.

In 2004, Lin and Yi extended the above result in view of the fixed point and they proved the following.

Theorem F. [12] Let f and g be two transcendental entire functions, and let $n \geq 7$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then $f = g$.

In 2010, Zhang got a analogue result in difference.

Theorem G. [19] Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer. If $f(z)^n(f(z)-1)f(z+c)$ and $g(z)^n(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.

In 2010, Qi, Yang and Liu obtained the difference counterpart of Theorem D by proving the following theorem.

Theorem H. [15] Let f and g be two transcendental entire functions of finite order, and c be a nonzero complex constant; let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share z CM, then $f \equiv t_1 g$ for a constant t_1 that satisfies $t_1^{n+1} = 1$.

Theorem I. [15] Let f and g be two transcendental entire functions of finite order, and c be a nonzero complex constant; let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share 1 CM, then $fg \equiv t_2$ or $f \equiv t_3 g$ for some constants t_2 and t_3 that satisfy $t_3^{n+1} = 1$.

X.M. Li et. al. [11] [Theorem 1.1] replaced the fixed point sharing in the above two theorems to sharing a polynomial with $deg < \frac{n+1}{2}$.

So we see that there are many generalization in terms of difference operator. The purpose of this paper is to study the uniqueness problem for more general difference polynomials namely $f^n P(f) f(z+c)$ and $g^n P(g) g(z+c)$ sharing a non-zero polynomial so that improved version of all the above results can be unified under a single result. We also relax the nature of sharing with the notion of weighted sharing introduced in [8]- [9]. The following theorem is the main result of the paper.

Theorem 1. Let f and g be two transcendental entire functions of finite order, c be a non-zero complex constant and let $p(z)$ be a nonzero polynomial with $deg(p) \leq n-1$, $n(\geq 1)$, $m^*(\geq 0)$ be two integers such that $n > m^* + 5$. Let $P(\omega) = a_m \omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1 \omega + a_0$ be a nonzero polynomial. If $f^n P(f) f(z+c) - p$ and $g^n P(g) g(z+c) - p$ share $(0, 2)$, then

(I) when $P(\omega) = a_m \omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1 \omega + a_0$ is a nonzero polynomial, one of the following three cases holds:

(I1) $f(z) \equiv t g(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,

(I2) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$,

(I3) $P(\omega)$ reduces to a nonzero monomial, namely $P(\omega) = a_i \omega^i \not\equiv 0$, for $i \in \{0, 1, \ldots, m\}$, if $p(z)$ is a nonzero constant b, then $f(z) = e^{\alpha(z)}$, $g = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $a_i^2 e^{(n+i+1)d} = b^2$;

(II) when $P(\omega) = \omega^m - 1$, then $f \equiv tg$ for some constant t such that $t^m = 1$;

(III) when $P(\omega) = (\omega - 1)^m (m \geq 2)$, one of the following two cases holds:

(III1) $f(z) \equiv g(z)$,

(III2) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n (\omega_1 - 1)^m \omega_1(z+c) - \omega_2^n (\omega_2 - 1)^m \omega_2(z+c)$;

(IV) when $P(\omega) \equiv c_0$, one of the following two cases holds:

(IV1) $f \equiv tg$ for some constant t such that $t^{n+1} = 1$,

(IV2) $f(z) = e^{\alpha(z)}$, $g = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $c_0^2 e^{(n+1)d} = b^2$.

We now explain following definitions and notations which are used in the paper.

Definition 1. [7] Let $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer p we denote by $N(r, a; f \mid \leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f\mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we can define $N(r,a;f\mid\geq p)$ and $\overline{N}(r,a;f\mid\geq p)$.

Definition 2. [9] Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$ and k times if $m>k$. Then

$$N_k(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+...+\overline{N}(r,a;f\mid\geq k).$$

Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

Definition 3. [8, 9] Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m\leq k$ and k+1 times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(>k)$ if and only if it is an a-point of g with multiplicity $n(>k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0\leq p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

2. Lemmas

Lemma 1. [16] Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r,a_i(z))=S(r,f)$ for $i=0,1,2,...,n$. Then

$$T(r,a_nf^n+a_{n-1}f^{n-1}+...+a_1f+a_0)=nT(r,f)+S(r,f).$$

Lemma 2. [2] Let $f(z)$ be a meromorphic function of finite order σ, and let c be a fixed nonzero complex constant. Then for each $\varepsilon>0$, we have

$$m(r,\frac{f(z+c)}{f(z)})+m(r,\frac{f(z)}{f(z+c)})=O(r^{\sigma-1+\varepsilon}).$$

Lemma 3. [2] Let f be a meromorphic function of finite order σ, $c\neq 0$ be fixed. Then for each $\varepsilon>0$, we have

$$T(r,f(z+c))=T(r,f)+O(r^{\sigma-1+\varepsilon})+O(\log r).$$

Lemma 4. Let f be an entire function of finite order σ, c be a fixed nonzero complex constant and let $n\in\mathbb{N}$ and $P(\omega)$ be defined as in Theorem 1. Then for each $\varepsilon>0$, we have

$$T(r,f^nP(f)f(z+c))=T(r,f^{n+1}P(f))+O(r^{\sigma-1+\varepsilon}).$$

Proof. By Lemma 2 we have

$$\begin{aligned} T(r,f^nP(f)f(z+c)) &= m(r,f^nP(f)f(z+c)) \\ &\le m(r,f^nP(f)f)+m(r,\frac{f(z+c)}{f(z)}) \\ &\le m(r,f^{n+1}P(f))+O(r^{\sigma-1+\varepsilon}) \\ &= T(r,f^{n+1}P(f))+O(r^{\sigma-1+\varepsilon}). \end{aligned}$$

Also we have

$$\begin{aligned} T(r,f^{n+1}P(f)) &= m(r,f^nP(f)f) \\ &\le m(r,f^nP(f)f(z+c))+m(r,\frac{f(z)}{f(z+c)}) \\ &\le m(r,f^nP(f)f(z+c))+O(r^{\sigma-1+\varepsilon}) \\ &\le T(r,f^nP(f)f(z+c))+O(r^{\sigma-1+\varepsilon}). \end{aligned}$$

Therefore $T(r,f^nP(f)f(z+c))=T(r,f^{n+1}P(f))+O(r^{\sigma-1+\varepsilon})$. □

Remark 1. Under the condition of Lemma 4, by Lemma 1 we have $S(r,f^nP(f)f(z+c))=S(r,f)$.

Lemma 5. [3] Let f be a non-constant meromorphic function of finite order and $c\in\mathbb{C}$. Then

$$N(r,0;f(z+c))\le N(r,0;f(z))+S(r,f),\quad N(r,\infty;f(z+c))\le N(r,\infty;f)+S(r,f),$$

$$\overline{N}(r,0;f(z+c))\le \overline{N}(r,0;f(z))+S(r,f),\quad \overline{N}(r,\infty;f(z+c))\le \overline{N}(r,\infty;f)+S(r,f),$$

Lemma 6. Let f be a transcendental entire function of finite order σ, c be a fixed nonzero complex constant, $n(\ge 1)$, $m^*(\ge 0)$ be two integers and let $a(z)(\not\equiv 0,\infty)$ be a small function of f. If $n>1$, then $f^nP(f)f(z+c)-a(z)$ has infinitely many zeros.

Proof. Let $\Phi=f^nP(f)f(z+c)$. Now in view of Lemma 5 and the second theorem for small functions (see [18]) we get

$$\begin{aligned} T(r,\Phi) &\le \overline{N}(r,0;\Phi)+\overline{N}(r,\infty;\Phi)+\overline{N}(r,a(z);\Phi)+(\varepsilon+o(1))\,T(r,f) \\ &\le \overline{N}(r,0;f^nP(f))+\overline{N}(r,0;f(z+c))+\overline{N}(r,a(z);\Phi)+(\varepsilon+o(1))\,T(r,f) \\ &\le 2\,\overline{N}(r,0;f)+\overline{N}(r,0;P(f))+\overline{N}(r,a(z);\Phi)+(\varepsilon+o(1))\,T(r,f) \\ &\le (2+m^*)\,T(r,f)+\overline{N}(r,a(z);\Phi)+(\varepsilon+o(1))\,T(r,f), \end{aligned}$$

for all $\varepsilon>0$.
From Lemmas 1 and 4 we get

$$(n+m^*+1)\,T(r,f)\le(2+m^*)\,T(r,f)+\overline{N}(r,a(z);\Phi)+(\varepsilon+o(1))\,T(r,f).$$

Take $\varepsilon<1$. Since $n>1$ from above one can easily say that $\Phi-a(z)$ has infinitely many zeros.
This completes the Lemma. □

Lemma 7. [9] Let f and g be two non-constant meromorphic functions sharing $(1,2)$. Then one of the following holds:

(i) $T(r,f)\le N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)+S(r,f)+S(r,g)$,

(ii) $fg\equiv 1$,

(iii) $f \equiv g$.

Lemma 8. [Hadamard Factorization Theorem] Let f be an entire function of finite order ρ with zeros $a_1, a_2, \ldots$, each zeros is counted as often as its multiplicity. Then f can be expressed in the form

$$f(z) = Q(z)e^{\alpha(z)},$$

where $\alpha(z)$ is a polynomial of degree not exceeding $[\rho]$ and $Q(z)$ is the canonical product formed with the zeros of f.

Lemma 9. Let f and g be two transcendental entire functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $p(z)$ be a nonzero polynomial such that $deg(p) \leq n-1$, where $n \in \mathbb{N}$. Let $P(\omega)$ be a nonzero polynomial defined as in Theorem 1. Suppose

$$f^n P(f) f(z+c) g^n P(g) g(z+c) \equiv p^2.$$

Then $P(\omega)$ reduces to a nonzero monomial, namely $P(\omega) = a_i\omega^i \not\equiv 0$, for $i \in \{0, 1, \ldots, m\}$. If $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = e^{\alpha(z)}$, $g = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $a_i^2 e^{(n+i+1)d} = b^2$.

Proof. Suppose

$$f^n P(f) f(z+c) g^n P(g) g(z+c) \equiv p^2. \tag{2.1}$$

We consider the following cases:
Case 1: Let $deg(p(z)) = l(\geq 1)$.
From the assumption that f and g are two transcendental entire functions, we deduce by (2.1) that $N(r, 0; f^n P(f)) = O(\log r)$ and $N(r, 0; g^n P(g)) = O(\log r)$.
First we suppose that $P(\omega)$ is not a nonzero monomial. For the sake of simplicity let $P(\omega) = \omega - a$ where $a \in \mathbb{C} \setminus \{0\}$. Clearly $\Theta(0; f) + \Theta(a; f) = 2$, which is impossible for an entire function. Thus $P(\omega)$ reduces to a nonzero monomial, namely $P(\omega) = a_i\omega^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$ and so (2.1) reduces to

$$a_i^2 f^{n+i} f(z+c) g^{n+i} g(z+c) \equiv p^2. \tag{2.2}$$

From (2.2) it follows that $N(r, 0; f) = O(\log r)$ and $N(r, 0; g) = O(\log r)$. Now by Lemma 8 we obtain that $f = h_1 e^{\alpha_1}$ and $f = h_2 e^{\beta_1}$, where h_1, h_2 are two nonzero polynomials and α_1 and β_1 are two non-constant polynomials.
By virtue of the polynomial $p(z)$, from (2.2) we arrive at a contradiction.
Case 2: Let $p(z) = b \in \mathbb{C} \setminus \{0\}$.
Then from (2.1) we have

$$f^n P(f) f(z+c) g^n P(f) g(z+c) \equiv b^2. \tag{2.3}$$

Now from the assumption that f and g are two non-constant entire functions, we deduce by (2.3) that $f^n P(f) \neq 0$ and $g^n P(g) \neq 0$. By Picard's theorem, we claim that $P(\omega) = a_i\omega^i \not\equiv 0$ for $i \in \{0, 1, \ldots, m\}$, otherwise the Picard's exception values are atleast three, which is a contradiction. Then (2.3) reduces to

$$a_i^2 f^{n+i} f(z+c) g^{n+i} g(z+c) \equiv b^2. \tag{2.4}$$

Hence by Lemma 8 we obtain that

$$f = e^{\alpha}, \quad g = e^{\beta}, \tag{2.5}$$

where $\alpha(z)$, $\beta(z)$ are two non-constant polynomials.
Now from (2.4) and (2.5) we obtain

$$(n+i)(\alpha(z)+\beta(z))+\alpha(z+c)+\beta(z+c)\equiv d_1,$$

where $d_1\in\mathbb{C}$, i.e.,

$$(n+i)(\alpha^{'}(z)+\beta^{'}(z))+\alpha^{'}(z+c)+\beta^{'}(z+c)\equiv 0. \tag{2.6}$$

Let $\gamma(z)=\alpha^{'}(z)+\beta^{'}(z)$. Then from (2.6) we have

$$(n+i)\,\gamma(z)+\gamma(z+c)\equiv 0. \tag{2.7}$$

We assert that $\gamma(z)\equiv 0$. It not suppose $\gamma(z)\not\equiv 0$. Note that if $\gamma(z)\equiv d_2\in\mathbb{C}$, from (2.7) we must have $d_2=0$. Suppose that $deg(\gamma)\geq 1$. Let $\gamma(z)=\sum\limits_{i=1}^{m} b_iz^i$, where $b_m\neq 0$. Therefore the co-efficient of z^m in $(n+i)\gamma(z)+\gamma(z+c)$ is $(n+1+i)b_m\neq 0$. Thus we arrive at a contradiction from (2.7). Hence $\gamma(z)\equiv 0$, i.e., $\alpha+\beta\equiv d\in\mathbb{C}$. Also from (2.4) we have $a_i^2e^{(n+i+1)d}=b^2$. This completes the proof. □

Lemma 10. Let f and g be two transcendental entire functions of finite order, $c\in\mathbb{C}\setminus\{0\}$ and $p(z)$ be a nonzero polynomial such that $deg(p)\leq n-1$, where $n\in\mathbb{N}$. Let $P(\omega)$ be defined as in Theorem 1 with at least two of a_i, $i=0,1,\ldots,m$ are nonzero. Then

$$f^nP(f)f(z+c)g^nP(g)g(z+c)\not\equiv p^2.$$

Proof. Proof of the Lemma follows from Lemma 9. □

Lemma 11. Let f, g be two transcendental entire functions of finite order, $c\in\mathbb{C}\setminus\{0\}$ and $n\in\mathbb{N}$ with $n>1$. If

$$f^nP(f)f(z+c)\equiv g^nP(g)g(z+c),$$

where $P(\omega)$ is defined as in Theorem 1 then

(I) when $P(\omega)=a_m\omega^m+a_{m-1}\omega^{m-1}+\ldots+a_1\omega+a_0$, one of the following two cases holds:

(I1) $f(z)\equiv tg(z)$ for a constant t such that $t^d=1$, where $d=GCD(n+m,\ldots,n+m-i,\ldots,n)$, $a_{m-i}\neq 0$ for some $i=1,2,\ldots,m$,

(I2) f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(\omega_1,\omega_2)=\omega_1^nP(\omega_1)\omega_1(z+c)-\omega_2^nP(\omega_2)\omega_2(z+c)$;

(II) when $P(\omega)=\omega^m-1$, then $f\equiv tg$ for some constant t such that $t^m=1$;

(III) when $P(\omega)=(\omega-1)^m(m\geq 2)$, one of the following two cases holds:

(III1) $f(z)\equiv g(z)$,

(III2) f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m\omega_1(z+c)-\omega_2^n(\omega_2-1)^m\omega_2(z+c)$;

(IV) when $P(w)\equiv c_0$, then $f\equiv tg$ for some constant t such that $t^{n+1}=1$.

Proof. Suppose

$$f^nP(f)f(z+c)\equiv g^nP(g)g(z+c). \tag{2.8}$$

Since g is transcendental entire function, hence $g(z),g(z+c)\not\equiv 0$.
We consider following two cases.

Case 1. $P(\omega) \not\equiv c_0$.
Let $h = \frac{f}{g}$. If h is a constant, by putting $f = hg$ in (2.8) we get

$$a_m g^m(h^{n+m+1}-1)+a_{m-1}g^{m-1}(h^{n+m}-1)+\ldots+a_1 g(h^{n+2}-1)+a_0(h^{n+1}-1)\equiv 0,$$

which implies that $h^d = 1$, where $d = GCD(n+m+1,\ldots,n+m+1-i,\ldots,n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0,1,\ldots,m\}$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$,where $d = GCD(n+m+1,\ldots,n+m+1-i,\ldots,n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0,1,\ldots,m\}$.
If h is not a constant, then we know by (2.8) that f and g satisfying the algebraic equation $R(f,g) = 0$, where $R(\omega_1,\omega_2) = \omega_1^n P(\omega_1)\omega_1(z+c) - \omega_2^n P(\omega_2)\omega_2(z+c)$.
We now discuss the following Subcases.
Subcase 1. $P(\omega) = \omega^m - 1$.
Then from (2.8) we have

$$f^n(f^m-1)f(z+c)\equiv g^n(g^m-1)g(z+c). \tag{2.9}$$

Let $h = \frac{f}{g}$. Clearly from (2.9) we get

$$g^m[h^{n+m}h(z+c)-1]\equiv h^n h(z+c)-1. \tag{2.10}$$

First we suppose that h is non-constant. We assert that $h^{n+m}h(z+c)$ is non-constant. If not let $h^{n+m}h(z+c) \equiv c_1 \in \mathbb{C}\setminus\{0\}$. Then we have

$$h^{n+m}\equiv\frac{c_1}{h(z+c)}.$$

Now by Lemmas 1 and 3 we get

$$(n+m)\,T(r,h)\leq T(r,h)+S(r,h),$$

which contradicts with $n > m+5$. Thus from (2.10) we have

$$g^m\equiv\frac{h^n h(z+c)-1}{h^{n+m}h(z+c)-1}. \tag{2.11}$$

Let z_0 be a zero of $h^{n+m}h(z+c)-1$. Since g is an entire function, it follows that z_0 is also a zero of $h^n h(z+c)-1$. Consequently z_0 is a zero of $h^m - 1$ and so

$$\overline{N}(r,0;h^{n+m}h(z+c))\leq\overline{N}(r,0;h^m)\leq m\,T(r,h)+O(1).$$

So in view of Lemmas 1, 4, 5 and the second fundamental theorem we get

$$\begin{aligned}(n+m+1)\,T(r,h) &= T(r,h^{n+m}h(z+c))+S(r,h)\\ &\leq \overline{N}(r,0;h^{n+m}h(z+c))+\overline{N}(r,1;h^{n+m}h(z+c))+S(r,h)\\ &\leq 2\,N(r,0;h)+m\,T(r,h)+S(r,h)\\ &\leq (m+2)\,T(r,h)+S(r,h),\end{aligned}$$

which contradicts with $n > 1$.
Hence h is a constant. Since g is transcendental entire function, from (2.10) we have

$$h^{n+m}h(z+c)-1\equiv 0 \Longleftrightarrow h^n h(z+c)-1\equiv 0$$

and so $h^m = 1$. Thus $f(z) \equiv tg(z)$ for a constant t such that $t^m = 1$.
Subcase 2. Let $P(\omega) = (\omega-1)^m$.
Then from (2.8) we have

$$f^n(f-1)^m f(z+c)\equiv g^n(g-1)^m g(z+c). \tag{2.12}$$

Let $h = \frac{f}{g}$. If $m = 1$, then the result follows from Subcase 1.
For $m \geq 2$: First we suppose that h is non-constant:
Then from (2.12) we can say that f and g satisfying the algebraic equation $R(f, g) = 0$, where

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m \omega_1(z + c) - \omega_2^n(\omega_2 - 1)^m \omega_2(z + c).$$

Next we suppose that h is a constant:
Then from (2.12) we get

$$f^n f(z + c) \sum_{i=0}^{m} (-1)^i\ {}^mC_{m-i}\ f^{m-i} \equiv g^n g(z + c) \sum_{i=0}^{m} (-1)^i\ {}^mC_{m-i} g^{m-i}. \qquad (2.13)$$

Now substituting $f = gh$ in (2.13) we get

$$\sum_{i=0}^{m} (-1)^i\ {}^mC_{m-i}\ g^{m-i}(h^{n+m+1-i} - 1) \equiv 0,$$

which implies that $h = 1$. Hence $f \equiv g$.
Case 2. $P(\omega) \equiv c_0$.
Let $h = \frac{f}{g}$. Then from (2.8) we have

$$h^n(z) \equiv \frac{1}{h(z + c)}. \qquad (2.14)$$

Thus from Lemmas 1 and 3 we get

$$n\, T(r, h) = T(r, h(z + c)) + O(1) = T(r, h) + S(r, h),$$

which is a contradiction since $n \geq 2$. Hence h must be a constant, which implies that $h^{n+1} = 1$, thus $f = tg$ and $t^{n+1} = 1$.
This completes the the proof. □

3. Proofs of the Theorem

Proof of Theorem 1. Let $F = \frac{f^n P(f) f(z+c)}{p}$ and $G = \frac{g^n P(g) g(z+c)}{p}$. Then F and G share $(1, 2)$ except the zeros of $p(z)$. Now applying Lemma 7 we see that one of the following three cases holds.
Case 1. Suppose

$$T(r, f) \leq N_2(r, 0; F) + N_2(r, 0; G) + S(r, F) + S(r, G).$$

Now by applying Lemmas 1 and 7 we have

$$\begin{aligned}
T(r,F) &\leq N_2(r,0;F)+N_2(r,0;G)+S(r,f)+S(r,g)\\
&= N_2(r,0;f^nP(f)f(z+c))+N_2(r,0;g^nP(g)g(z+c))+S(r,f)+S(r,g)\\
&\leq N_2(r,0;f^nP(f))+N_2(r,0;f(z+c))+N_2(r,0;g^nP(g))+N_2(r,0;g(z+c))\\
&\quad +S(r,f)+S(r,g)\\
&\leq 2\,N(r,0;f)+N(r,0;P(f))+N(r,0;f(z+c))+2\,N(r,0;g)+N(r,0;P(g))\\
&\quad +N(r,0;g(z+c))+S(r,f)+S(r,g)\\
&\leq (2+m^*)\,T(r,f)+N(r,0;f)+(2+m^*)\,T(r,g)+N(r,0;g)+S(r,f)+S(r,g)\\
&\leq (3+m^*)\,T(r,f)+(3+m^*)\,T(r,g)+S(r,f)+S(r,g)\\
&\leq (6+2m^*)\,T(r)+S(r)
\end{aligned}$$

From Lemmas 1 and 4 we have

$$(n+m^*+1)\,T(r,f)\leq (6+2m^*)\,T(r)+S(r). \tag{3.1}$$

Similarly we have

$$(n+m^*+1)\,T(r,g)\leq (6+2m^*)\,T(r)+S(r). \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$(n+m^*+1)\,T(r)\leq (6+2m^*)T(r)+S(r),$$

which contradicts with $n>5+m^*$.

Case 2. $F\equiv G$.

Then we have

$$f^nP(f)f(z+c)\equiv g^nP(g)g(z+c)$$

and so the result follows from Lemma 11.

Case 3. $FG\equiv 1$.

Then we have

$$f^nP(f)f(z+c)g^nP(g)g(z+c)\equiv p^2$$

and so the result follows from Lemma 9.

This completes the proof. □

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