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Algebraic dependences of Gauss maps of algebraic complete minimal surfaces sharing hyperplanes without counting multiplicities

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Abstract. The aim of this paper is to give some algebraic dependences theorems for the Gauss maps of algebraic complete minimal surfaces sharing hyperplanes in projective without counting multiplicity, where all zeros with multiplicities more than a certain number are omitted. As a consequence, we obtain some results on uniqueness problem of Gauss maps of algebraic complete minimal surfaces which generalize and improve a known result of L. Jin - M. Ru [5].

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1. Introduction

Value distribution theory of the Gauss map of complete regular minimal surfaces has a long history, in particular, much attention has been given to this theory from the viewpoint of the Nevanlinna Theory. Over the last few decades, there have been several results on the unicity of the Gauss maps of the complete regular minimal surfaces.

In 1993, H. Fujimoto [2] showed some unicity theorems of the Gauss maps of the complete regular minimal surfaces immersed in $\mathbb{R}^3$. After that, in [2], he also extended these results to the generalized Gauss maps of complete minimal surfaces in $\mathbb{R}^m$.

As we know, when the minimal surface is of finite total curvature, the surface is conformally equivalent to a compact Riemann surface (after the surface is equipped with a complex structure) punctured at a finite number of points and the (generalized) Gauss map is holomorphically extended to the compact Riemann surface. For this reason, the minimal surfaces with finite total curvatures are called algebraic minimal surfaces and the theory of algebraic curves can be applied in this case. For instance, in 2007, by using the Riemann-Hurwitz theorem and the Plücker formula, L. Jin - M. Ru [5] established the following second main theorem of algebraic curves for hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$.

Theorem A [5, Theorem 2.4] *Let S be a compact complex Riemann surface of genus g. Let $f : S \to \mathbb{P}^n(\mathbb{C})$ be non-constant algebraic curve. Assume that $f(S)$ is contained in some k-dimensional projective subspace of $\mathbb{P}^n(\mathbb{C})$, but not in any subspace of dimension lower than k, where $1 \le k \le n$. Let $H_1, \ldots, H_q$ be the hyperplanes in $\mathbb{P}^n(\mathbb{C})$,*

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located in general position and let $L_1, \ldots, L_q$ be the corresponding linear forms. Let E be a finite subset of S. Then

$$(q-2n+k-1)\deg(f) \leq \sum_{j=1}^{q}\sum_{P\notin E}\min\{k, \nu_P(L_j(f))\}+\frac{1}{2}k(2n-k+1)\{2(g-1)+|E|\},$$

where $\nu_P(L_j(f))$ is the vanishing order of $L_j(f)$ at the point P.

Here by an algebraic curve, we mean a holomorphic map $f : S \to \mathbb{P}^n(\mathbb{C})$, where S is a compact complex Riemann surface of genus g.

We now recall some notations.

Let S be a complete immersed minimal surface in $\mathbb{R}^m$. Take an immersion $x = (x_0, \ldots, x_{m-1}) : S \to \mathbb{R}^m$. Then S has the structure of a Riemann surface and any local isothermal coordinate (x, y) of S gives a local holomorphic coordinate $z = x + \sqrt{-1}y$. The generalized Gauss map of S is defined to be

$$G : S \to \mathbb{P}^{m-1}(\mathbb{C}), G = \mathbb{P}\Big(\frac{\partial x}{\partial z}\Big) = \Big(\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_{m-1}}{\partial z}\Big).$$

Since $x : S \to \mathbb{R}^m$ is immersed, it implies that

$$g = g_z := (g_0, \ldots, g_{m-1}) = ((g_0)_z, \ldots, (g_{m-1})_z) = \Big(\frac{\partial x_0}{\partial z}, \ldots, \frac{\partial x_{m-1}}{\partial z}\Big)$$

is a (local) reduced representation of G. Moreover, for another local holomorphic coordinate ξ on S, we have $g_\xi = g_z \cdot \Big(\dfrac{dz}{d\xi}\Big)$ and hence, g is well defined (independently of the local holomorphic coordinate). Since S is minimal, G is a holomorphic map.

Let $x : S \to \mathbb{R}^m$ be a complete regular minimal surface with finite total curvature. Let $G : S \to \mathbb{P}^{m-1}(\mathbb{C})$ be its generalized Gauss map. By the result of S. S. Chern - R. Osserman (see [1]), S is conformally equivalent to a compact surfaces $\bar{S}$ punctured at a finite number of points $P_1, \ldots, P_r$ and the generalized Gauss map G extends holomorphically to $\bar{G} : \bar{S} \to \mathbb{P}^{m-1}(\mathbb{C})$. Hence, $G : S = \bar{S} \setminus \{P_1, \ldots, P_r\} \to \mathbb{P}^{m-1}(\mathbb{C})$ is algebraic. We call S the basic domain of the minimal surface.

Using Theorem A, L. Jin - M. Ru [5] showed the following theorem on unicity of generalized Gauss maps of the complete regular minimal surfaces immersed in $\mathbb{R}^m$ with finite total curvature.

Theorem B [5, Theorem 4.1] *Consider two algebraic minimal surfaces S_1, S_2 immersed in $\mathbb{R}^m$ with the same basic domain $S = \bar{S} \setminus \{P_1, \ldots, P_r\}$. Let G_1, G_2 be the generalized Gauss maps of S_1, S_2 respectively. Assume that G_1, G_2 are linearly nondegenerate and $G_1 \not\equiv G_2$. Let $\{H_i\}_{i=1}^q$ be the hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position. Assume that*

(i) $\min\{\nu_P(L_j(G_1)), 1\} = \min\{\nu_P(L_j(G_2)), 1\}$ *for all* $P \in S$ *and* $1 \leq j \leq q$,

(ii) for every $i \neq j$, $G_1^{-1}(H_i) \bigcap G_1^{-1}(H_j) = \emptyset$,

(iii) $G_1 \equiv G_2$ on $\bigcup_{i=1}^q G_1^{-1}(H_j)$. Then

$$q < \frac{1}{2}(m^2 + 5m - 4).$$

The main aim of this paper is to give some algebraic dependency theorems of Gauss maps of the minimal surfaces immersed in $\mathbb{R}^m$. From these results, we try to generalize and improve Theorem B.

We would like to note that the algebraic dependency theory which was studied by W. Stoll [12]. After that, Stoll's result has been developed by M. Ru. We refer readers to the articles [9, 11, 12, 13, 14] and the references therein for the development of related subjects.

In order to state some of our results, we first recall the following.

Let $f_t : S \to \mathbb{P}^n(\mathbb{C})$ $(1 \leqslant t \leqslant \lambda)$ be algebraic curves with local reduced representations $f_t := (f_{t0} : \cdots : f_{tn})$. Let $H_j : a_{j0}z_0 + \cdots + a_{jn}z_n = 0$ $(1 \leqslant j \leqslant q)$ be hyperplanes located in general position in $\mathbb{P}^n(\mathbb{C})$. Assume that $H_j(f_t) = (f_t, H_j) := \sum_{i=0}^{n} f_{ti}a_{ji} \neq 0$ for each $1 \leq t \leq \lambda$, $1 \leq j \leq q$. Let k_j $(1 \leq k_j \leq q)$ be positive integers or $+\infty$. Assume that $\min\{1, \nu_{H_j(f_1),\leq k_j}\} = \cdots = \min\{1, \nu_{H_j(f_\lambda),\leq k_j}\}$. Put $A_j = Supp(\nu_{H_j(f_1),\leq k_j})$. For each $z \in S$, we define $\rho(z) = \sharp\{j | z \in A_j\}$. Then $\rho(z) \leq n$. Indeed, suppose that $z \in A_j$ for each $1 \leq j \leq n+1$. Then $\sum_{i=0}^{n} f_{1i}(z) \cdot a_{ji} = 0$ for each $1 \leq j \leq n+1$. Since the family $\{H_j\}_{j=1}^q$ is in general position, it implies that $\text{rank}(a_{ij})_{1\leq i\leq n+1, 0\leq j\leq n} = n+1$. Therefore, $f_{1i}(z) = 0$ for each $0 \leq i \leq n$. This is impossible. We define $d = \sup\{\rho(z) | z \in S\}$. Then $d \leq n$. If for each $i \neq j$, $A_i \cap A_j = \emptyset$, then $d = 1$.

With above notations, we will prove the followings.

Theorem 1.1. *Consider λ algebraic minimal surfaces $S_1, \ldots, S_\lambda$ immersed in $\mathbb{R}^m$ with the same basic domain $S = \bar{S} \setminus \{P_1, \ldots, P_r\}$. Let $G_1, \ldots, G_\lambda$ be the generalized Gauss maps of $S_1, \ldots, S_\lambda$ respectively. Assume that $G_1, \ldots, G_\lambda$ are linearly non-degenerate. Let $\{H_i\}_{i=1}^q$ be the hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position. Let k_j $(1 \leq k_j \leq q)$ be positive integers or $+\infty$. Assume that*

(i) $\min\{\nu_{H_j(G_1),\leq k_j)}(P), 1\} = \cdots = \min\{\nu_{H_j(G_\lambda),\leq k_\lambda}(P), 1\}$ for all $P \in S$ and $1 \leq j \leq q$,

(ii) there exists an integer number $l, 2 \leq l \leq \lambda$, such that for any increasing sequence $1 \leq i_1 < \cdots < i_l \leq \lambda$, $G_1(P) \wedge \cdots \wedge G_\lambda(P) = 0$ for every point $P \in \bigcup_{i=1}^q G_1^{-1}(H_j)$.

Then $G_1 \wedge \cdots \wedge G_\lambda \equiv 0$ on S, i.e., $G_1, \ldots, G_\lambda$ are algebraically dependent on S if

$$\sum_{j=1}^{q} \frac{1}{k_j} \leq \frac{2q - m(m+1)}{2(m-1)} - \frac{\lambda d}{\lambda - l + 1}.$$

From the above result, letting $k_j = +\infty, (1 \leq j \leq q)$ and $l = \lambda = 2$, we get an unicity theorem as follows.

Corollary 1.2. *Consider two algebraic minimal surfaces S_1, S_2 immersed in $\mathbb{R}^m$ with the same basic domain $S = \bar{S} \setminus \{P_1, \ldots, P_r\}$. Let G_1, G_2 be the generalized Gauss maps of S_1, S_2 respectively. Assume that G_1, G_2 are linearly non-degenerate. Let $\{H_i\}_{i=1}^q$ be the hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position. Assume that*

(i) $\min\{\nu_{H_j(G_1)}(P), 1\} = \min\{\nu_{(H_j(G_2)}(P), 1\}$ for all $P \in S$ and $1 \leq j \leq q$,

(ii) $G_1(P) = G_2(P)$ for every point $P \in \bigcup_{i=1}^q G_1^{-1}(H_j)$.

Then $G_1 \equiv G_2$ on S if

$$q \geq \frac{m^2 + (4d+1)m - 4d}{2}.$$

In the case $d = 1$, the condition of the above Corollary 1.2 is fulfilled with $q \geq \dfrac{m^2+5m-4}{2}$. We will get an uniqueness theorem for the Gauss maps sharing hyperplanes in general position without multiplicity. This is also the result of Theorem B. Therefore, our result generalizes L. Jin - M. Ru's result.

Now, we assume further on the images of the Gauss maps G_i in Theorem 1.1, we will get a better results as follows.

Theorem 1.3. *Consider λ algebraic minimal surfaces $S_1, \dots, S_\lambda$ immersed in $\mathbb{R}^m$ with the same basic domain $S = \bar{S} \setminus \{P_1, \dots, P_r\}$. Let $G_1, \dots, G_\lambda$ be the generalized Gauss maps of $S_1, \dots, S_\lambda$ respectively. Assume that $G_1, \dots, G_\lambda$ are linearly non-degenerate. Let $\{H_i\}_{i=1}^q$ be the hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position. Let k_j $(1 \leq k_j \leq q)$ be positive integers or $+\infty$. Assume that*

(i) $\min\{\nu_{H_j(G_1),\leq k_j}(P), 1\} = \cdots = \min\{\nu_{H_j(G_\lambda),\leq k_\lambda}(P), 1\}$ *for all $P \in S$ and $1 \leq j \leq q$,*

(ii) for every $i \neq j$, $G_1^{-1}(H_i) \bigcap G_1^{-1}(H_j) = \emptyset$,

(iii) there exists an integer number $l, 2 \leq l \leq \lambda$, such that for any increasing sequence $1 \leq i_1 < \cdots < i_l \leq \lambda$, $G_1(P) \wedge \cdots \wedge G_\lambda(P) = 0$ for every point $P \in \bigcup_{i=1}^q G_1^{-1}(H_j)$.

Then $G_1 \wedge \cdots \wedge G_\lambda \equiv 0$ on S, i.e., $G_1, \dots, G_\lambda$ are algebraically dependent on S if

$$\sum_{j=1}^{q} \frac{1}{k_j - m + 2} \leq \frac{2q - m(m+1)}{2(m-1)} - \frac{\lambda q}{(\lambda - l + 1)q + \lambda(m-2)}.$$

Letting $k_j = +\infty, (1 \leq j \leq q)$ and $l = \lambda = 2$, we get an unicity theorem as follows.

Corollary 1.4. *Consider two algebraic minimal surfaces S_1, S_2 immersed in $\mathbb{R}^m$ with the same basic domain $S = \bar{S}\setminus\{P_1, \dots, P_r\}$. Let G_1, G_2 be the generalized Gauss maps of S_1, S_2 respectively. Assume that G_1, G_2 are linearly non-degenerate. Let $\{H_i\}_{i=1}^q$ be the hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position. Assume that*

(i) $\min\{\nu_{H_j(G_1)}(P), 1\} = \min\{\nu_{H_j(G_2)}(P), 1\}$ *for all $P \in S$ and $1 \leq j \leq q$,*

(ii) for every $i \neq j$, $G_1^{-1}(H_i) \bigcap G_1^{-1}(H_j) = \emptyset$,

(iii) $G_1(P) = G_2(P)$ for every point $P \in \bigcup_{i=1}^q G_1^{-1}(H_j)$.

Then $G_1 \equiv G_2$ on S if

$$q \geq \frac{m^2 + m + 4 + \sqrt{m^4 + 18m^3 - 7m^2 - 24m + 16}}{4}.$$

Now let $q_1 = \dfrac{m^2+5m-4}{2}$ as in Theorem B and

$$q_2 = \frac{m^2 + m + 4 + \sqrt{m^4 + 18m^3 - 7m^2 - 24m + 16}}{4}$$

as in Corollary 1.4. We compare q_1 with q_2 as $m \geq 3$. We have

$$\begin{aligned}(m^2 - 9m - 12)^2 &- (\sqrt{m^4 + 18m^3 - 7m^2 - 24m + 16})^2 \\ &= 64(m^2 - 3m + 2) \\ &= 64(m-1)(m-2) > 0, \ \forall m \geq 3.\end{aligned}$$

This implies that $m^2 - 9m - 12 > \sqrt{m^4 + 18m^3 - 7m^2 - 24m + 16}$, $\forall m \geq 3$. Therefore,

$$q_1 - q_2 = \frac{m^2 - 9m - 12 - \sqrt{m^4 + 18m^3 - 7m^2 - 24m + 16}}{4} > 0, \ \forall m \geq 3.$$

Thus, Corollary 1.4 is an improvement of L. Jin - M. Ru's result.

2. Auxiliary lemmas

We now recall auxiliary results in the theory of algebraic curves and in the theory of algebraic dependency in the projective space which will be used later.

2.1. Theory of algebraic curves. Assume that $f : S \to \mathbb{P}^n(\mathbb{C})$ is a linearly nondegenerate algebraic curve (that is, $f(S)$ is not contained in any hyperplane in $\mathbb{P}^n(\mathbb{C})$). For every point $P \in S$, in a neighborhood of P, let $\mathbf{f}(z) = (f_0(z), \ldots, f_n(z))$ be a reduced representation of f at P with $z(P) = 0$, where z is a local parameter for S at P and $f_0, \ldots, f_n$ are holomorphic functions without common zeros. Take a hyperplane $H : a_0 z_0 + \cdots + a_n z_n = 0$ in $\mathbb{P}^n(\mathbb{C})$ and put

$$H(f) = a_0 f_0 + \cdots + a_n f_n.$$

Then $\sum_{z \in S} \nu_{H(f)}(z)$ does not depend on a choice of H, where $\nu_{H(f)}(z)$ is the intersection multiplicity of the images of f and H at $f(z)$. We define degree of f by

$$\deg(f) = \sum_{P \in S} \nu_{H(f)}(P).$$

It is easy to see that if $f^{-1}(H) = \{P_1, \ldots, P_r\}$, then

$$\deg(f) = \sum_{j=1}^{r} \nu_{H(f)}(P_j) \geq r. \tag{1}$$

2.2. Divisor. For a divisor ν on S and for positive integers m, k or $m, k = +\infty$, we define the truncated divisor of ν by

$$\nu^{[m]}(z) = \min\{m, \nu(z)\}, \quad \nu^{[m]}_{\leq k}(z) = \begin{cases} \nu^{[m]}(z) & \text{if } \ \nu^{[m]}(z) \leq k \\ 0 & \text{if } \ \nu^{[m]}(z) > k \end{cases}.$$

Similarly, we define $\nu^{[M]}_{>k}(z)$.

Lemma 2.1. *For a divisor ν on S and for positive integers m, k, $(k \geq m)$ or $m, k = +\infty$. We have*

$$\nu^{[m]}_{\leq k} \geq \frac{k+1}{k+1-m} \nu^{[m]} - \frac{m}{k+1-m} \nu.$$

Proof. For each $z \in S$, we have

$$\begin{aligned} \nu^{[m]}_{\leq k}(z) &= \nu^{[m]}(z) - \nu^{[M]}_{>k}(z) \\ &\geq \nu^{[m]}(z) - \frac{m}{k+1} \nu_{>k}(z) \\ &= \nu^{[m]}(z) - \frac{m}{k+1} \nu(z) + \frac{m}{k+1} \nu_{\leq k}(z) \\ &\geq \nu^{[m]}(z) - \frac{m}{k+1} \nu(z) + \frac{m}{k+1} \nu^{[m]}_{\leq k}(z). \end{aligned}$$

Therefore,

$$\nu^{[m]}_{\leq k}(z) \geq \frac{k+1}{k+1-m}\nu^{[m]}(z) - \frac{m}{k+1-m}\nu(z).$$

The lemma is proved. □

2.3. Theory of algebraic dependency. Let V be a complex vector space of dimension $N \geq 1$. The vectors $\{v_1, \dots, v_k\}$ are said to be in general position if for each selection of integers $1 \leq i_1 < \cdots < i_p \leq k$ with $p \leq N$, then $v_{i_1} \wedge \cdots \wedge v_{i_p} \not\equiv 0$. The vectors $\{v_1, \dots, v_k\}$ are said to be in special position if they are not in general position. Take $1 \leq p \leq k$. Then $\{v_1, \dots, v_k\}$ are said to be in p-special position if for each selection of integers $1 \leq i_1 < \cdots < i_p \leq k$, the vectors $v_{i_1}, \dots, v_{i_p}$ are in special position.

Assume that $f_1, \dots, f_\lambda : S \to \mathbb{P}^n(\mathbb{C})$ are not in special position. Let $F_t : U \to \mathbb{C}^{n+1}$ be a local reduced representation of f_t on U for $1 \leq t \leq \lambda$. Then $F_1 \wedge \cdots \wedge F_\lambda : U \to \bigwedge_\lambda \mathbb{C}^{n+1}$ is not identically zero, there exists one and only divisor define by

$$\nu_{f_1\wedge\cdots\wedge f_\lambda}\big|_U = \nu_{F_1\wedge\cdots\wedge F_\lambda}.$$

Obviously $\nu_{f_1\wedge\cdots\wedge f_\lambda} \geq 0$. Also, we can define a holomorphic maps $f_1 \wedge \cdots \wedge f_\lambda : S \to \mathbb{P}(\bigwedge_\lambda \mathbb{C}^{n+1})$ by $f_1 \wedge \cdots \wedge f_\lambda = \mathbb{P}(F_1 \wedge \cdots \wedge F_\lambda)$ on U.

Theorem 2.2 (The Second Main Theorem for general position [12, Theorem 2.1, p.320])**.** *Let M be a connected complex manifold of dimension m. Let A be a pure $(m-1)$-dimensional analytic subset of M. Let V be a complex vector space of dimension $n+1 > 1$. Let p and k be integers with $1 \leq p \leq k \leq n+1$. Let $f_i : M \to P(V), 1 \leq i \leq k$, be meromorphic mappings. Assume that $f_1, \dots, f_k$ are in general position. Also assume that $f_1, \dots, f_k$ are in p-special position on A. Then we have*

$$\nu_{f_1\wedge\cdots\wedge f_k} \geq (k-p+1)\nu_A.$$

Lemma 2.3. *Let $f_1, \dots, f_k : S \to \mathbb{P}(\mathbb{C})$ be algebraic curves. Assume that $\{f_t\}_{t=1}^{\lambda}$ are not in special position. Then*

$$\sum_{P\in S} \nu_{f_1\wedge\cdots\wedge f_\lambda}(P) \leq \sum_{t=1}^{\lambda} \deg(f_t).$$

Proof. For each $P \in S$, we take z is a local parameter for S at P, defined on a subset U of S. Let $F_t = (f_{t0} : \cdots : f_{tn}) : U \to \mathbb{C}^{n+1}$ be a local reduced representation of f_t on U for $1 \leq t \leq \lambda$. Then

$$F_1 \wedge \cdots \wedge F_\lambda = \sum_{0\leq i_0<i_1\cdots<i_{\lambda-1}\leq n} \det(f_{ti_j})_{1\leq t\leq\lambda, 0\leq j\leq\lambda-1} E_{i_1} \wedge \cdots \wedge E_{i_{\lambda-1}},$$

where $\{E_0, \dots, E_n\}$ is a standard basic of $\mathbb{C}^{n+1}$. By the assumption, without loss of generality, we may assume that $\det(f_{tj})_{0\leq j\leq\lambda-1, 1\leq t\leq\lambda} \not\equiv 0$. Let $H_0, \dots, H_{\lambda-1}$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ defined by $H_j : \omega_j = 0$ for $0 \leq j \leq \lambda - 1$. Obviously,

$$\det(H_j(f_t))_{0\leq j\leq\lambda-1, 1\leq t\leq\lambda} = \det(f_{tj})_{0\leq j\leq\lambda-1, 1\leq t\leq\lambda}$$

on U. Hence, for each $P \in S$, we have

$$\nu_{f_1\wedge\cdots\wedge f_\lambda}(P) \leq \nu_{\det(H_j(f_t))_{0\leq j\leq\lambda-1, 1\leq t\leq\lambda}}(P). \tag{2}$$

We define
$$\varphi = \frac{\det(H_j(f_t))_{0\le j\le\lambda-1}}{H_0(f_1)\cdots H_0(f_\lambda)}.$$
It is easy to see that such definition is independent of the choice of the representations of f_t and of the parameter z. Hence φ is well defined. It implies that
$$\nu_{\det(H_j(f_t))_{0\le j\le\lambda-1}}(P) - \nu_{H_0(f_1)\cdots H_0(f_\lambda)}(P) = \nu_\varphi(P) = 0.$$
Therefore,
$$\nu_{\det(H_j(f_t))_{0\le j\le\lambda-1}}(P) = \sum_{t=1}^{\lambda}\nu_{H_0(f_t)}(P).$$
By the definition, we get
$$\sum_{P\in S}\nu_{\det(H_j(f_t))_{0\le j\le\lambda-1}}(P) = \sum_{P\in S}\sum_{t=1}^{\lambda}\nu_{H_0(f_t)}(P) = \sum_{t=1}^{\lambda}\deg(f_t). \tag{3}$$
Combining (2) with (3), we get
$$\sum_{P\in S}\nu_{f_1\wedge\cdots\wedge f_\lambda}(P) \le \sum_{t=1}^{\lambda}\deg(f_t).$$
The lemma is proved. □

3. Proof of Theorem 1.1

It suffices to prove Theorem 1.1 in the cace of $\lambda \le m$. Suppose that $G_1\wedge\cdots\wedge G_\lambda \not\equiv 0$ on S.

Since the minimal surface S has finite total curvature, S is conformally equivalent to a compact surface $\bar{S}$ punctured at a finite mumber of points $P_1,\ldots,P_r$ and the generalized Gauss map G_i extends holomorphically to $\bar{G_i} : \bar{S} \to \mathbb{P}^{m-1}(\mathbb{C})$ for $1 \le i \le \lambda$ (see [1]). Hence, $\bar{G_1}\wedge\cdots\wedge\bar{G_\lambda} \not\equiv 0$ on $\bar{S}$.

The first, we claim that for each $1\le t\le\lambda$ and $P\in S$,
$$\sum_{j=1}^{q}\min\{1, \nu_{H_j(\bar{G_t}),\le k_j}(P)\} \le \frac{d}{\lambda-l+1}\nu_{\bar{G_1}\wedge\cdots\wedge\bar{G_\lambda}}(P). \tag{4}$$

Indeed, for $P \notin \bigcup_{j=1}^{q} {G_1}^{-1}(H_j)$, $P \notin \bigcup_{j=1}^{q} \bar{G_1}^{-1}(H_j)$. Hence, $\nu_{H_j(\bar{G_t}),\le k_j}(P) = 0$ for all j, $1\le j\le q$. The inequality (4) is true.

For $P \in \bigcup_{j=1}^{q} {G_1}^{-1}(H_j)$, and for each increasing sequence $1\le i_1<\cdots<i_l\le\lambda$, by our assumption, we have $G_{i_1}(P)\wedge\cdots\wedge G_{i_l}(P) = 0$. Hence, $\bar{G_{i_1}}(P)\wedge\cdots\wedge\bar{G_{i_l}}(P) = 0$.

By the Second Main Theorem for general position [12, Theorem 2.1, p.320] (Theorem 2.2), we have
$$\nu_{\bar{G_1}\wedge\cdots\wedge\bar{G_\lambda}}(P) \ge \lambda-(l-1).$$
This implies that
$$\sum_{j=1}^{q}\min\{1, \nu_{H_j(\bar{G_t}),\le k_j}(P)\} \le \sum_{j=1}^{q}\min\{1, \nu_{H_j(\bar{G_t})}(P)\} \le d \le \frac{d}{\lambda-l+1}\nu_{\bar{G_1}\wedge\cdots\wedge\bar{G_\lambda}}(P).$$
Therefore, the inequality (4) is proved.

By Lemma 2.1 and (4), for $p \in S$, we have

$$\sum_{j=1}^{q}\left(\min\{1,\nu_{H_j(\bar{G}_t)}(P)\} - \frac{1}{k_j}\nu_{H_j(\bar{G}_t)}(P)\right) \leq \frac{d}{\lambda - l + 1}\nu_{\bar{G}_1\wedge\cdots\wedge\bar{G}_\lambda}(P).$$

Hence,

$$\sum_{P\in S}\sum_{j=1}^{q}\min\{1,\nu_{H_j(\bar{G}_t)}(P)\} \leq \sum_{P\in\bar{S}}\frac{d}{\lambda - l + 1}\nu_{\bar{G}_1\wedge\cdots\wedge\bar{G}_\lambda}(P) + \sum_{P\in\bar{S}}\sum_{j=1}^{q}\frac{1}{k_j}\nu_{H_j(\bar{G}_t)}(P). \tag{5}$$

Then from (5) and Lemma 2.3, we have

$$\sum_{P\in S}\sum_{j=1}^{q}\min\{1,\nu_{H_j(\bar{G}_t)}(P)\} \leq \frac{d}{\lambda - l + 1}\sum_{i=1}^{\lambda}\deg(\bar{G}_i) + \sum_{j=1}^{q}\frac{1}{k_j}\deg(\bar{G}_t).$$

Therefore, we get

$$\sum_{t=1}^{\lambda}\sum_{P\in S}\sum_{j=1}^{q}\min\{1,\nu_{H_j(\bar{G}_t)}(P)\} \leq \left(\frac{d\lambda}{\lambda - l + 1} + \sum_{j=1}^{q}\frac{1}{k_j}\right)\sum_{i=1}^{\lambda}\deg(\bar{G}_i). \tag{6}$$

By the result of S. S. Chern - R. Osserman (see [1]), we have

$$C(S) = -2\pi\deg(\bar{G}_t) \leq 2\pi(\mathcal{X} - r) = 2\pi(2 - 2g - r - r),$$

where $\mathcal{X}$ is the Euler characteristic of $\bar{S}$ and g is genus of $\bar{S}$. Hence,

$$2(g-1) \leq \deg(\bar{G}_t) - 2r, \ \forall 1 \leq t \leq \lambda.$$

This implies that

$$2(g-1) + \sharp E \leq \deg(\bar{G}_t) - r < \deg(\bar{G}_t), \ \forall 1 \leq t \leq \lambda. \tag{7}$$

By the Second Main Theorem of L. Jin - M. Ru (Theorem A) for algebraic curves with $E = \{P_1, \ldots, P_r\}$ and since (7), we have

$$\begin{aligned}(q-m)\deg(\bar{G}_t) &\leq \sum_{P\notin E}\sum_{j=1}^{q}\min\{m-1,\nu_{H_j(\bar{G}_t)}(P)\} + \frac{1}{2}(m-1)m\{2(g-1) + \sharp E\}\\ &< (m-1)\sum_{P\notin E}\sum_{j=1}^{q}\min\{1,\nu_{H_j(\bar{G}_t)}(P)\} + \frac{1}{2}(m-1)m\deg(\bar{G}_t).\end{aligned}$$

Hence,

$$(q-m)\sum_{t=1}^{\lambda}\deg(\bar{G}_t) < (m-1)\sum_{t=1}^{\lambda}\sum_{P\notin E}\sum_{j=1}^{q}\min\{1,\nu_{H_j(\bar{G}_t)}(P)\} + \frac{(m-1)m}{2}\sum_{t=1}^{\lambda}\deg(\bar{G}_t).$$

It implies that

$$\frac{2q - m^2 - m}{2(m-1)}\sum_{t=1}^{\lambda}\deg(\bar{G}_t) < \sum_{t=1}^{\lambda}\sum_{P\notin E}\sum_{j=1}^{q}\min\{1,\nu_{H_j(\bar{G}_t)}(P)\}. \tag{8}$$

Combining (6) with (8), we get

$$\frac{2q - m^2 - m}{2(m-1)} < \frac{d\lambda}{\lambda - l + 1} + \sum_{j=1}^{q}\frac{1}{k_j}.$$

This is a contradiction. Hence, $G_1 \wedge \cdots \wedge G_\lambda \equiv 0$ on S. The proof of Theorem 1.1 is completed. □

4. Proof of Theorem 1.3

We prove the following lemma.

Lemma 4.1. *Let $h_i : S \to \mathbb{P}^n(\mathbb{C})$, $(1 \le i \le p \le n+1)$ be algebraic curves. Let $H_i : a_{i0}z_0 + \cdots + a_{in}z_n = 0, (1 \le i \le n+1)$ be hyperplanes in $P^n(\mathbb{C})$. Put $\tilde{h_i} := (H_1(h_i) : \cdots : H_{n+1}(h_i))$. Assume that $H_1, \dots, H_{n+1}$ are located in general position such that $H_j(h_i) \not\equiv 0$ $(1 \le i \le p, 1 \le j \le n+1)$. Let M be an analytic subset of S. Then $h_1 \wedge \cdots \wedge h_p \equiv 0$ on M if and only if $\tilde{h_1} \wedge \cdots \wedge \tilde{h_p} \equiv 0$ on M.*

Proof. Consider $P_0 \in M$. Take z is a local parameter for M at P_0, defined on a subset U of S. Let $F_t = (h_{t0} : \cdots : h_{tn}) : U \to \mathbb{C}^{n+1}$ be a local reduced representation of h_t on U for $1 \le t \le p$. We have $\tilde{h_1}(P_0) \wedge \cdots \wedge \tilde{h_p}(P_0) = 0$ if and only if the following matrix is of rank $\le p-1$

$$\begin{pmatrix} H_1(h_1)(P_0) & \cdots & H_1(h_p)(P_0) \\ H_2(h_1)(P_0) & \cdots & H_2(h_p)(P_0) \\ \vdots & \vdots & \vdots \\ H_{n+1}(h_1)(P_0) & \cdots & H_{n+1}(h_p)(P_0) \end{pmatrix}$$

$$= \begin{pmatrix} a_{10} & \cdots & a_{1n} \\ a_{20} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n+10} & \cdots & a_{n+1n} \end{pmatrix} \cdot \begin{pmatrix} h_{10}(P_0) & \cdots & h_{p0}(P_0) \\ h_{11}(P_0) & \cdots & h_{p1}(P_0) \\ \vdots & \vdots & \vdots \\ h_{1n}(P_0) & \cdots & h_{pn}(P_0) \end{pmatrix}$$

Hence, the matrix

$$\begin{pmatrix} h_{10}(P_0) & \cdots & h_{p0}(P_0) \\ h_{11}(P_0) & \cdots & h_{p1}(P_0) \\ \vdots & \vdots & \vdots \\ h_{1n}(P_0) & \cdots & h_{pn}(P_0) \end{pmatrix}$$

is of rank $\le p-1$, i.e., $h_1(P_0) \wedge \cdots \wedge h_p(P_0) = 0$. Thus, Lemma 4.1 is proved. □

We now prove Theorem 1.3. It suffices to prove the theorem in the case of $\lambda \le m$.

Suppose that $G_1 \wedge \cdots \wedge G_\lambda \not\equiv 0$ on S. Denote by $\bar{G_1}, \dots, \bar{G_\lambda}$ the extended maps of $G_1, \dots, G_\lambda$ respectively. Then, we have $\bar{G_1} \wedge \cdots \wedge \bar{G_\lambda} \not\equiv 0$ on $\bar{S}$. For each λ indices $1 \le j_1 < \cdots < j_m \le q$, there exists indices, for instance it is $\{j_1, \dots, j_\lambda\}$ such that

$$B_J = \begin{pmatrix} H_{j_1}(\bar{G_1}) & \cdots & H_{j_1}(\bar{G_\lambda}) \\ H_{j_2}(\bar{G_1}) & \cdots & H_{j_2}(\bar{G_\lambda}) \\ \vdots & \vdots & \vdots \\ H_{j_\lambda}(\bar{G_1}) & \cdots & H_{j_\lambda}(\bar{G_\lambda}) \end{pmatrix}.$$

has rank of λ.

Put $J = \{j_1, \dots, j_\lambda\}$ and $J^c = \{1, \dots, q\} \setminus J$. We prove the following lemma.

Lemma 4.2.

$$\sum_{P\in S}\Big(\sum_{i\in J}\big(\min_{1\le t\le\lambda}\{\nu_{H_i(\bar{G_t}),\le k_i}(P)\}-\min\{1,\nu_{H_i(\bar{G_1}),\le k_i}(P)\}\big)$$
$$+\sum_{i=1}^{q}(\lambda-l+1)\min\{1,\nu_{H_i(\bar{G_1}),\le k_i}(P)\}\Big)\le\sum_{i=1}^{\lambda}\deg(\bar{G_i}).$$

Proof. Denote $\mathcal{A}=\bigcup_{j\in J}G_1^{-1}(H_j)$ and $\mathcal{A}^c=\bigcup_{j\in J^c}G_1^{-1}(H_j)$. We consider the following two cases.

Case 1. Let $P_0\in\mathcal{A}$. Then by our assumption (ii), P_0 is a zero of one of the holomorphic functions $\{H_j(\bar{G_1})\}_{j\in J}$. Without loss of generality, we may assume that P_0 is a zero of $H_{j_0}(\bar{G_1})$. Let $M\subset S$ be an irreducible component of $\mathcal{A}$ containing P_0. Let U be an open neighborhood of P_0 in M such that $U\cap(\mathcal{A}\setminus M)=\emptyset$. Choose a holomorphic function h on a neighborhood $U'\subset U$ of P_0 such that $\nu_h(P)=\min_{1\le t\le\lambda}\{\nu_{H_{j_1}(G_t),\le k_{j_1}}(P)\}$ if $P\in M$ and $\nu_h=0$ if $P\notin M$. Then $H_{j_1}(\bar{G_i})=a_ih$, $(1\le i\le\lambda)$, where a_i are holomorphic functions. By the matrix

$$B=\begin{pmatrix} H_{j_2}(\bar{G_1}) & \cdots & H_{j_2}(\bar{G_\lambda})\\ H_{j_3}(\bar{G_1}) & \cdots & H_{j_3}(\bar{G_\lambda})\\ \vdots & \vdots & \vdots\\ H_{j_\lambda}(\bar{G_1}) & \cdots & H_{j_\lambda}(\bar{G_\lambda})\end{pmatrix}$$

has rank of $\lambda-1$, there exist λ holomorphic functions $b_1,\dots,b_\lambda$, not all zeros, such that

$$\sum_{i=1}^{\lambda}b_i\cdot H_{j_k}(\bar{G_i}),\ (2\le k\le\lambda).$$

Without loss of generality, we may assume that the set of common zeros of $\{b_i\}_{i=1}^{\lambda}$ is an empty set. Then there exists an index $t_1, 1\le t_1\le\lambda$ such that $M\not\subset b_{t_1}^{-1}(0)$. We can assume that $t_1=\lambda$.

Put $\tilde{G_i}=(H_{j_1}(\bar{G_i}):\cdots:H_{j_\lambda}(\bar{G_i}))$, $(1\le i\le\lambda)$. Then, for each $P\in(U'\cap S)\setminus b_\lambda^{-1}(0)$, we have

$$\begin{aligned}\tilde{G}_1(z)\wedge\cdots\wedge\tilde{G}_\lambda(P)&=\tilde{G}_1(P)\wedge\cdots\wedge\tilde{G}_{\lambda-1}(P)\wedge\Big(\tilde{G}_\lambda(P)+\sum_{t=1}^{\lambda-1}\frac{b_t}{b_\lambda}\tilde{G}_t(P)\Big)\\&=\tilde{G}_1(P)\wedge\cdots\wedge\tilde{G}_{\lambda-1}(z)\wedge(V(P)h(P))\\&=h(P)\cdot(\tilde{G}_1(P)\wedge\cdots\wedge\tilde{G}_{\lambda-1}(P)\wedge V(P)),\end{aligned}$$

where $V(z):=(a_\lambda+\sum_{t=1}^{\lambda-1}\dfrac{b_t}{b_\lambda}a_t,0,\dots,0)$.

By the assumption, for any increasing sequence $1\le j_1<\cdots<j_l\le\lambda-1$, we have $\bar{G_{j_1}}\wedge\cdots\wedge\bar{G_{j_l}}\equiv 0$ on M. It easily follows from Lemma 4.1 that $\tilde{G}_{j_1}\wedge\cdots\wedge\tilde{G}_{j_l}\equiv 0$ on M. This implies that the family $\{\tilde{G}_1,\dots\tilde{G}_{\lambda-1},V\}$ is in $(l+1)$-special position on M. By using Theorem 2.2, we have

$$\nu_{\tilde{G}_1\wedge\cdots\wedge\tilde{G}_{\lambda-1}\wedge V}(z)\ge\lambda-l,\forall P\in M.$$

Hence $\nu_{\tilde{G}_1\wedge\cdots\wedge\tilde{G}_\lambda}(P) \geq \nu_h(P) + \lambda - l = \min_{1\leq i\leq\lambda}\{\nu_{H_{j_1}(\bar{G}_i)),\leq k_{j_0}}(P)\} + \lambda - l, \forall P \in (U' \cup M) \setminus b_\lambda^{-1}(0)$. In particular, we have

$$\nu_{\tilde{G}_1\wedge\cdots\wedge\tilde{G}_\lambda}(P_0) \geq \min_{1\leq i\leq\lambda}\{\nu_{H_{j_1}(\bar{G}_i)),\leq k_{j_0}}(P_0)\} + \lambda - l.$$

This implies that

$$\sum_{j\in J}(\min_{1\leq i\leq\lambda}\{\nu_{H_j(\bar{G}_i),\leq k_j}(P_0)\} - \min\{1, \nu_{H_j(\bar{G}_i),\leq k_j}(P_0)\})$$
$$+\sum_{j=1}^{q}(\lambda - l + 1)\min\{1, \nu_{H_j(\bar{G}_i),\leq k_j}(P_0)\} \leq \nu_{\tilde{G}_1\wedge\cdots\wedge\tilde{G}_\lambda}(P_0).$$

Case 2. Let $P_0 \in \mathcal{A}^c$. Then P_0 is a zero of one of the meromorphic mappings $\{H_i(\bar{G}_1)\}_{i\in J^c}$. By the assumption and by Lemma 4.1, the family $\{\tilde{G}_1, \ldots, \tilde{G}_\lambda\}$ is in l-special position on an irreducible analytic subset of $\mathcal{A}^c$ which contains P_0. By using Theorem 2.2 again, we have

$$\nu_{\tilde{G}_1\wedge\cdots\wedge\tilde{G}_\lambda}(P_0) \geq \lambda - l + 1.$$

Hence,

$$\sum_{i\in J}(\min_{1\leq t\leq\lambda}\{\nu_{H_i(\bar{G}_t),\leq k_i}(P_0)\} - \min\{1, \nu_{H_i(\bar{G}_1),\leq k_i}(P_0)\})$$
$$+\sum_{i=1}^{q}(\lambda - l + 1)\min\{1, \nu_{H_i(\bar{G}_1),\leq k_i}(P_0)\} = \lambda - l + 1 \leq \nu_{\tilde{G}_1\wedge\cdots\wedge\tilde{G}_\lambda}(P_0).$$

From the above cases, we get

$$\sum_{p\in S}\Big(\sum_{i\in J}(\min_{1\leq t\leq\lambda}\{\nu_{H_i(\bar{G}_t),\leq k_i}(P)\} - \min\{1, \nu_{H_i(\bar{G}_1),\leq k_i}(P)\})$$
$$+\sum_{i=1}^{q}(\lambda - l + 1)\min\{1, \nu_{H_i(\bar{G}_1),\leq k_i}(P)\}\Big) \leq \sum_{P\in S}\nu_{\tilde{G}_1\wedge\cdots\wedge\tilde{G}_\lambda}(P). \quad (9)$$

By our assumption (ii), $\tilde{G}_i = (H_{j_1}(\bar{G}_i) : \cdots : H_{j_\lambda}(\bar{G}_i))$, $(1 \leq i \leq \lambda)$ is a local reduced representation around $P \in S$ of $\tilde{G}_i$. For $P \notin S$, $\tilde{G}_i = \left(\frac{H_{j_1}(\bar{G}_i)}{h} : \cdots : \frac{H_{j_\lambda}(\bar{G}_i)}{h}\right)$, $(1 \leq i \leq \lambda)$ is a local reduced representation around P of $\tilde{G}_i$, with h is some holomorphic function. Therefore, by Lemma 2.3, we have

$$\sum_{P\in S}\nu_{\tilde{G}_1\wedge\cdots\wedge\tilde{G}_\lambda}(P) \leq \sum_{P\in\bar{S}}\nu_{\tilde{G}_1\wedge\cdots\wedge\tilde{G}_\lambda}(P) \leq \sum_{i=1}^{\lambda}\deg(\tilde{G}_i) = \sum_{i=1}^{\lambda}\sum_{P\in\bar{S}}\nu_{H_0(\tilde{G}_i)}(P)$$
$$= \sum_{i=1}^{\lambda}\sum_{P\in S}\nu_{H_{j_1}(\bar{G}_i)}(P) + \sum_{i=1}^{\lambda}(\sum_{P\notin S}\nu_{H_{j_1}(\bar{G}_i)}(P) - \nu_h(P))$$
$$\leq \sum_{i=1}^{\lambda}\deg(\bar{G}_i), \quad (10)$$

where $H_0 : \omega_0 = 0$ is the hyperplane in $\mathbb{P}^{m-1}(\mathbb{C})$. Combining (9) with (10), we get

$$\sum_{P\in S}\big(\sum_{i\in J}\big(\min_{1\le t\le\lambda}\{\nu_{H_i(\bar{G}_t),\le k_i}(P)\} - \min\{1,\nu_{H_i(\bar{G}_1),\le k_i}(P)\}\big)$$
$$+\sum_{i=1}^{q}(\lambda-l+1)\min\{1,\nu_{H_i(\bar{G}_1),\le k_i}(P)\}\big) \le \sum_{i=1}^{\lambda}\deg(\bar{G}_i).$$

Lemma 4.2 is proved. □

We now continue to prove the theorem. For each $j, 1 \le j \le q$, we set

$$\nu_j = \sum_{P\in S}\big(\sum_{i=1}^{\lambda}\min\{m-1,\nu_{H_j(\bar{G}_i),\le k_j}(P)\} - ((\lambda-1)(m-1)+1)\min\{1,\nu_{H_j(\bar{G}_1),\le k_j}(P)\}\big). \tag{11}$$

Without loss of generality, we can assume that

$$\nu_1 \ge \cdots \ge \nu_q.$$

By the assumption for $G_1 \wedge \cdots \wedge G_\lambda \not\equiv 0$ on S, there exists indices, for instance $J = \{j_1, \ldots, j_\lambda\}$ such that $1 = j_1 < j_2 < \cdots < j_\lambda \le j_m$.

We see that $\min_{1\le i\le\lambda}\{a_i\} \ge \sum_{i=1}^{\lambda}\min\{m-1, a_i\} - (\lambda-1)(m-1)$ for every λ non-negative integers $a_1, \ldots, a_\lambda$. Then by Lemma 4.2, we have

$$\sum_{P\in S}\sum_{j\in J}\big(\sum_{j=1}^{\lambda}\min\{m-1,\nu_{H_j(\bar{G}_i),\le k_j}(P)\} - ((\lambda-1)(m-1)+1)\min\{1,\nu_{H_j(\bar{G}_1),\le k_j}(P)\}\big)$$
$$+\sum_{P\in S}\sum_{j=1}^{q}(\lambda-l+1)\min\{1,\nu_{H_j(\bar{G}_1),\le k_j}(P)\} \le \sum_{i=1}^{\lambda}\deg(\bar{G}_i).$$

Therefore, from (11), we have

$$\sum_{j\in J}\nu_j + \sum_{P\in S}\sum_{j=1}^{q}(\lambda-l+1)\min\{1,\nu_{H_j(\bar{G}_1),\le k_j}(P)\} \le \sum_{i=1}^{\lambda}\deg(\bar{G}_i). \tag{12}$$

Note that

$$\sum_{j\in J}\nu_j = \sum_{i=1}^{\lambda}\nu_{j_i} \ge \frac{\lambda}{q}\sum_{i=1}^{q}\nu_{j_i} = \frac{\lambda}{q}\sum_{j=1}^{q}\nu_j.$$

Combining this with (12), we have

$$\sum_{i=1}^{\lambda}\deg(\bar{G}_i) \ge \frac{\lambda}{q}\sum_{j=1}^{q}\nu_j + \sum_{P\in S}\sum_{j=1}^{q}(\lambda-l+1)\min\{1,\nu_{H_j(\bar{G}_1),\le k_j}(P)\}$$
$$= \sum_{P\in S}\sum_{j=1}^{q}\left(\lambda-l+1-\frac{\lambda((\lambda-1)(m-1)+1)}{q}\right)\min\{1,\nu_{H_j(\bar{G}_1),\le k_j}(P)\}$$
$$+\sum_{P\in S}\frac{\lambda}{q}\sum_{j=1}^{q}\sum_{i=1}^{\lambda}\min\{m-1,\nu_{H_j(\bar{G}_i),\le k_j}(P)\}.$$

$$\geq \sum_{P\in S}\sum_{i=1}^{\lambda}\sum_{j=1}^{q}\left(\frac{\lambda}{q}+\frac{\lambda-l+1}{\lambda(m-1)}-\frac{\lambda((\lambda-1)(m-1)+1)}{q(m-1)\lambda}\right)\min\{m-1,\nu_{H_j(\bar{G}_i),\leq k_j}(P)\}$$

$$\geq \sum_{P\in S}\sum_{i=1}^{\lambda}\sum_{j=1}^{q}\frac{q(\lambda-l+1)+\lambda(m-2)}{q\lambda(m-1)}\min\{m-1,\nu_{H_j(\bar{G}_i),\leq k_j}(P)\}.$$

This implies that

$$\frac{q\lambda(m-1)}{q(\lambda-l+1)+\lambda(m-2)}\sum_{i=1}^{\lambda}\deg(\bar{G}_i)\geq \sum_{P\in S}\sum_{i=1}^{\lambda}\sum_{j=1}^{q}\min\{m-1,\nu_{H_j(\bar{G}_i),\leq k_j}(P)\}.$$

Applying Lemma 2.1, we have

$$\frac{q\lambda(m-1)}{q(\lambda-l+1)+\lambda(m-2)}\sum_{i=1}^{\lambda}\deg(\bar{G}_i)$$

$$\geq \sum_{P\in S}\sum_{i=1}^{\lambda}\sum_{j=1}^{q}\left(\min\{m-1,\nu_{H_j(\bar{G}_i)}(P)\}-\frac{m-1}{k_j+2-m}\nu_{H_j(\bar{G}_i)}(P)\right)$$

$$\geq \sum_{P\in S}\sum_{i=1}^{\lambda}\sum_{j=1}^{q}\min\{m-1,\nu_{H_j(\bar{G}_i)}(P)\}-\sum_{P\in \bar{S}}\sum_{i=1}^{\lambda}\sum_{j=1}^{q}\frac{m-1}{k_j+2-m}\nu_{H_j(\bar{G}_i)}(P).$$

Hence,

$$\left(\frac{q\lambda(m-1)}{q(\lambda-l+1)+\lambda(m-2)}+\sum_{j=1}^{q}\frac{m-1}{k_j+2-m}\right)\sum_{i=1}^{\lambda}\deg(\bar{G}_i)$$

$$\geq \sum_{i=1}^{\lambda}\sum_{P\in S}\sum_{j=1}^{q}\min\{m-1,\nu_{H_j(G_i)}(P)\}. \tag{13}$$

Applying Theorem A for algebraic curves with $E=\{P_1,\dots,P_r\}$ and together (13) with (7), we have

$$\frac{q\lambda(m-1)}{q(\lambda-l+1)+\lambda(m-2)}+\sum_{j=1}^{q}\frac{m-1}{k_j+2-m}>q-m-\frac{1}{2}m(m-1)$$

$$=\frac{2q-m(m+1)}{2}.$$

Therefore, we get

$$\frac{q\lambda}{q(\lambda-l+1)+\lambda(m-2)}+\sum_{j=1}^{q}\frac{1}{k_j+2-m}>\frac{2q-m(m+1)}{2(m-1)}.$$

This is a contradiction. Hence, $G_1\wedge\cdots\wedge G_\lambda\equiv 0$ on S. Theorem 1.3 is proved. □

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