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Lacunary Ideal quasi Cauchy sequences

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Abstract. A real function is lacunary ideal ward continuous if it preserves lacunary ideal quasi Cauchy sequences where a sequence (x_n) is said to be lacunary ideal quasi Cauchy (or I_θ-quasi Cauchy) when $(\Delta x_n) = (x_{n+1} - x_n)$ is lacunary ideal convergent to 0. i.e. a sequence (x_n) of points in $\mathbb{R}$ is called lacunary ideal quasi Cauchy (or I_θ-quasi Cauchy) for every $\varepsilon > 0$ if

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in J_r} |x_{n+1} - x_n| \geq \varepsilon \right\} \in I.$$

Also we introduce the concept of lacunary ideal ward compactness and obtain results related to lacunary ideal ward continuity, lacunary ideal ward compactness, ward continuity, ward compactness, ordinary compactness, uniform continuity, ordinary continuity, δ-ward continuity, and slowly oscillating continuity. Finally we introduce the concept of ideal Cauchy continuous function in metric space and prove some results related to this notion.

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1. Introduction

The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. A sequence (x_n) of points in $\mathbb{R}$ is called quasi-Cauchy if (Δx_n) is a null sequence where $\Delta x_n = x_{n+1} - x_n$. In [1] Burton and Coleman named these sequences as "quasi-Cauchy" and in [7] Çakallı used the term "ward convergent to 0" sequences. From now on in this paper we also prefer to using the term "quasi-Cauchy" to using the term "ward convergent to 0" for simplicity. In terms of quasi-Cauchy we restate the definitions of ward compactness and ward continuity as follows: a function f is ward continuous if it preserves quasi-Cauchy sequences, i.e. $(f(x_n))$ is quasi-Cauchy whenever (x_n) is, and a subset E of $\mathbb{R}$ is ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x}$.

A Cauchy regular function is a special kind of continuous function between metric spaces. Cauchy continuous (or regular) functions have the useful property that they can always be extended to the Cauchy completion of their domain (see [14]).

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2. Preliminaries and Notations

It is known that a sequence (x_n) of points in $\mathbb{R}$, the set of real numbers, is slowly oscillating if

$$\lim_{\lambda\to 1^+} \overline{\lim}_n \max_{n+1\le k\le[\lambda n]} |x_k - x_n| = 0$$

where $[\lambda n]$ denotes the integer part of λn. This is equivalent to the following if $(x_m - x_n) \to 0$ whenever $1 \le \frac{m}{n} \to 1$ as, $m, n \to \infty$. Using $\varepsilon > 0$ and $\delta > 0$ this is also equivalent to the case when for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \ge N(\varepsilon)$ and $n \le m \le (1+\delta)n$.

A function defined on a subset E of $\mathbb{R}$ is called slowly oscillating continuous if it preserves slowly oscillating sequences, i.e. $(f(x_n))$ is slowly oscillating whenever (x_n) is.

The concept of statistical convergence is a generalization of the usual notion of convergence that, for real-valued sequences, parallels the usual theory of convergence. For a subset E of $\mathbb{N}$ the asymptotic density of E, denoted by $\delta(E)$, is given by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in E\}|,$$

if this limit exists, where $|\{k \le n : k \in E\}|$ denotes the cardinality of the set $\{k \le n : k \in E\}$. A sequence (x_n) is statistically convergent to ℓ (see [17]) if

$$\delta(\{n \in \mathbb{N} : |x_n - \ell| \ge \epsilon\}) = 0,$$

for every $\epsilon > 0$. In this case ℓ is called the statistical limit of $\mathbf{x}$. Schoenberg [29] studied some basic properties of statistical convergence and also studied the statistical convergence as a summability method. Fridy [19] gave characterizations of statistical convergence.

By a lacunary sequence $\theta = (k_r)_{r\in\mathbb{N}\cup\{0\}}$, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 \ne 0$ and $h_r : k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$, and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. Freedman et al., [18] introduced the notion of lacunary convergence as follows:

A sequence (x_n) of points in $\mathbb{R}$ is called lacunary convergent (or N_θ-convergent) to ℓ in $\mathbb{R}$ if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in J_r} |x_n - \ell| = 0,$$

and it is denoted by N_θ-$\lim_n x_n = \ell$. This defines a method of sequential convergence, i.e. $G(\mathbf{x}) := N_\theta - \lim x_n$. Any convergent sequence is N_θ-convergent, but the converse is not always true.

The notion of lacunary statistical convergence was introduced, and studied by Fridy and Orhan in [20] and [21] (see also [18]). A sequence (x_k) of points in $\mathbb{R}$ is called lacunary statistically convergent to an element ℓ of $\mathbb{R}$ if

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in J_r : |x_k - \ell| \ge \varepsilon\}| = 0,$$

for every positive real number ε. In this case we write $S_\theta - \lim x_n = \ell$. In 1937, Cartan [13] introduced the notion of the ideal convergence is the dual (equivalent) to the notion of filter convergence. The notion of the filter convergence is a generalization of the classical notion of convergence of a sequence and it has been an important tool in general topology and functional analysis. Nowadays many authors use an equivalent dual notion of the ideal convergence. Kostyrko et al. [26] and Nuray and Ruckle [27] independently studied in details about the notion of ideal convergence which is based on the structure of the admissible ideal I of subsets of natural numbers $\mathbb{N}$. Although an ideal is defined as a hereditary and additive family of subsets of a non-empty set X, here in our study it suffices to take I as a family of sets $I \subset P(\mathbb{N})$ (the power sets of $\mathbb{N}$) such that for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset P(\mathbb{N})$ is a *filter* on $\mathbb{N}$ if and only if $\phi \notin F$, for each $A, B \in F$, we have $A \cap B \in F$ and each $A \in F$ and each $A \subset B$, we have $B \in F$. An ideal I is called *non-trivial* ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A non-trivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals can be found in Kostyrko, et.al (see [26]). Recall that a sequence $\mathbf{x} = (x_n)$ of points in $\mathbb{R}$ is said to be I-convergent to the number ℓ if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\} \in I$. In this case we write $I\text{-}\lim x_n = \ell$. We see that I-convergence of a sequence (x_n) implies I-quasi-Cauchyness of (x_n). The notion of lacunary ideal convergence of real sequences was introduced in [32] and Hazarika [22, 23], introduced the lacunary ideal convergent sequences of fuzzy real numbers and studied some properties. Cakalli and Hazarika [2] introduced the concept of ideal quasi Cauchy sequences and proved some results related to ideal ward continuity and ideal ward compactness. For more details on ideal convergence we refer to [24, 25, 31].

Throughout this paper we assume I is a non-trivial admissible ideal in $\mathbb{N}$, also, $I(\mathbb{R})$ and ΔI will denote the set of all I-convergent sequences, and the set of all I-quasi-Cauchy sequences of points in $\mathbb{R}$, respectively. If we take, $I = I_f = \{A \subseteq \mathbb{N} : A$ is a finite subset $\}$, then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence, and $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ denote the asymptotic density of the set A, then I_δ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

Connor and Grosse-Erdman [15] gave sequential definitions of continuity for real functions calling G-continuity instead of A-continuity and their results cover the earlier works related to A-continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of s, space of all sequences, denoted by c_G, into $\mathbb{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$. In particular lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space c and st-lim denotes the statistical limit function $st\text{-}\lim \mathbf{x} = st\text{-}\lim_n x_n$ on the linear space $st(\mathbb{R})$. Also I-lim denotes the I-limit function $I\text{-}\lim \mathbf{x} = I\text{-}\lim_n x_n$ on the linear space $I(\mathbb{R})$.

A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$.

Recently, Cakalli gave new sequential definitions of compactness and slowly oscillating compactness in [8, 9, 10].

3. Lacunary ideal sequential compactness

First we recall the definition of G-sequentially compactness of a subset E of $\mathbb{R}$. A subset E of $\mathbb{R}$ is called G-sequentially compact if whenever (x_n) is a sequence of points in E there is subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) whose $G(\mathbf{y}) = \lim \mathbf{y}$ in E (see [11]). For regular methods any sequentially compact subset E of $\mathbb{R}$ is also G-sequentially compact and the converse is not always true. For any regular subsequential method G, a subset E of $\mathbb{R}$ is G-sequentially compact if and only if it is sequentially compact in the ordinary sense.

Although I_θ-sequential compactness is a special case of G-sequential compactness when $G = \lim$, we state the definition of I_θ-sequential compactness of a subset E of $\mathbb{R}$ as follows.

Definition 3.1. A subset E of $\mathbb{R}$ is called I_θ-sequentially compact if whenever (x_n) is a sequence of points in E there is I_θ-convergent subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) such that I_θ-$\lim \mathbf{y}$ is in E.

Lemma 3.1. *Sequential method I_θ is regular.*

Proof. The proof follows from the fact that I is admissible (see also [30]). □

Lemma 3.2. *Any I_θ-convergent sequence of points in $\mathbb{R}$ with a I_θ-limit ℓ has a convergent subsequence with the same limit ℓ in the ordinary sense.*

Proof. See Proposition 3.2. in [28] for a proof. □

Theorem 3.3. *The sequential method I_θ is regular and subsequential.*

Proof. Regularity of I_θ follows from Lemma 3.1, and subsequentiality of I_θ follows from Lemma 3.2. □

Theorem 3.4. *A subset of $\mathbb{R}$ is sequentially compact if and only if it is I_θ-sequentially compact.*

Proof. The proof easily follows from Corollary 3 on page 597 in [9] and Lemma 3.2, so is omitted. □

Although I_θ-sequential continuity is a special case of G-sequential continuity when $G = \lim$ (see also Definition 2 in [30]), we state the definition of I_θ-sequential continuity of a function defined on a subset E of $\mathbb{R}$ as follows.

Definition 3.2. A function $f : E \to \mathbb{R}$ is I_θ-sequentially continuous at a point x_0 if, given a sequence (x_n) of points in E, I_θ-$\lim \mathbf{x} = x_0$ implies that I_θ-$\lim f(\mathbf{x}) = f(x_0)$.

Theorem 3.5. *Any I_θ-sequentially continuous function at a point x_0 is continuous at x_0 in the ordinary sense.*

Proof. Let f be any I_θ-sequentially continuous function at point x_0. Since any proper admissible ideal is a regular subsequential method, it follows from Theorem 13 on page 316 in [11] that f is continuous in the ordinary sense. □

Theorem 3.6. *Any continuous function at a point x_0 is I_θ-sequentially continuous at x_0.*

Proof. For the proof of the theorem see Theorem 2.2 [30]. □

Combining Theorem 3.5 and Theorem 3.6 we have the following.

Corollary 3.7. *A function is I_θ-sequentially continuous at a point x_0 if and only if it is continuous at x_0.*

Corollary 3.8. *For any regular subsequential method G, a function is G-sequentially continuous at a point x_0, then it is I_θ-sequentially continuous at x_0.*

Proof. The proof follows from Theorem 13 on page 316 in [11]. □

Corollary 3.9. *Any ward continuous function on a subset E of $\mathbb{R}$ is I_θ-sequentially continuous on E.*

Theorem 3.10. *If a function is slowly oscillating continuous on a subset E of $\mathbb{R}$, then it is I_θ-sequentially continuous on E.*

Proof. Let f be any slowly oscillating continuous on E. It follows from Theorem 2.1 in [8] that f is continuous. By Theorem 3.6 we see that f is I_θ-sequentially continuous on E. This completes the proof. □

Theorem 3.11. *If a function is δ-ward continuous on a subset E of $\mathbb{R}$, then it is I_θ-sequentially continuous on E.*

Proof. Let f be any δ-ward continuous function on E. It follows from Corollary 2 on page 399 in [12] that f is continuous. By Theorem 3.6 we obtain that f is I_θ-sequentially continuous on E. This completes the proof. □

Corollary 3.12. *If a function is quasi-slowly oscillating continuous on a subset E of $\mathbb{R}$, then it is I_θ-sequentially continuous on E.*

Proof. Let f be any quasi-slowly oscillating continuous on E. It follows from Theorem 3.2 in [16] that f is continuous. By Theorem 3.6 we deduce that f is I_θ-sequentially continuous on E. This completes the proof. □

4. Lacunary ideal quasi Cauchy sequences

We say that a sequence $\mathbf{x} = (x_n)$ is I_θ-ward convergent to a number ℓ if I_θ-$\lim_{n\to\infty} \Delta x_n = \ell$ where $\Delta x_n = x_{n+1} - x_n$. For the special case $\ell = 0$ we say that $\mathbf{x}$ is lacunary ideal quasi-Cauchy, or I_θ-quasi-Cauchy, in place of I_θ-ward convergent to 0. Thus a sequence (x_n) of points of $\mathbb{R}$ is I_θ-quasi-Cauchy if (Δx_n) is I_θ-convergent to 0. We denote ΔI_θ the set of all lacunary ideal quasi Cauchy sequences of points in $\mathbb{R}$.

Now we give the definitions of I_θ-ward compactness of a subset of $\mathbb{R}$.

Definition 4.1. A subset E of $\mathbb{R}$ is called I_θ-ward compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in E there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ such that I_θ-$\lim_{k\to\infty} \Delta z_k = 0$.

We note that this definition of I_θ-ward compactness can not be obtained by any G-sequential compactness, i.e. by any summability matrix A, even by the summability matrix $A = (a_{nk})$ defined by $a_{nk} = -1$ if $k = n$ and $a_{kn} = 1$ if $k = n + 1$ and

$$G(x) = I_\theta - \lim A\mathbf{x} = I_\theta - \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{kn} x_n = I_\theta - \lim_{k\to\infty} \Delta x_k$$

(see [9] for the definition of G-sequential compactness). Despite that G-sequential compact subsets of $\mathbb{R}$ should include the singleton set $\{0\}$, I_θ-ward compact subsets of $\mathbb{R}$ do not have to include the singleton $\{0\}$.

Theorem 4.1. *A subset E of $\mathbb{R}$ is ward compact if and only if it is I_θ-ward compact.*

Proof. Let us suppose first that E is ward compact. It follows from Lemma 2 on page 1725 in [6] that E is bounded. Then for any sequence (x_n), there exists a convergent subsequence (x_{n_k}) of (x_n) whose limit may be in E or not. Then the sequence (Δx_{n_k}) is a null sequence. Since I is a regular method, (Δx_{n_k}) is I_θ-convergent to 0, so it is I_θ-quasi-Cauchy. Thus E is I_θ-ward compact. Now to prove the converse suppose that E is I_θ-ward compact. Take any sequence (x_n) of points in E. Then there exists an I_θ-quasi-Cauchy subsequence (x_{n_k}) of (x_n). Since I_θ is subsequential there exists a convergent subsequence $(x_{n_{k_m}})$ of (x_{n_k}). Therefore $(x_{n_{k_m}})$ is a quasi-Cauchy subsequence of the sequence (x_n). Thus E is ward compact. This completes the proof of the theorem. □

Theorem 4.2. *A subset E of $\mathbb{R}$ is bounded if and only if it is I_θ-ward compact.*

Proof. Using an idea in the proof of Lemma 2 on page 1725 in [6] and the preceding theorem the proof can be obtained easily so is omitted. □

Now we give the definition of I_θ-ward continuity of a real function.

Definition 4.2. A function f is called I_θ-ward continuous on E if $I_\theta\text{-}\lim_{n\to\infty} \Delta f(x_n) = 0$ whenever $I_\theta\text{-}\lim_{n\to\infty} \Delta x_n = 0$, for a sequence $\mathbf{x} = (x_n)$ of terms in E.

We note that sum of two I_θ-ward continuous functions is I_θ-ward continuous but the product of two I_θ-ward continuous functions need not be I_θ-ward continuous as it can be seen by considering product of the I_θ-ward continuous function $f(x) = x$ with itself.

In connection with I_θ-quasi-Cauchy sequences and I_θ-convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbb{R}$.

(δi_θ) $(x_n) \in \Delta I_\theta \Rightarrow (f(x_n)) \in \Delta I_\theta$

$(\delta i_\theta c)$ $(x_n) \in \Delta I_\theta \Rightarrow (f(x_n)) \in c$

(c) $(x_n) \in c \Rightarrow (f(x_n)) \in c$

$(c\delta i_\theta)$ $(x_n) \in c \Rightarrow (f(x_n)) \in \Delta I_\theta$

(i_θ) $(x_n) \in I_\theta \Rightarrow (f(x_n)) \in I_\theta$.

We see that (δi_θ) is I_θ-ward continuity of f, (i_θ) is a I_θ-continuity of f and (c) states the ordinary continuity of f. It is easy to see that $(\delta i_\theta c)$ implies (δi_θ), and (δi_θ) does not imply $(\delta i_\theta c)$, and (δi_θ) implies $(c\delta i_\theta)$, and $(c\delta i_\theta)$ does not imply (δi_θ); $(\delta i_\theta c)$ implies (c) and (c) does not imply $(\delta i_\theta c)$; and (c) is equivalent to $(c\delta i_\theta)$.

Now we give the implication (δi_θ) implies (i_θ), i.e. any I_θ-ward continuous function is I_θ-sequentially continuous.

Theorem 4.3. *If f is I_θ-ward continuous on a subset E of $\mathbb{R}$, then it is I_θ-sequentially continuous on E.*

Proof. Suppose that f is an I_θ-ward continuous function on a subset E of $\mathbb{R}$. Let (x_n) be an I_θ-quasi-Cauchy sequence of points in E. Then the sequence

$$(x_1, x_0, x_2, x_0, x_3, x_0, ..., x_{n-1}, x_0, x_n, x_0, ...)$$

is an I_θ-quasi-Cauchy sequence. Since f is I_θ-ward continuous, the sequence

$$(y_n) = (f(x_1), f(x_0), f(x_2), f(x_0), ..., f(x_n), f(x_0), ...)$$

is a I_θ-quasi-Cauchy sequence. Therefore I_θ-$\lim_{n\to\infty} \Delta y_n = 0$.
Hence I_θ-$\lim_{n\to\infty}[f(x_n) - f(x_0)] = 0$. It follows that the sequence $(f(x_n))$ I_θ-converges to $f(x_0)$. This completes the proof of the theorem. $\square$

The converse is not always true for the function $f(x) = x^2$ is an example since $I_\theta - \lim_{n\to\infty} \Delta x_n = 0$ for the sequence $(x_n) = (\sqrt{n})$. But I_θ-$\lim_{n\to\infty} \Delta f(x_n) \neq 0$, because $(f(\sqrt{n})) = (n)$.

Theorem 4.4. *If f is I_θ-ward continuous on a subset E of $\mathbb{R}$, then it is continuous on E in the ordinary sense.*

Proof. Let f be an I_θ-ward continuous function on E. By Theorem 4.3, f is I_θ-sequentially continuous on E. It follows from Theorem 3.5 that f is continuous on E in the ordinary sense. Thus the proof is completed. $\square$

Theorem 4.5. *If f is N_θ-ward continuous on a subset E of $\mathbb{R}$, then it is I-continuous on E.*

Proof. The proof of the theorem follows from Lemma 3.1 and [3], Corollary 1. $\square$

Theorem 4.6. *Let I be an admissible ideal of $\mathbb{N}$. If f is N_θ-continuous on a subset E of $\mathbb{R}$, then it is I-continuous on E.*

Proof. The proof of the theorem is straightforward from the definitions. $\square$

Theorem 4.7. *An I_θ-ward continuous image of any I_θ-ward compact subset of $\mathbb{R}$ is I_θ-ward compact.*

Proof. Suppose that f is an I_θ-ward continuous function on a subset E of $\mathbb{R}$ and E is an I_θ-ward compact subset of $\mathbb{R}$. Let (y_n) be a sequence of points in $f(E)$. Write $y_n = f(x_n)$ where $x_n \in E$ for each $n \in \mathbb{N}$. I_θ-ward compactness of E implies that there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of (x_n) with I_θ-$\lim_{k\to\infty} \Delta z_k = 0$. Write $(t_k) = (f(z_k))$. As f is I_θ-ward continuous, so we have $I_\theta - \lim_{k\to\infty} \Delta f(z_k) = 0$. Thus we have obtained a subsequence (t_k) of the sequence $(f(x_n))$ with I_θ-$\lim_{k\to\infty} \Delta t_k = 0$. Thus $f(E)$ is I_θ-ward compact. This completes the proof of the theorem. $\square$

Corollary 4.8. *An I_θ-ward continuous image of any compact subset of $\mathbb{R}$ is compact.*

Proof. The proof of this theorem follows from Theorem 3.5. $\square$

Corollary 4.9. *An I_θ-ward continuous image of an I_θ-sequentially compact subset of $\mathbb{R}$ is G-sequentially compact for any regular subsequential method G.*

It is a well known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case of I_θ-ward continuity, i.e. uniform limit of a sequence of I_θ-ward continuous functions is I_θ-ward continuous.

Theorem 4.10. *If (f_n) is a sequence of I_θ-ward continuous functions defined on a subset E of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f, then f is I_θ-ward continuous on E.*

Proof. Let $\varepsilon > 0$ and (x_n) be a sequence of points in E such that $I_\theta\text{-}\lim_{n\to\infty} \Delta x_n = 0$. By the uniform convergence of (f_n) there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. By the definition of ideal for all $x \in E$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in J_r} |f_n(x) - f(x)| \geq \frac{\varepsilon}{3}\right\} \in I.$$

As f_N is I_θ-ward continuous on E we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in J_r} |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\} \in I.$$

On the other hand we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n\in J_r}|f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n\in J_r}|f(x_{n+1}) - f_N(x_{n+1})| \geq \frac{\varepsilon}{3}\right\}$$
$$\cup\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n\in J_r}|f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\} \cup \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n\in J_r}|f_N(x_n) - f(x_n)| \geq \frac{\varepsilon}{3}\right\}. \tag{1}$$

Since I is an admissible ideal, so the right hand side of the relation (1) belongs to I, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in J_r} |f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\right\} \in I.$$

This completes the proof of the theorem. □

Theorem 4.11. *The set of all I_θ-ward continuous functions on a subset E of $\mathbb{R}$ is a closed subset of the set of all continuous functions on E, i.e. $\overline{\Delta i_\theta wc(E)} = \Delta i_\theta wc(E)$ where $\Delta i_\theta wc(E)$ is the set of all I_θ-ward continuous functions on E, $\overline{\Delta i_\theta wc(E)}$ denotes the set of all cluster points of $\Delta i_\theta wc(E)$.*

Proof. Let f be an element in $\overline{\Delta i_\theta wc(E)}$. Then there exists sequence (f_n) of points in $\Delta i_\theta wc(E)$ such that $\lim_{n\to\infty} f_n = f$. To show that f is I_θ-ward continuous consider a sequence (x_n) of points in E such that $I_\theta\text{-}\lim_{n\to\infty} \Delta x_n = 0$. Since (f_n) converges to f, there exists a positive integer N such that for all $x \in E$ and for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. By the definition of ideal for all $x \in E$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in J_r} |f_n(x) - f(x)| \geq \frac{\varepsilon}{3}\right\} \in I.$$

As f_N is I_θ-ward continuous on E we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in J_r} |f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\} \in I.$$

On the other hand we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n \in J_r}|f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n \in J_r}|f(x_{n+1}) - f_N(x_{n+1})| \geq \frac{\varepsilon}{3}\right\}$$
$$\cup \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n \in J_r}|f_N(x_{n+1}) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\} \cup \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n \in J_r}|f_N(x_n) - f(x_n)| \geq \frac{\varepsilon}{3}\right\}. \tag{2}$$

Since I is an admissible ideal, so the right hand side of the relation (2) belongs to I; we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{n \in J_r}|f(x_{n+1}) - f(x_n)| \geq \frac{\varepsilon}{3}\right\} \in I.$$

This completes the proof of the theorem. □

Corollary 4.12. *The set of all I_θ-ward continuous functions on a subset E of $\mathbb{R}$ is a complete subspace of the space of all continuous functions on E.*

Proof. The proof follows from the preceding theorem. □

Cakalli [5] introduced the concept G-sequentially connected as, a non-empty subset E of $\mathbb{R}$ is called G-sequentially connected if there are non-empty and disjoint G-sequentially closed subsets U and V such that $A \subseteq U \cup V$, and $A \cap U$ and $A \cap V$ are empty. As far as G-sequentially connectedness is considered, then we get the following results.

Theorem 4.13. *Any I_θ-sequentially continuous image of any I_θ-sequentially connected subset of $\mathbb{R}$ is I_θ-sequentially connected.*

Proof. The proof follows from the Theorem 1 in [5]. □

Theorem 4.14. *A subset of $\mathbb{R}$ is I_θ-sequentially connected if and only if it is connected in ordinary sense and so is an interval.*

Proof. The proof follows from the Corollary 1 in [5]. □

Remark 4.1. If we take, $I = I_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset }\}$, then I_θ-quasi-Cauchy sequences coincides with N_θ-quasi Cauchy sequences (see [3, 4]).

5. Ideal Cauchy continuous function

In this section we introduce the concept of ideally Cauchy continuous function in metric space and prove some results.

Definition 5.1. A sequence $\mathbf{x} = (x_n)$ of points in a metric space X is said to be ideally Cauchy, for every $\varepsilon > 0$ and $m \in \mathbb{N}$ such that the set

$$\{n \in \mathbb{N} : d(x_n, x_m) \geq \varepsilon\} \in I.$$

Definition 5.2. Let X and Y be metric spaces, and let f be a function from X to Y. Then f is said to be ideally Cauchy continuous if and only if, given any ideally Cauchy sequence $\mathbf{x} = (x_n)$ in X, the sequence $f(\mathbf{x}) = (f(x_n))$ is an ideally Cauchy sequence in Y.

Theorem 5.1. *Every uniformly continuous function is ideally Cauchy continuous.*

Proof. Let $f : (X, d_X) \longrightarrow (Y, d_Y)$ be an uniformly continuous function. Then for $\delta > 0$ there exists $\varepsilon > 0$ such that for all $x, y \in X$ we have

$$d_Y(f(x), f(y)) < \varepsilon \text{ whenever } d_X(x, y) < \delta. \tag{3}$$

Let $\mathbf{x} = (x_n)$ be an ideally Cauchy sequence in X. For $\delta > 0$ and $m \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : d_X(x_n, x_m) \geq \delta\} \in I.$$

By the relation (3) we have

$$\{n \in \mathbb{N} : d_Y(f(x_n), f(x_m)) \geq \varepsilon\} \in I.$$

Hence f is ideally Cauchy continuous. □

The proof of the following results are straightforward.

Theorem 5.2. *Every ideally Cauchy continuous function is ideally continuous.*

Corollary 5.3. *Every ideally Cauchy continuous function is continuous.*

Theorem 5.4. *If X is totally bounded, then every ideally Cauchy continuous function is uniformly continuous.*

Theorem 5.5. *If X is complete, then every ideally continuous function on X is ideally Cauchy continuous.*

Theorem 5.6. *If X is not complete, as long as Y is complete, then any ideally Cauchy continuous function from X to Y can be extended to a function defined on the Cauchy completion of X; and this extension is necessarily unique.*

Example 5.1. If $X = \mathbb{R}$, then every ideally Cauchy continuous functions on $\mathbb{R}$ are the same as the ideally continuous ones. But on the subspace $\mathbb{Q}$ of rational numbers, however the matters are different. For example, define a two-valued function

$$f(x) = \begin{cases} 0, & \text{when } x^2 < 2; \\ 1, & \text{when } x^2 > 2. \end{cases}$$

Note that x^2 never equal to 2 for any rational number x. This function is ideally continuous on $\mathbb{Q}$ but not ideally Cauchy continuous, since it cannot be extended to $\mathbb{R}$ as an ideal continuous function. On the other hand, any uniformly continuous function on $\mathbb{Q}$ must be ideally Cauchy continuous.

Example 5.2. Let $f(x) = 2^x$ for all $x \in \mathbb{Q}$. This function is not uniformly continuous on $\mathbb{Q}$, but it is ideally Cauchy continuous on $\mathbb{Q}$.

Example 5.3. An ideal Cauchy sequence $(y_1, y_2, y_3, ...)$ in Y can be identified with a ideally Cauchy continuous function from $\{1, \frac{1}{2}, \frac{1}{3}, ...\}$ to Y, defined by $f(\frac{1}{n}) = y_n$. If Y is complete, then this function can be extended to $\{1, \frac{1}{2}, \frac{1}{3}, ..., 0\}$; $f(0)$ will be the limit of the ideal Cauchy sequence.

Finally we note the following further investigation problems arise.

1. For further study we suggest to investigate I_θ-quasi-Cauchy sequences of fuzzy points and I_θ-ward continuity for the fuzzy functions. However due to the change in settings, the definitions and methods of proofs will not always be analogous to these of the present work.

2. For another further study we suggest to introduce a new concept in dynamical systems using I_θ-ward continuity.
3. For another further study we suggest to introduce and give an investigation of I_θ-quasi-Cauchy sequences in abstract spaces.

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