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Weak solutions of one-dimensional pollutant transport model

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Abstract. We consider a one-dimensional bilayer model coupling shallow water and Reynolds lubrication equations that is a similar model derived in [European J. Applied Mathematics 24(6) (2013), 803-833]. The model considered is represented by the two superposed immiscible fluids. Under an hypothesis about the unknowns, we show the existence of global weak solution in time with a periodic domain.

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1. Introduction

In this paper, we study the existence of global weak solutions in time for the following one dimensional model of transport of pollutant derived in [5] :

$$\begin{cases} \partial_t h_1 + \partial_x(h_1 u_1) = 0, \\ \partial_t(h_1 u_1) + \partial_x(h_1 {u_1}^2) + \dfrac{1}{2} g \partial_x h_1^2 - 4\nu_1 \partial_x(h_1 \partial_x u_1) + \dfrac{\alpha}{\rho_1}\gamma(h_1)u_1 - \dfrac{\delta_\xi}{\rho_1} h_1 \partial_x^3 h_1 \\ +r_1 h_1 |u|^2 u + rgh_1 \partial_x h_2 + rgh_2 \partial_x(h_1 + h_2) = 0, \\ \partial_t h_2 + \partial_x(h_2 u_1) + \partial_x\left(-{h_2}^2 \dfrac{1}{\rho_2}\Big(\dfrac{1}{c} + \dfrac{1}{3\nu_2} h_2\Big)\partial_x p_2\right) = 0, \end{cases} \tag{1}$$

with

$$\partial_x p_2 = \rho_2 g \partial_x(h_1 + h_2) \quad \text{and} \quad \gamma(h_1) = \left(1 + \frac{\alpha}{3\nu_1} h_1\right)^{-1}. \tag{2}$$

Subscript 1 will correspond to the layer located below and subscript 2 to that located on the top. In this model, we denote by h_1, h_2 respectively, the water and the pollutant heights, u_1 is the water velocity, ρ_1 and ρ_2 the densities of each layer of fluid (we also introduce the ratio of densities $r = \dfrac{\rho_2}{\rho_1}$), ν_i is the kinematic viscosity, p_2 the pressure of the pollutant layer and g is the constant gravity. The coefficients δ_ξ, α, r_1, c, are respectively the coefficients of the intrefaz tension, friction at the bottom, quadratic friction and friction at the interfaz. This model is derived from a two-dimensional Navier-Stokes bilayer equations with capillary and friction effects at the interfaz. It is used to simulate the evolution of a thin viscous pollutant over water (see [5]). Let us recall some results about the existence of weak solution for a

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system composed by three equations (Shallow-water and transport equations). The case with viscosity term of the form $-\nu\Delta u$ was investigated in [8] in which existence of weak solutions for a viscous sedimentation model is obtained by assuming smallness of the data. In their analysis the authors considered a transport equation with Grass model of the form $q_b = hu$ and used Brower fixed point theorem to get the result. In [14], the authors studied the stability of global weak solutions for a sediment transport model in two- dimensional case. In this model, the viscosity coefficient is of the form $-\nu\mathtt{div}(hD(u))$ and the sediment transport equation considered is $\partial_t z + \mathtt{div}(h|u|^k u) - \frac{\nu}{2}\Delta u = 0$. The stability result is obtained without any restriction on the data and by using a mathematical entropy introduced firstly in [4] namely BD entropy . We note that it's the BD entropy inequality which allows the authors in [1, 3, 4, 6, 7] to get existence results of global weak solutions for Shallow-Water and viscous compressible Navier-Stokes equations.

In [12], the authors obtained a result of existence of global weak solution of similar model in a two dimensional case. To have this result, the authors needed of some additional regularizing terms such as a quadratic friction term $h_1|u|^2u$, a cold pressure $h_1^{1-\alpha}$ with $\alpha > 1$ and a capillarity term of the form $h_1\nabla\Delta h_1$. They used a transport equation of the form $\partial_t h_2 + \mathrm{div}(h_2 u) - g\nabla\cdot\left((1+\dfrac{h_2}{h_1})\nabla(h_1+h_2)\right) = 0.$ The key point with the BD entropy is that, with the structure of the diffusive term, we get an extra regularity for the water height. In our analysis, we consider in one-dimensional, a periodic domain $\Omega = (0,1)$ to simplify. We assume that the pollutant layer is smaller than that of the water:

$$h_2 \le h_1. \tag{3}$$

Notice that, to deduce the model, we make this hypothesis for the caracteristic heights (see [5]). We will intend in the future to study the present model without this condition. We complete system (1) with initial conditions :

$$h_1(0,x) = h_{1_0}(x), \quad h_2(0,x) = h_{2_0}(x), \quad (h_1u_1)(0,x) = \mathbf{m}_0(x) \quad \text{in } (0,1). \tag{4}$$

$$\begin{gathered} h_{1_0} \in L^2(0,1), \quad h_{1_0} + h_{2_0} \in L^2(0,1), \quad \partial_x(h_{1_0} \in L^2(0,1), \\ \partial_x\mathbf{m}_0 \in L^1(0,1), \quad \mathbf{m}_0 = 0 \quad \text{if } h_{1_0} = 0, \\ \frac{|\mathbf{m}_0|^2}{h_{1_0}} \in L^1(0,1), \quad f(h_{1_0}) \in L^1(0,1), \end{gathered} \tag{5}$$

where f will be defined later on (see (16)).

The paper is organized as follows : in the Section 2, we will start by giving the definition of global weak solutions, then we will establish a classical energy equality and the "mathematical BD entropy", which give some regularities on the unknowns. We will also give an existence theorem of global weak solutions. In section 3, we will give the proof of the existence theorem.

2. Main results

Definition 2.1. We shall say that (h_1, h_2, u_1) is a weak solution on $(0,T)$ of (1), with initial conditions (4) if the following conditions are satisfied :

- (4) holds in $\mathcal{D}'(\Omega)$;

- (h_1, h_2, u_1) verified the energy inequalities (2.1) and (2.2) for a.e. non negative t;
- for all smooth test function $\varphi = \varphi(t,x)$ with $\varphi(T,) = 0$, we have:

$$h_{1_0}\varphi(0,.) - \int_0^T\int_0^1 h_1\partial_t\varphi - \int_0^T\int_0^1 h_1u_1\partial_x\varphi = 0, \tag{6}$$

$$-h_{2_0}\varphi(0,.) - \int_0^T\int_0^1 h_2\partial_t\varphi - \int_0^T\int_0^1 h_2u_1\partial_x\varphi$$
$$+\int_0^T\int_0^1 {h_2}^2\frac{1}{\rho_2}\Big(\frac{1}{c}+\frac{1}{3\nu_2}h_2\Big)\partial_x p_2\partial_x\varphi = 0, \tag{7}$$

$$h_{1_0}u_{1_0}\varphi(0,.) - \int_0^T\int_0^1 h_1u_1\partial_t\varphi - \int_0^T\int_0^1 h_1{u_1}^2\partial_x\varphi - \frac{1}{2}g\int_0^T\int_0^1 h_1^2\partial_x\varphi$$
$$+4\nu_1\int_0^T\int_0^1 h_1\partial_x u_1\partial_x\varphi + \frac{\alpha}{\rho_1}\int_0^T\int_0^1 \gamma(h_1)u_1\varphi + \frac{\delta_\xi}{\rho_1}\int_0^T\int_0^1 h_1\partial_x^2 h_1\partial_x\varphi$$
$$+\frac{\delta_\xi}{\rho_1}\int_0^T\int_0^1 \partial_x h_1\partial_x^2 h_1\varphi - rg\int_0^T\int_0^1 h_2\partial_x h_1\varphi - rg\int_0^T\int_0^1 h_1h_2\partial_x\varphi$$
$$-rg\int_0^T\int_0^1 (h_1+h_2)h_2\varphi - rg\int_0^T\int_0^1 (h_1+h_2)\partial_x h_2\varphi + r_1\int_0^T\int_0^1 |u_1|^2u_1\varphi = 0. \tag{8}$$

Before giving the main theorem, we give the following two important lemmas. We firstly give the classical energy associated with system (1) and secondly the mathematical BD entropy.

Lemma 2.1. The model defined by (1) admits an entropy equality

$$\int_0^1 \Big[\frac{1}{2}h_1|u_1|^2 + \frac{1}{2}g(1-r)|h_1|^2 + \frac{1}{2}rg|h_1+h_2|^2 + \frac{1}{2}\frac{\delta_\xi}{\rho_1}|\partial_x h_1|^2\Big]$$
$$+r_1\int_0^T\int_0^1 h_1|u_1|^4 + 4\nu_1\int_0^T\int_0^1 h_1|\partial_x u_1|^2 + \frac{\alpha}{\rho_1}\int_0^T\int_0^1 \gamma(h_1)|u_1|^2$$
$$+rg^2\int_0^T\int_0^1 h_2^2\big(\frac{1}{c}+\frac{1}{3\nu_2}h_2\big)\Big(\partial_x(h_1+h_2)\Big)^2$$
$$= \int_0^1 \Big[\frac{1}{2}h_{1_0}|u_{1_0}|^2 + \frac{1}{2}g(1-r)|h_{1_0}|^2 + \frac{1}{2}rg|h_{1_0}+h_{2_0}|^2 + \frac{1}{2}\frac{\delta_\xi}{\rho_1}|\partial_x h_{1_0}|^2\Big]. \tag{9}$$

Proof. Firstly, we multiply the momentum equation by u_1 and we integrate from 0 to 1. We use the mass conservation equation of the first layer for simplification. Then, we obtain

$$\frac{d}{dt}\int_0^1 \Big[\frac{1}{2}(h_1u_1^2+gh_1^2)\Big] - \frac{\delta_\xi}{\rho_1}\int_0^1 \partial_t h_1\partial_x^2 h_1 + r_1\int_0^1 h_1|u_1|^4 + rg\int_0^1 h_2\partial_t h_1$$
$$+rg\int_0^1 h_2u_1\partial_x(h_1+h_2) + 4\nu_1\int_0^1 h_1(\partial_x u_1)^2 + \frac{\alpha}{\rho_1}\int_0^1 \gamma(h_1)u_1^2 = 0. \tag{10}$$

Secondly, we multiply the equation for the thin film flow by $\rho_2 g(h_1+h_2)$ and integrate to obtain

$$\frac{1}{2}rg\frac{d}{dt}\int_0^1 h_2^2 + rg\int_0^1 h_1\partial_t h_2 + rg\int_0^1 (h_1+h_2)\partial_x(h_2u_1)$$

$$= rg^2 \int_0^1 h_2^2(\frac{1}{c} + \frac{1}{3\nu_2} h_2)\Big(\partial_x(h_1+h_2)\Big)^2. \tag{11}$$

We use the mass conservation equation to write

$$\int_0^1 h_2 \partial_t h_1 + \int_0^1 h_1 \partial_t h_2 = \frac{d}{dt}\int_0^1 h_1 h_2, \tag{12}$$

and to develop the following product affecting the terms with δ_ξ

$$\int_0^1 \partial_x(h_1 u_1)\partial_x^2 h_1 = \int_0^1 h_1 \partial_t(h_1)\partial_x^2 h_1 = -\frac{1}{2}\frac{d}{dt}\int_0^1 |\partial_x h_1|^2. \tag{13}$$

By adding (10) and (11), and taking into account (12) and (13), we obtain

$$\frac{d}{dt}\int_0^1 \Big[\frac{1}{2}h_1 u_1^2 + \frac{1}{2} g h_1^2 + rgh_2(h_1 + \frac{h_2}{2})\Big] + \frac{1}{2}\frac{\delta_\xi}{\rho_1}\frac{d}{dt}\int_0^1 (\partial_x h_1)^2 + r_1 \int_0^1 h_1 |u_1|^4$$

$$+4\nu_1 \int_0^1 h_1(\partial_x u_1)^2 + rg^2 \int_0^1 h_2^2(\frac{1}{c} + \frac{1}{3\nu_2}h_2)\Big(\partial_x(h_1+h_2)\Big)^2 + \frac{\alpha}{\rho_1}\int_0^1 \gamma(h_1) u_1^2 = 0. \tag{14}$$

To end, we integrate from 0 to t to have the equality (9). □

Corollary 2.1. Let (h_1, h_2, u_1) be a solution of model (1). Then, thanks to Lemma 2.1 we have:

$$\begin{array}{ccc}
h_1 & \text{is bounded in} & L^\infty(0,T;L^2(0,1)),\\
h_2 & \text{is bounded in} & L^\infty(0,T;L^2(0,1)),\\
\partial_x h_1 & \text{is bounded in} & L^\infty(0,T;L^2(0,1)),\\
\sqrt{h_1}u_1 & \text{is bounded in} & L^\infty(0,T;L^2(0,1)),\\
\sqrt{h_1}\partial_x u_1 & \text{is bounded in} & L^2(0,T;L^2(0,1)),\\
u_1 & \text{is bounded in} & L^2(0,T;L^2(0,1)),\\
h_1^{\frac{1}{4}}u_1 & \text{is bounded in} & L^2(0,T;L^2(0,1)),\\
h_2\sqrt{\frac{1}{c}+\frac{1}{3\nu_2}h_2}\Big(\partial_x(h_1+h_2)\Big) & \text{is bounded in} & L^2(0,T;L^2(0,1)).
\end{array}$$

Remark 2.1. (1) In the Corollary 2.1, the estimate

$$\sqrt{h_1}u_1 \quad \text{is bounded in} \quad L^\infty(0,T;L^2(0,1))$$

implies,

$$h_1 u_1 \quad \text{is bounded in} \quad L^\infty(0,T;L^2(0,1))$$

this leads us

$$\partial_t h_1 \quad \text{is bounded in} \quad L^\infty(0,T;W^{-1,2}(0,1)).$$

(2) We have the additional regularities thanks to **Corollary** 2.1:
   (a) h_1 is bounded in $L^2(0,T;H^1(0,1))$,
   (b) $h_1 u_1$ is bounded in $L^3(0,T;L^3(0,1)) \cap L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;W^{1,1}(0,1))$,
   (c) $\gamma(h_1)$ is bounded in $L^\infty(0,T;H^1(0,1)) \cap L^\infty(0,T;L^\infty(0,1))$.

Remark 2.2. We have the following additional regularities:
(1) h_2 is bounded in $L^\infty(0,T;L^\infty(0,1))$,
(2) $\partial_x(h_1+h_2)$ is bounded in $L^2(0,T;L^2(0,1))$.

We will need in the following some additional regularity on h_1 and this will be achieved through an additional BD entropy inequality presented in the next lemma.

Lemma 2.2. For smooth solutions (h_1, h_2, u_1) of model (1) satisfying the classical energy equality of the **Lemma** 2.1, we have the following mathematical BD entropy inequality:

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 \left[h_1|u_1+4\nu_1\partial_x\log h_1|^2+rg|h_1+h_2|^2+g(1-r)|h_1|^2|-8\nu_1 f(h_1)+\frac{\delta_\xi}{\rho_1}|\partial_x h_1|^2\right]\\
&\quad+\frac{\alpha}{\rho_1}\int_0^T\int_0^1\gamma(h_1)|u_1|^2+r_1\int_0^T\int_0^1 h_1|u_1|^4+4\nu_1 r_1\int_0^T\int_0^1|u_1|^2u_1\partial_x h_1\\
&\quad+2g\nu_1\int_0^T\int_0^1(1+2r\frac{h_2}{h_1})|\partial_x h_1|^2+4rg\nu_1\int_0^1(1+\frac{h_2}{h_1})\partial_x h_1\partial_x h_2+\frac{\delta_\xi}{\rho_1}\int_0^T\int_0^1|\partial_x^2 h_1|^2\\
&\quad+rg^2\int_0^T\int_0^1 h_2^2(\frac{1}{c}+\frac{1}{3\nu_2}h_2)\Big(\partial_x(h_1+h_2)\Big)^2+4\frac{\nu_1\alpha}{\rho_1}\int_0^T\int_0^1\gamma'(h_1)u_1\partial_x h_1\\
&\quad\leqslant 4\nu_1\int_0^1 f(h_{1_0})+\int_0^1\left[h_{1_0}|u_{1_0}|^2+128\nu_1^2|\partial_x\sqrt{h_{1_0}}|^2+\frac{1}{2}g(1-r)|h_{1_0}|^2\right]\\
&\qquad+\int_0^1\left[\frac{1}{2}rg|h_{1_0}+h_{2_0}|^2+\frac{1}{2}\frac{\delta_\xi}{\rho_1}|\partial_x h_{1_0}|^2\right],
\end{aligned}
\tag{15}
$$

where

$$
f(h_1)=\alpha\log\left(\frac{h_1}{3+\alpha\nu_1^{-1}h_1}\right). \tag{16}
$$

Proof. Let us consider the mass equation

$$\partial_t h_1+\partial_x h_1 u_1=0.$$

When we use both the transport equation and the renormalized technical, we get:

$$\partial_t(\partial_x h_1)+\partial_x(h_1\partial_x u_1)+\partial_x(u_1\partial_x h_1)=0.$$

Replacing $\partial_x h_1$ by $h_1\partial_x\log h_1$ and introducing the viscosity $4\nu_1$, this becomes

$$4\nu_1\partial_t(h_1\partial_x\log h_1)+4\nu_1\partial_x(h_1\partial_x u_1)+4\nu_1\partial_x(h_1u_1\partial_x\log h_1)=0.$$

Then, we add the momentum equation to obtain

$$
\begin{aligned}
&\partial_t[h_1(u_1+4\nu_1\partial_x\log h_1)]+\partial_x[h_1u_1(u_1+4\nu_1\partial_x\log h_1)]+\tfrac{1}{2}g\partial_x h_1^2+\tfrac{\alpha}{\rho_1}\gamma(h_1)u_1\\
&\qquad-h_1\frac{\delta_\xi}{\rho_1}\partial_x^3 h_1+r_1h_1|u_1|^2u_1+rgh_1\partial_x h_2+rgh_2\partial_x(h_1+h_2)=0.
\end{aligned}
$$

We multiply this equation by $(u_1+4\nu_1\partial_x\log h_1)$ and we integrate between 0 and 1. Now, we transform each term of the resulting identity separately

$$
\begin{aligned}
&\int_0^1[\partial_t[h_1(u_1+4\nu_1\partial_x\log h_1)]+\partial_x[h_1u_1(u_1+4\nu_1\partial_x\log h_1)]](u_1+4\nu_1\partial_x\log h_1)\\
&\qquad=\frac{1}{2}\frac{d}{dt}\int_0^1 h_1|u_1+4\nu_1\partial_x\log h_1|^2.
\end{aligned}
$$

Next, we only study the terms which do not appear in (9).
The pressure terms become:

$$\frac{1}{2}g\int_0^1\partial_x h_1^2(4\nu_1\partial_x\log h_1)=2g\nu_1\int_0^1|\partial_x h_1|^2,$$

$$rg\int_0^1 [h_1\partial_x h_2 + h_2\partial_x(h_1+h_2)](4\nu_1\partial_x \log h_1)$$
$$= 4rg\nu_1\int_0^1 \frac{h_2}{h_1}|\partial_x h_1|^2 + 4rg\nu_1\int_0^1 (1+\frac{h_2}{h_1})\partial_x h_1\partial_x h_2.$$

Adding these two terms, we have:

$$\frac{1}{2}g\int_0^1 \partial_x h_1^2(4\nu_1\partial_x \log h_1 + rg\int_0^1 [h_1\partial_x h_2 + h_2\partial_x(h_1+h_2)](4\nu_1\partial_x \log h_1)$$
$$= 2g\nu_1\int_0^1 (1+2r\frac{h_2}{h_1})|\partial_x h_1|^2 + 4rg\nu_1\int_0^1 (1+\frac{h_2}{h_1})\partial_x h_1\partial_x h_2.$$

For the friction term at the bottom, we have

$$\frac{\alpha}{\rho_1}\int_0^1 \gamma(h_1)u_1(4\nu_1\partial_x \log h_1) = \frac{4\nu_1}{\rho_1}\int_0^1 \frac{3\nu_1\alpha}{3\nu_1+\alpha h_1}u_1\partial_x \log h_1$$
$$= -\frac{4\nu_1}{\rho_1}\int_0^1 \frac{3\nu_1\alpha}{3\nu_1+\alpha h_1}\Big(\frac{\partial_t h_1}{h_1} + \partial_x u_1\Big).$$

Considering that Lemma 2.2 gives $f'(h_1) = \dfrac{3\nu_1\alpha}{3\nu_1+\alpha h_1}\dfrac{1}{h_1}$,

therefore,

$$4\frac{\nu_1\alpha}{\rho_1}\int_0^1 \gamma(h_1)u_1\partial_x \log h_1 = -4\frac{\nu_1}{\rho_1}\frac{d}{dt}\int_0^1 f(h_1) + 4\frac{\nu_1\alpha}{\rho_1}\int_0^1 \gamma'(h_1)u_1\partial_x h_1. \qquad \square$$

Remark 2.3. (1) The term including $\log\left(\dfrac{h_1}{3+\alpha\nu_1^{-1}h_1}\right)$ is bounded, see [12].

(2) In Lemma 2.2 all the terms, except $-\displaystyle\int_0^T\int_0^1 |u_1|^2 u_1\partial_x h_1$ and $\displaystyle\int_0^T\int_0^1 (1+\frac{h_2}{h_1})\partial_x h_1\partial_x h_2$ are controlled since they have the good sign. But the control of the both terms takes inspiration in [12].

(3) If (h_1, h_2, u_1) is solution of the model (1), then, thanks to Lemma 2.2, we have that:

$$\partial_x\sqrt{h_1} \text{ is bounded in } L^\infty(0,T;L^2(0,1)) \text{ and } \partial_x^2 h_1 \text{ is bounded in } L^2(0,T;L^2(0,1)).$$

Theorem 2.1. There exists global weak solutions to system (1) with initial data (4), (5) and satisfying energy equality (9) and energy inequality (15).

3. Convergences

This section is devoted to the proof of Theorem 2.1. Let (h_1^k, h_2^k, u_1^k) be a sequence of weak solutions with initial data

$$h^k_{1|t=0} = h^k_{1_0}, \quad h^k_{2|t=0} = h^k_{2_0}, \quad (h_1^k u_1^k)_{|t=0} = m_0^k$$

such as

$$h^k_{1_0} \longrightarrow h_{1_0} \text{ in } L^1(0,1), \quad h^k_{2_0} \longrightarrow h_{2_0} \text{ in } L^1(0,1), \quad m_0^k \longrightarrow m_0 \text{ in } L^1(0,1),$$

and satisfies

$$4\nu_1\int_0^1 f(h_{1_0}) + \int_0^1 \Big[h_{1_0}|u_{1_0}|^2 + 128\nu_1^2|\partial_x\sqrt{h_{1_0}}|^2 + \frac{1}{2}g(1-r)|h_{1_0}|^2 + \frac{1}{2}rg|h_{1_0}+h_{2_0}|^2\Big]$$

$$+\frac{1}{2}\frac{\delta_\xi}{\rho_1}\int_0^1 |\partial_x h_{1_0}|^2 \leq C.$$

Such approximate solutions can be built by a regularization of capillary effect.

3.1. Strong convergence of $\left(\sqrt{h_1^k}\right)_k$. Here, we are going to establish the spaces in which $\left(\sqrt{h_1^k}\right)_k$ is bounded.

In this sense we are going to integrate the mass equation and we directly get $\sqrt{h_1^k}$ in $L^\infty(0,T;L^2(0,1))$, the Remark 2.3 gives us $\left|\partial_x\sqrt{h_1^k}\right|$ in $L^\infty(0,T;L^2(0,1))$. So we obtain:

$$\sqrt{h_1^k} \quad \text{is bounded in } L^\infty(0,T;H^1(0,1)). \qquad (*)$$

Moreover, we use the mass equation again to have the following equality:

$$\partial_t\sqrt{h_1^k} = \frac{1}{2}\sqrt{h_1^k}\partial_x u^k - \partial_x(\sqrt{h_1^k}u^k),$$

which gives that $\partial_t\sqrt{h_1^k}$ is bounded in $L^2(0,T;H^{-1}(0,1))$.

Applying Aubin-Simon lemma ([9, 13]), we can extract a subsequence, still denoted $(h_1^k)_{1\leq k}$, such as

$$\left(\sqrt{h_1^k}\right)_k \quad \text{strongly converges to} \quad \sqrt{h_1} \text{ in } L^2(0,T;L^2(0,1)).$$

3.2. Strong convergence of h_1 and h_2. Let now study the subsequence $(h_1^k)_k$. According to the property $(*)$ and Sobolev embeddings, we know that, for any finite s,

$$(h_1^k)_k \quad \text{is bounded in} \quad L^\infty(0,T;L^s(0,1)).$$

In the following, we will assume that $4 \leq s$ in order to simplify our expressions and ensure that

$$(h_1^k)_k \quad \text{is bounded in} \quad L^\infty(0,T;L^2(0,1)).$$

The equality $\partial_x h_1^k = 2\sqrt{h_1^k}\partial_x\sqrt{h_1^k}$ enables us to bound the sequence $\partial_x h_1^k$ in $L^\infty(0,T;(L^{\frac{2s}{s+2}}(0,1))^2)$ and consequently the sequence

$$(h_1^k)_k \quad \text{is bounded in} \quad L^\infty(0,T;W^{1,\frac{2s}{s+2}}(0,1)).$$

Moreover, we have some properties on the time derivative of (h_1^k); actually the mass equation can be written as: $\partial_t h_1^k = -\partial_x(h_1^k u_1^k)$. Splitting the product $h_1^k u_1^k$ into $h_1^k u_1^k = \sqrt{h_1^k}\sqrt{h_1^k}u_1^k$, we get

$$h_1^k u_1^k \quad \text{in} \quad L^\infty(0,T;(L^{\frac{2s}{s+2}}(0,1))^2) \quad \text{and} \quad \partial_t h_1^k \quad \text{in} \quad L^\infty(0,T;W^{-1,\frac{2s}{s+2}}(0,1)).$$

Thanks to Aubin-Simon lemma again, we find:

$$h_1^k \longrightarrow h_1 \quad \text{in} \quad C^0(0,T;L^{\frac{2s}{s+2}}(0,1))$$

We have $h_2^k \in L^2(0,T;H^1(0,1))$.

Moreover, we have $\partial_t h_2^k = -\partial_x(h_2^k u_1^k) + g\partial_x\left[-{h_2^k}^2(\frac{1}{c}+\frac{1}{3\nu_1}h_2^k)\partial_x(h_1^k+h_2^k)\right]$.

According to the Sobolev embeddings, we show that the first term is in $W^{-1,1}(0,1)$,

since
$h_2^k \in L^2(0,1)$ and $u_1^k \in L^2(0,1)$. By analogy we prove that the last term is in the same space and we get $\partial_t h_2^k$ also in this space. Thanks to the Aubin-Simon lemma, we find:

$$(h_2^k)_k \quad \text{converges strongly to} \quad h_2 \quad \text{in} \quad L^2(0,T;W^{-1,\frac{2s}{s+2}}(0,1)).$$

3.3. Strong convergence of $(h_1^k u_1^k)_k$. Let us write $h_1^k u_1^k$ as follow:
$h_1^k u_1^k = \sqrt{h_1^k}\sqrt{h_1^k}u_1^k$, we have

$$\left(\sqrt{h_1^k}\right)_k \text{ bounded in } L^\infty(0,T;L^4(0,1))$$

and

$$\left(\sqrt{h_1^k}u_1^k\right)_k \text{ bounded in } L^\infty(0,T;L^2(0,1)).$$

Thus we have:

$$(h_1^k u_1^k)_k \text{ bounded in } L^\infty(0,T;L^{\frac{4}{3}}(0,1)).$$

Let's write the gradient as follows:

$$\partial_x(h_1^k u_1^k) = h_1^k\partial_x u_1^k + u_1^k\partial_x h_1^k = \sqrt{h_1^k}\sqrt{h_1^k}\partial_x u_1^k + u_1^k\partial_x h_1^k,$$

since the first term is in $L^2(0,T;L^{\frac{4}{3}}(0,1))$ and thanks to the Corollary 2.1, second one belongs to $L^\infty(0,T;W^{-1,\frac{4}{3}}(0,1)) \cap L^2(0,T;L^1(0,1))$, we have

$$(h_1^k u_1^k)_k \text{ bounded in } L^2(0,T;W^{1,1}(0,1)).$$

Moreover, the momentum equation of (1) enables us to write the time derivation of the water discharge:
$\partial_t(h_1^k u_1^k) = -\partial_x(h_1^k {u_1^k}^2)) - \frac{1}{2}g\partial_x[(h_1^k)^2] - 4\nu_1\partial_x(h_1^k\partial_x u_1^k) - \frac{\alpha}{\rho_1}\gamma(h_1^k)u_1^k + \alpha(h_1^k)h_1^k u_1^k|u_1^k|^2$

$$+\frac{\delta_\xi}{\rho_1}h_1^k\partial_x^3 h_1^k - rgh_1^k\partial_x h_2^k - rgh_2^k\partial_x(h_1^k + h_2^k) = 0.$$

we then study each term:
- $\partial_x(h_1^k(u_1^k)^2) = \partial_x(\sqrt{h_1^k}\sqrt{h_1^k}(u_1^k)^2)$ which is in $L^2(0,T;W^{-1,\frac{4}{3}}(0,1))$.
- as $(h_1^k)_k$ is bounded in $L^\infty(0,T;W^{1,1}(0,1))$, it is also bounded in $L^\infty(0,T;L^2(0,1))$ and we can write the following relation:

$$\left(\partial_x[(h_1^k)^2]\right)_k \text{ is bounded in } L^\infty(0,T;W^{-1,1}(0,1)).$$

- $\left(\partial_x(h_1^k\partial_x u_1^k)\right)_k$ is bounded in $L^2(0,T;W^{-1,\frac{4}{3}}(0,1))$.
- Let us write $h_1^k u_1^k(u_1^k)^2 = \sqrt{h_1^k}u_1^k\sqrt{h_1^k}(u_1^k)^2$, which is in $L^2(0,T;W^{1,1}(0,1))$.
- The last three terms are bounded in $L^\infty(0,T;W^{-1,2}(0,1))$.

Then, applying Aubin-Simon lemma, we obtain,

$$(h_1^k u_1^k)_k \text{ converges stongly to } \mathbf{m} \text{ in } C^0(0,T;W^{-1,1}(0,1)).$$

3.4. Strong convergence of $\left(\sqrt{h_1^k}u_1^k\right)_k$. Setting $\mathbf{m}^k = h_1^k u_1^k$, so, we have $\sqrt{h_1^k}u^k = \dfrac{\mathbf{m}^k}{\sqrt{h_1^k}}$. We want to prove the strong convergence for this term. We know that

$$\left(\frac{\mathbf{m}^k}{\sqrt{h_1^k}}\right)_k \text{ is bounded in } L^\infty(0,T;(L^2(0,1))^2);$$

consequently Fatou lemma reads:

$$\int_0^1 \liminf \frac{(\mathbf{m}^k)^2}{h_1^k} \le \liminf \int_0^1 \frac{(\mathbf{m}^k)^2}{h_1^k} < +\infty.$$

In particular, $\mathbf{m}$ is equal to zero for almost every x where $h_1(t,x)$ vanishes. Then, we can define the limit velocity taking $u_1(t,x) = \frac{\mathbf{m}(t,x)}{h_1(t,x)}$ if $h_1(t,x) \neq 0$ or else $u_1(t,x) = 0$. So we have a link between the limits $\mathbf{m}(t,x) = h_1(t,x)u_1(t,x)$ and:

$$\int_0^1 \frac{(\mathbf{m})^2}{h_1} = \int_0^1 h_1|u_1|^2 < +\infty.$$

Moreover, we can use Fatou lemma again to write

$$\int_0^T \int_0^1 h_1|u_1|^4 \le \int_0^T \int_{]0,1[} \liminf h_1|u_1|^4 \le \liminf \int_0^T \int_0^1 h_1|u_1|^4$$

$$= \liminf \int_0^T \int_0^1 \sqrt{h_1}|u_1|^2 \sqrt{h_1}|u_1|^2,$$

which gives $\sqrt{h_1}|u_1|^2$ in $L^2(0,T;L^2(0,1))$.

As $\mathbf{m}^k$ and h_1^k converge almost everywhere, the sequence of $\sqrt{h_1^k}u_1^k = \dfrac{\mathbf{m}^k}{\sqrt{h_1^k}}$ converges almost everywhere to $\sqrt{h_1}u_1 = \dfrac{\mathbf{m}}{\sqrt{h_1}}$. Moreover, for all M positive $\sqrt{h_1^k}u_1^k 1_{|u_1^k|\le M}$ converges to $\sqrt{h_1}u_1 1_{|u|\le M}$ (still assuming that h_1^k does not vanish). If h_1 vanishes, we can write $\sqrt{h_1^k}u^k_{1\,|u_1^k|\le M} \le M\sqrt{h_1^k}$ and then have convergence towards zero. Then, almost everywhere, we obtain the convergence of $(\sqrt{h_1^k}u_1^k 1_{|u_1^k|\le M})_k$.

Finally, let us consider the following norm:

$$\int_0^T \int_0^1 \left|\sqrt{h_1^k}u_1^k - \sqrt{h_1}u_1\right|^2 \le$$

$$\int_0^T \int_0^1 \left(\left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|\le M} - \sqrt{h_1}u_1 1_{|u_1|\le M}\right| + \left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|>M}\right| + \left|\sqrt{h_1}u_1 1_{|u_1|>M}\right|\right)^2$$

$$\le 3\int_0^T \int_0^1 \left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|\le M} - \sqrt{h_1}u_1 1_{|u_1|\le M}\right|^2 + 3\int_0^T \int_0^1 \left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|>M}\right|^2$$
$$+3\int_0^T \int_0^1 \left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|>M}\right|^2.$$

Since $\left(\sqrt{h_1^k}\right)_k$ is bounded in $L^2(0,T;L^4(0,1))$, it follows

$$\left(\sqrt{h_1^k}u_1^k 1_{|u_1^k|\le M}\right)_k \text{ is bounded in this space.}$$

So, as we have seen previously, the first integral tends to zero. Let us study the other two terms:

$$\int_0^1 \left|\sqrt{h_1^k}u_1^k 1_{|u_1^k|>M}\right|^2 \le \frac{1}{M^2}\int_0^1 h_1^k(u_1^k)^4 \le \frac{c}{M^2}$$

and

$$\int_0^1 \left|\sqrt{h_1}u_1 1_{|u_1|>M}\right|^2 \le \frac{1}{M^2}\int_0^1 h_1 u_1^4 \le \frac{c}{M^2},$$

for all $M>0$. When M tends to the infinity, our two integrals tend to zero. Then

$$\left(\sqrt{h_1^k}u_1^k\right)_k \quad \text{converges strongly to} \quad \sqrt{h_1}u_1 \quad \text{in} \quad L^2(0,T;(L^2(]0,1[))^2).$$

3.5. Convergence of $(\partial_x h_1^k)_k$, $(h_1^k\partial_x h_1^k)_k$, $(h_2^k\partial_x h_1^k)_k$, $(\partial_x^2 h_1^k)_k$, $(h_1^k\partial_x^2 h_1)_k$ **and** $(\partial_x h_1^h\partial_x^2 h_1^k)_k$. • We have $(\partial_x h_1^k)_k$ bounded in $L^2(0,T;H^1(0,1))$ and $(\partial_t\partial_x h_1^k)_k$ is bounded in $L^\infty(0,T;H^{-2}(0,1))$ since $(\partial_t h_1^k)_k$ is bounded in $L^\infty(0,T;H^{-1}(0,1))$. Thanks to compact injection of $H^1(0,1)$ in $L^2(0,1)$ in one dimension, we have:

$$(\partial_x h_1^k)_k \quad \text{converges strongly to} \quad \partial_x h_1 \quad \text{in } L^2(0,T;L^2(0,1))$$

• The bound of $\partial_x^2 h_1^k$ in $L^2(0,T;L^2(0,1))$ and $\partial_x h_2^k$ in $L^2(0,T;L^2(0,1))$ gives us:

$$(\partial_x^2 h_1^k)_k \quad \text{converges weakly to} \quad \partial_x^2 h_1 \quad \text{in } L^2(0,T;L^2(0,1)),$$

$$(\partial_x h_2^k)_k \quad \text{converges weakly to} \quad \partial_x h_2 \quad \text{in } L^2(0,T;L^2(0,1)).$$

• Thanks to the strong convergence of $(h_1^k)_k$, $(h_2^k)_k$, $(\partial_x h_1^k)_k$ and the weak convergence of $(\partial_x^2 h_1^k)_k$, we have:

$$(h_1^k\partial_x h_1^k)_k \quad \text{converges strongly to} \quad h_1\partial_x h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_2^k\partial_x h_1^k)_k \quad \text{converges strongly to} \quad h_2\partial_x h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_1^k\partial_x^2 h_1^k)_k \quad \text{converges weakly to} \quad h_1\partial_x^2 h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(\partial_x h_1^k\partial_x^2 h_1^k)_k \quad \text{converges weakly to} \quad \partial_x h_1\partial_x^2 h_1 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_1^k\partial_x h_2^k)_k \quad \text{converges strongly to} \quad h_1\partial_x h_2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_2^k\partial_x h_2^k)_k \quad \text{converges strongly to} \quad h_2\partial_x h_2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$\left((h_1^k)^2\right)_k \quad \text{converges strongly to} \quad {h_1}^2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$\left((h_2^k)^2\right)_k \quad \text{converges strongly to} \quad {h_2}^2 \quad \text{in } L^1(0,T;L^1(0,1)),$$

$$(h_1^k h_2^k)_k \quad \text{converges strongly to} \quad h_1 h_2 \quad \text{in } L^1(0,T;L^1(0,1)).$$

3.6. Convergence of $(h_1^k\partial_x u_1^k)_k$, $(\gamma(h_1^k)u_1^k)_k$ and $(h_1^k|u_1^k|^2u_1^k)_k$. As $(u_1^k)_k$ is bounded in $L^2(0,T;L^2(0,1))$, then $(\partial_x u_1^k)_k$ is bounded in $L^2(0,T;W^{-1,2}(0,1))$.
Moreover, we have $(\gamma(h_1^k))_k$ bounded in $L^\infty(0,T;H^1(0,1))$.

Then,

$$(\gamma(h_1^k))_k \quad \text{converges strongly to} \quad \gamma(h_1) \quad \text{in } C^0(0,T;L^2(0,1)),$$

$$(u_1^k)_k \quad \text{converges weakly to} \quad u_1 \quad \text{in } L^2(0,T;L^2(0,1)).$$

So,

$$(\gamma(h_1^k)u_1^k)_k \quad \text{converges weakly to} \quad \gamma(h_1)u_1 \text{ in } L^2(0,T;L^2(0,1)).$$

However, the function $(h_1^k,\partial_x h_1^k) \longmapsto h_1^k\partial_x h_1^k$ is a continuous in $L^\infty(0,T;H^1(0,1)) \times L^2(0,T;W^{-1,2}(0,1))$ to $L^2(0,T;W^{-1,2}(0,1))$.
So,

$$(h_1^k\partial_x u_1^k)_k \quad \text{converges weakly to} \quad h_1\partial_x u_1 \quad \text{in } L^2(0,T;H^{-1}(0,1)).$$

Finally, thanks to the strong convergence of $\left(\sqrt{h_1^k}u_1^k\right)_k$ to $\sqrt{h_1}u_1$ in $L^2(0,T;L^2(0,1))$ and the weak convergence of $(u_1^k)_k$ to u_1 mentioned above, we have :

$$(h_1^k|u_1^k|^2u_1^k)_k \quad \text{converges weakly to} \quad h_1|u_1|^2u_1 \quad \text{in } L^1(0,T;L^1(0,1)).$$

3.7. Convergences of $(h_2^ku_1^k)_k$ and $\left((h_2^k)^2(\frac{1}{c}+\frac{1}{3\nu_2}h_2^k)\partial_x(h_1^k+h_2^k)\right)_k$. We know that $(\partial_x(h_1^k+h_2^k))_k$ converges weakly to $\partial_x(h_1+h_2)$ in $L^2(0,T;L^2(0,1))$ and $\left((h_2^k)^2(\frac{1}{c}+\frac{1}{3\nu_2}h_2^k)\right)_k$ converges strongly to $h_2^2(\frac{1}{c}+\frac{1}{3\nu_2})h_2$ in $L^1(0,T;L^1(0,1))$. So, $\left((h_2^k)^2(\frac{1}{c}+\frac{1}{3\nu_2}h_2^k)\partial_x(h_1^k+h_2^k)\right)_k$ converges weakly to $(h_2)^2(\frac{1}{c}+\frac{1}{3\nu_2}h_2)\partial_x(h_1+h_2)$ in $L^1(0,T;L^1(0,1))$. To conclude, we have:

$$(u_1^k)_k \quad \text{converges weakly to} \quad u_1 \text{ in } L^2(0,T;L^2(0,1))$$

and the strong convergence of$(h_2^k)_k$ to h_2, both give us:

$$(h_2^ku_1^k)_k \quad \text{converges weakly to} \quad h_2u_1 \quad \text{in } L^1(0,T;L^1(0,1)).$$

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