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Some inequalities in inner product spaces related to Buzano's and Grüss' results

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Abstract. Some inequalities in inner product spaces related to Buzano's and Grüss' results are given. Applications for discrete and integral inequalities are provided as well.

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1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\| \|y\| \geq |\langle x, y \rangle| \text{ for any } x, y \in H. \tag{1}$$

The equality case holds in (1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [4] (see also [19]) established the following refinement of (1):

$$\|x\| \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| \tag{2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \geq |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|$$

and by (2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \\ &\geq 2 |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|, \end{aligned}$$

which implies the *Buzano inequality* [2]

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y \rangle|] \geq |\langle x, e \rangle \langle e, y \rangle| \tag{3}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

In [5], the author has proved the following Grüss' type inequality in real or complex inner product spaces.

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Theorem 1.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{4}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma| . \tag{5}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

For other Schwarz, Buzano and Grüss related inequalities in inner product spaces, see [1]-[3], [4]-[13], [17]-[20], [22]-[29], and the monographs [14], [15] and [16].

2. Main Results

The following results hold:

Theorem 2.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. If $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$, then*

$$\|x\| \|y\| - |\langle x, e\rangle \langle f, y\rangle| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle + \langle x, e\rangle \langle f, y\rangle \langle e, f\rangle| . \tag{6}$$

Proof. Using Schwarz inequality we have

$$\|x - \langle x, e\rangle e\|^2 \|y - \langle y, f\rangle f\|^2 \geq |\langle x - \langle x, e\rangle e, y - \langle y, f\rangle f\rangle|^2 \tag{7}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

Since

$$\|x - \langle x, e\rangle e\|^2 = \|x\|^2 - |\langle x, e\rangle|^2 , \quad \|y - \langle y, f\rangle f\|^2 = \|y\|^2 - |\langle y, f\rangle|^2$$

and

$$\langle x - \langle x, e\rangle e, y - \langle y, f\rangle f\rangle = \langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle + \langle x, e\rangle \langle f, y\rangle \langle e, f\rangle ,$$

then by (7) we get

$$\begin{aligned} &\left(\|x\|^2 - |\langle x, e\rangle|^2\right)\left(\|y\|^2 - |\langle y, f\rangle|^2\right) \\ &\geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle + \langle x, e\rangle \langle f, y\rangle \langle e, f\rangle|^2 \end{aligned} \tag{8}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

Using the elementary inequality

$$(ac - bd)^2 \geq \left(a^2 - b^2\right)\left(c^2 - d^2\right)$$

that holds for any real numbers $a, b, c, d \in \mathbb{R}$, we have

$$(\|x\| \|y\| - |\langle x, e\rangle| |\langle y, f\rangle|)^2 \geq \left(\|x\|^2 - |\langle x, e\rangle|^2\right)\left(\|y\|^2 - |\langle y, f\rangle|^2\right) \tag{9}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

By Schwarz inequality for the pairs (x, e) and (y, f) we have

$$\|x\| \geq |\langle x, e\rangle| \text{ and } \|y\| \geq |\langle y, f\rangle| ,$$

which shows that

$$\|x\| \|y\| - |\langle x, e\rangle| |\langle y, f\rangle| \geq 0,$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

Making use of (8) and (9) we get

$$
\begin{aligned}
&\left(\|x\| \, \|y\| - |\langle x, e\rangle| \, |\langle y, f\rangle|\right)^2 \qquad (10)\\
&\geq \left|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle + \langle x, e\rangle \langle f, y\rangle \langle e, f\rangle\right|^2
\end{aligned}
$$

and by taking the square root in (10) we get the desired result. □

Corollary 2.2. *With the assumptions of Theorem 2.1 and if* $e \perp f$, *i.e.* $\langle e, f\rangle = 0$, *then we have the inequality*

$$\|x\| \, \|y\| - |\langle x, e\rangle \langle f, y\rangle| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle| . \qquad (11)$$

Remark 2.1. From the inequality (11) we have

$$
\begin{aligned}
\|x\| \, \|y\| &\geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle| + |\langle x, e\rangle \langle f, y\rangle| \qquad (12)\\
&\geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle \pm \langle x, e\rangle \langle f, y\rangle| .
\end{aligned}
$$

By the triangle inequality we also have

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle| \geq |\langle x, e\rangle \langle e, y\rangle + \langle x, f\rangle \langle f, y\rangle| - |\langle x, y\rangle|$$

and by the first inequality in (14) we get

$$\|x\| \, \|y\| \geq |\langle x, e\rangle \langle e, y\rangle + \langle x, f\rangle \langle f, y\rangle| - |\langle x, y\rangle| + |\langle x, e\rangle \langle f, y\rangle| ,$$

which implies

$$
\begin{aligned}
\|x\| \, \|y\| + |\langle x, y\rangle| &\geq |\langle x, e\rangle \langle e, y\rangle + \langle x, f\rangle \langle f, y\rangle| + |\langle x, e\rangle \langle f, y\rangle| \qquad (13)\\
&\geq |\langle x, e\rangle \langle e, y\rangle + \langle x, f\rangle \langle f, y\rangle + \langle x, e\rangle \langle f, y\rangle|
\end{aligned}
$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$ and $e \perp f$.

Corollary 2.3. *With the assumptions of Theorem 2.1 we have*

$$\|x\| \, \|y\| - |\langle x, e\rangle \langle f, y\rangle| \left(1 - |\langle e, f\rangle|\right) \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle| \qquad (14)$$

and

$$\|x\| \, \|y\| + |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle| \geq |\langle x, e\rangle \langle f, y\rangle| \left(|\langle e, f\rangle| + 1\right) . \qquad (15)$$

Indeed, by the triangle inequality we have

$$
\begin{aligned}
&|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle + \langle x, e\rangle \langle f, y\rangle \langle e, f\rangle|\\
&\geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle| - |\langle x, e\rangle \langle f, y\rangle \langle e, f\rangle|
\end{aligned}
$$

and by (6) we get (14).

By the triangle inequality we also have

$$
\begin{aligned}
&|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle + \langle x, e\rangle \langle f, y\rangle \langle e, f\rangle|\\
&\geq |\langle x, e\rangle \langle f, y\rangle \langle e, f\rangle| - |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle|
\end{aligned}
$$

and by (6) we get (15).

Remark 2.2. With the assumptions of Theorem 2.1 and if $|\langle e, f\rangle| = 1$, then we have

$$\|x\| \, \|y\| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle| \qquad (16)$$

and

$$\frac{1}{2} \left[\|x\| \, \|y\| + |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle - \langle x, f\rangle \langle f, y\rangle|\right] \geq |\langle x, e\rangle \langle f, y\rangle| . \qquad (17)$$

If we take $f = e$ in (16) and (17), then we get the inequalities

$$\|x\|\,\|y\| \geq |\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle| \tag{18}$$

and

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle|\right] \geq |\langle x,e\rangle\langle e,y\rangle| \tag{19}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality we have

$$|\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle| \geq 2|\langle x,e\rangle\langle e,y\rangle| - |\langle x,y\rangle|$$

and by (18) we get

$$\|x\|\,\|y\| \geq |\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle| \geq 2|\langle x,e\rangle\langle e,y\rangle| - |\langle x,y\rangle|. \tag{20}$$

The inequality between the first and last term in (20) is equivalent to Buzano's inequality (3).

The following lemma holds, see [6]:

Lemma 2.4. *Let a, x, A be vectors in the inner product space $(H, \langle\cdot,\cdot\rangle)$ over $\mathbb{K}$ with $a \neq A$. Then*

$$\operatorname{Re}\langle A - x, x - a\rangle \geq 0 \tag{21}$$

if and only if

$$\left\|x - \frac{a+A}{2}\right\| \leq \frac{1}{2}\|A - a\|. \tag{22}$$

Proof. Define

$$I_1 := \operatorname{Re}\langle A - x, x - a\rangle \text{ and } I_2 := \frac{1}{4}\|A - a\|^2 - \left\|x - \frac{a+A}{2}\right\|^2.$$

A simple calculation shows that

$$I_1 = I_2 = \operatorname{Re}\left[\langle x,a\rangle + \langle A,x\rangle\right] - \operatorname{Re}\langle A,a\rangle - \|x\|^2$$

and thus, obviously, $I_1 \geq 0$ iff $I_2 \geq 0$ showing the required equivalence. □

The following corollary is obvious:

Corollary 2.5. *Let $x, e \in H$ with $\|e\| = 1$ and $\delta, \Delta \in \mathbb{K}$ with $\delta \neq \Delta$. Then*

$$\operatorname{Re}\langle \Delta e - x, x - \delta e\rangle \geq 0 \tag{23}$$

iff

$$\left\|x - \frac{\delta+\Delta}{2}\cdot e\right\| \leq \frac{1}{2}|\Delta - \delta|. \tag{24}$$

Remark 2.3. If $H = \mathbb{C}$, then $\operatorname{Re}\left[(A - x)(\bar{x} - \bar{a})\right] \geq 0$ if and only if $\left|x - \frac{a+A}{2}\right| \leq \frac{1}{2}|A - a|$, where $a, x, A \in \mathbb{C}$. If $H = \mathbb{R}$, and $A > a$ then $a \leq x \leq A$ if and only if $\left|x - \frac{a+A}{2}\right| \leq \frac{1}{2}(A - a)$.

The following lemma is of interest [6].

Lemma 2.6. *Let $x, e \in H$ with $\|e\| = 1$. Then one has the following representation*

$$\|x\|^2 - |\langle x,e\rangle|^2 = \inf_{\lambda\in\mathbb{K}} \|x - \lambda e\|^2 \geq 0. \tag{25}$$

Proof. Observe, for any $\lambda \in \mathbb{K}$, that

$$\begin{aligned}\langle x-\lambda e, x-\langle x,e\rangle e\rangle &= \|x\|^2 - |\langle x,e\rangle|^2 - \lambda\left[\langle e,x\rangle - \langle e,x\rangle \|e\|^2\right]\\ &= \|x\|^2 - |\langle x,e\rangle|^2 .\end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned}\left[\|x\|^2 - |\langle x,e\rangle|^2\right]^2 &= |\langle x-\lambda e, x-\langle x,e\rangle e\rangle|^2 \le \|x-\lambda e\|^2 \|x-\langle x,e\rangle e\|^2\\ &= \|x-\lambda e\|^2\left[\|x\|^2 - |\langle x,e\rangle|^2\right],\end{aligned}$$

giving the bound

$$\|x\|^2 - |\langle x,e\rangle|^2 \le \|x-\lambda e\|^2, \qquad \lambda \in \mathbb{K}. \tag{26}$$

Taking the infimum in (26) over $\lambda \in \mathbb{K}$, we deduce

$$\|x\|^2 - |\langle x,e\rangle|^2 \le \inf_{\lambda\in\mathbb{K}} \|x-\lambda e\|^2 .$$

Since, for $\lambda_0 = \langle x,e\rangle$, we get $\|x-\lambda_0 e\|^2 = \|x\|^2 - |\langle x,e\rangle|^2$, then the representation (25) is proved. □

The following result also holds:

Theorem 2.7. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e, f \in H, \|e\| = \|f\| = 1$. If $\varphi,\gamma,\Phi,\Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x-\varphi e\rangle \ge 0,\ \operatorname{Re}\langle \Gamma f - y, y-\gamma f\rangle \ge 0 \tag{27}$$

hold, or, equivalently, the following assumptions

$$\left\|x - \frac{\varphi+\Phi}{2}e\right\| \le \frac{1}{2}|\Phi-\varphi|, \quad \left\|y - \frac{\gamma+\Gamma}{2}f\right\| \le \frac{1}{2}|\Gamma-\gamma| \tag{28}$$

are valid, then one has the inequality

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle f,y\rangle\langle e,f\rangle| \le \frac{1}{4}|\Phi-\varphi||\Gamma-\gamma| . \tag{29}$$

Proof. Using the inequality (8) and Lemma 2.6 we have

$$\begin{aligned}&|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle f,y\rangle\langle e,f\rangle|^2 \qquad (30)\\ &\le \left(\|x\|^2 - |\langle x,e\rangle|^2\right)\left(\|y\|^2 - |\langle y,f\rangle|^2\right) = \inf_{\lambda\in\mathbb{K}}\|x-\lambda e\|^2 \inf_{\eta\in\mathbb{K}}\|y-\eta f\|^2\\ &\le \left\|x - \frac{\varphi+\Phi}{2}e\right\|^2 \left\|y - \frac{\gamma+\Gamma}{2}f\right\|^2 \le \frac{1}{4}|\Phi-\varphi|^2 \frac{1}{4}|\Gamma-\gamma|^2,\end{aligned}$$

which is equivalent to the desired inequality (29). □

Corollary 2.8. *With the assumptions of Theorem 2.7 and if $e \perp f$, then we have the simpler inequality*

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle| \le \frac{1}{4}|\Phi-\varphi||\Gamma-\gamma| . \tag{31}$$

Remark 2.4. If we take $f = e$ in Theorem 2.7, then we get the result from Theorem 1.1.

3. Applications

Consider the Hilbert space $\mathbb{C}^n$ endowed with the inner product $\langle\cdot,\cdot\rangle_{\mathbf{p}}:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ defined by

$$\langle\mathbf{x},\mathbf{y}\rangle_{\mathbf{p}}:=\sum_{j=1}^{n}p_jx_j\overline{y}_j,$$

where $\mathbf{p}=(p_1,...,p_n)$ is a probability distribution, i.e. $p_j\geq 0,\ j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$ and

$$\mathbf{x}=(x_1,...,x_n),\ \mathbf{y}=(y_1,...,y_n)\in\mathbb{C}^n.$$

Assume that $\mathbf{e}=(e_1,...,e_n),\mathbf{f}=(f_1,...,f_n)\in\mathbb{C}^n$ with

$$\sum_{j=1}^{n}p_j\left|e_j\right|^2=\sum_{j=1}^{n}p_j\left|f_j\right|^2=1. \tag{32}$$

Then for any $\mathbf{x}=(x_1,...,x_n),\mathbf{y}=(y_1,...,y_n)\in\mathbb{C}^n$ we have the inequality

$$\begin{aligned}&\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2}-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j\right| \\ &\geq\left|\sum_{j=1}^{n}p_jx_j\overline{y}_j-\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_je_j\overline{y}_j\right. \\ &\qquad\left.-\sum_{j=1}^{n}p_jx_j\overline{f}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j+\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j\sum_{j=1}^{n}p_je_j\overline{f}_j\right|.\end{aligned} \tag{33}$$

Moreover, if $\mathbf{e}=(e_1,...,e_n),\mathbf{f}=(f_1,...,f_n)\in\mathbb{C}^n$ satisfy the additional condition

$$\sum_{j=1}^{n}p_je_j\overline{f}_j=0, \tag{34}$$

then from (33) we get

$$\begin{aligned}&\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2}-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j\right| \\ &\geq\left|\sum_{j=1}^{n}p_jx_j\overline{y}_j-\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_je_j\overline{y}_j-\sum_{j=1}^{n}p_jx_j\overline{f}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j\right|.\end{aligned} \tag{35}$$

If we denote by $\mathcal{C}(0,1)$ the unit circle of radius 1 in $\mathbb{C}$, namely $\mathcal{C}(0,1)=\{z\in\mathbb{C}||z|=1\}$, then for $\mathbf{e}=(e_1,...,e_n),\mathbf{f}=(f_1,...,f_n)\in\mathbb{C}^n$ with $e_j,f_j\in\mathcal{C}(0,1)$ for any $j\in\{1,...,n\}$ we have that the condition (32) holds true and therefore the inequality (33) is valid.

If we consider the nonnegative weights $w_j\geq 0,j\in\{1,...,n\}$ with $W_n=\sum_{k=1}^{n}w_k>0$ and if we assume that

$$\frac{1}{W_n}\sum_{j=1}^{n}w_j\left|e_j\right|^2=\frac{1}{W_n}\sum_{j=1}^{n}w_j\left|f_j\right|^2=1 \tag{36}$$

then by (33) we get

$$\left(\frac{1}{W_n}\sum_{j=1}^{n} w_j |x_j|^2\right)^{1/2}\left(\frac{1}{W_n}\sum_{j=1}^{n} w_j |y_j|^2\right)^{1/2} - \left|\frac{1}{W_n}\sum_{j=1}^{n} w_j x_j \bar{e}_j \frac{1}{W_n}\sum_{j=1}^{n} w_j f_j \overline{y}_j\right| \tag{37}$$
$$\geq \left|\frac{1}{W_n}\sum_{j=1}^{n} w_j x_j \overline{y}_j - \frac{1}{W_n}\sum_{j=1}^{n} w_j x_j \bar{e}_j \frac{1}{W_n}\sum_{j=1}^{n} w_j e_j \overline{y}_j\right.$$
$$\left. - \frac{1}{W_n}\sum_{j=1}^{n} w_j x_j \overline{f}_j \frac{1}{W_n}\sum_{j=1}^{n} w_j f_j \overline{y}_j + \frac{1}{W_n}\sum_{j=1}^{n} w_j x_j \bar{e}_j \frac{1}{W_n}\sum_{j=1}^{n} w_j f_j \overline{y}_j \frac{1}{W_n}\sum_{j=1}^{n} w_j e_j \overline{f}_j\right|.$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0, R)$ with $R > 0$ or $R = \infty$.

The most important power series with nonnegative coefficients that can be used to illustrate the above results are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,\ z \in \mathbb{C},\ \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,\ z \in D(0,1), \tag{38}$$
$$\ln \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{1}{n} z^n,\ z \in D(0,1),\ \ \cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n},\ z \in \mathbb{C},$$
$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1},\ z \in \mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1},\ z \in D(0,1), \tag{39}$$
$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)\, n!} z^{2n+1},\ z \in D(0,1),$$
$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1},\ z \in D(0,1),$$
$${}_2F_1(\alpha,\beta,\gamma,z) := \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0,\ z \in D(0,1),$$

where Γ is *Gamma function.*

Proposition 3.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0, R)$ with $R > 0$ or $R = \infty$. If $0 < p < R$,*

$u,v\in\mathcal{C}(0,1)$ and $x,y\in\mathbb{C}$ with $p|x|^2, p|y|^2<R$ then we have the inequality

$$\left(\frac{f\left(p|x|^2\right)}{f(p)}\right)^{1/2}\left(\frac{f\left(p|y|^2\right)}{f(p)}\right)^{1/2}-\left|\frac{f(px\overline{u})}{f(p)}\frac{f(pv\overline{y})}{f(p)}\right|$$
$$\geq\left|\frac{f(px\overline{y})}{f(p)}-\frac{f(px\overline{u})}{f(p)}\frac{f(pu\overline{y})}{f(p)}-\frac{f(px\overline{v})}{f(p)}\frac{f(pv\overline{y})}{f(p)}+\frac{f(px\overline{u})}{f(p)}\frac{f(pv\overline{y})}{f(p)}\frac{f(pu\overline{v})}{f(p)}\right|. \quad (40)$$

Proof. If $u,v\in\mathcal{C}(0,1)$ then for any $n\geq 0$ we have $u^n,v^n\in\mathcal{C}(0,1)$. Observe that for any $m\geq 1$ we have that

$$\frac{\sum_{n=0}^m a_np^n|u^n|^2}{\sum_{n=0}^m a_np^n}=\frac{\sum_{n=0}^m a_np^n|v^n|^2}{\sum_{n=0}^m a_np^n}=\frac{\sum_{n=0}^m a_np^n}{\sum_{n=0}^m a_np^n}=1.$$

Using the inequality (37) we have

$$\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}\left(\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}-\left|\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\frac{\sum_{n=0}^m a_np^n(v\overline{y})^n}{\sum_{n=0}^m a_np^n}\right| \quad (41)$$
$$\geq\left|\frac{\sum_{n=0}^m a_np^n(x\overline{y})^n}{\sum_{n=0}^m a_np^n}-\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\frac{\sum_{n=0}^m a_np^n(u\overline{y})^n}{\sum_{n=0}^m a_np^n}\right.$$
$$\left.+\left(-\frac{\sum_{n=0}^m a_np^n(x\overline{v})^n}{\sum_{n=0}^m a_np^n}+\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\frac{\sum_{n=0}^m a_np^n(u\overline{v})^n}{\sum_{n=0}^m a_np^n}\right)\frac{\sum_{n=0}^m a_np^n(v\overline{y})^n}{\sum_{n=0}^m a_np^n}\right|.$$

Since all the series whose partial sums are involved in inequality (41) are convergent, then by letting $m\to\infty$ in (41) we get the desired result (40). □

Remark 3.1. The inequality (40) can provide some particular inequalities of interest. For instance, if we take $f(z)=\exp(z)$, $z\in\mathbb{C}$, then we get

$$\exp\left[p\left(\frac{|x|^2+|y|^2}{2}-1\right)\right]-|\exp[p(x\overline{u}+v\overline{y}-2)]| \quad (42)$$
$$\geq|\exp[p(x\overline{y}-1)]-\exp[p(x\overline{u}+u\overline{y}-2)]-\exp[p(x\overline{v}+v\overline{y}-2)]$$
$$+\exp[p(x\overline{u}+v\overline{y}+u\overline{v}-3)]|$$

for any $p>0, u,v\in\mathcal{C}(0,1)$ and $x,y\in\mathbb{C}$.

If we take $u=v=1$, then from (42) we get

$$\exp\left[p\left(\frac{|x|^2+|y|^2}{2}-1\right)\right]-|\exp[p(x+\overline{y}-2)]| \quad (43)$$
$$\geq|\exp[p(x\overline{y}-1)]-\exp[p(x+\overline{y}-2)]|$$

for any $p>0$ and $x,y\in\mathbb{C}$.

Moreover, if we take in (43) $x=\overline{y}=z\in\mathbb{C}$, then we get

$$\exp\left[p\left(|z|^2-1\right)\right]-|\exp[2p(z-1)]|\geq\left|\exp\left[p\left(z^2-1\right)\right]-\exp[2p(z-1)]\right| \quad (44)$$

for any $p>0$ and $z\in\mathbb{C}$.

Consider $L^2[a,b]$ the Hilbert space of all complex valued functions f with $\int_a^b |f(t)|^2 dt < \infty$. The inner product is given by

$$\langle f,g\rangle_2 := \int_a^b f(t)\overline{g(t)}dt.$$

Assume that $h,k \in L^2[a,b]$ with

$$\int_a^b |h(t)|^2 dt = \int_a^b |k(t)|^2 dt = 1. \tag{45}$$

For instance, if $h(t) = \frac{1}{\sqrt{b-a}}\rho(t), k(t) = \frac{1}{\sqrt{b-a}}\varphi(t)$ with $\rho(t), \varphi(t) \in \mathcal{C}(0,1)$ for almost any $t \in [a,b]$, then $h,k \in L^2[a,b]$ and the condition (45) is satisfied.

Proposition 3.2. *Assume that $h,k \in L^2[a,b]$ with the property (45). Then for any $f,g \in L^2[a,b]$ we have the inequality*

$$\begin{aligned}\left(\int_a^b |f(t)|^2 dt\right)^{1/2}&\left(\int_a^b |g(t)|^2 dt\right)^{1/2} - \left|\int_a^b f(t)\overline{h(t)}dt\int_a^b k(t)\overline{g(t)}dt\right| \\ \geq &\left|\int_a^b f(t)\overline{g(t)}dt - \int_a^b f(t)\overline{h(t)}dt\int_a^b h(t)\overline{g(t)}dt\right. \\ &\left.-\int_a^b f(t)\overline{k(t)}dt\int_a^b k(t)\overline{g(t)}dt + \int_a^b f(t)\overline{h(t)}dt\int_a^b k(t)\overline{g(t)}dt\int_a^b h(t)\overline{k(t)}dt\right|.\end{aligned} \tag{46}$$

The proof follows by Theorem 2.1 for the inner product $\langle\cdot,\cdot\rangle_2$.

Remark 3.2. If $\rho(t), \varphi(t) \in \mathcal{C}(0,1)$ for almost any $t \in [a,b]$, then we have the following inequalities for integral means

$$\begin{aligned}&\left(\frac{1}{b-a}\int_a^b |f(t)|^2 dt\right)^{1/2}\left(\frac{1}{b-a}\int_a^b |g(t)|^2 dt\right)^{1/2} \\ &\quad - \left|\frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\frac{1}{b-a}\int_a^b \varphi(t)\overline{g(t)}dt\right| \\ &\geq \left|\frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt - \frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\frac{1}{b-a}\int_a^b \rho(t)\overline{g(t)}dt\right. \\ &\quad -\frac{1}{b-a}\int_a^b f(t)\overline{\varphi(t)}dt\frac{1}{b-a}\int_a^b \varphi(t)\overline{g(t)}dt \\ &\quad \left.+\frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\frac{1}{b-a}\int_a^b \varphi(t)\overline{g(t)}dt\frac{1}{b-a}\int_a^b \rho(t)\overline{\varphi(t)}dt\right|,\end{aligned} \tag{47}$$

for any $f,g \in L^2[a,b]$.

If we take $\rho(t) = 1$, $\varphi(t) = sgn\left(t - \frac{a+b}{2}\right)$, $t \in [a,b]$, then $\rho(t), \varphi(t) \in \mathcal{C}(0,1)$ for almost any $t \in [a,b]$ and since

$$\int_a^b \rho(t)\overline{\varphi(t)} = \int_a^b sgn\left(t - \frac{a+b}{2}\right)dt = 0,$$

then we get from (47)

$$\left(\frac{1}{b-a}\int_a^b |f(t)|^2\,dt\right)^{1/2}\left(\frac{1}{b-a}\int_a^b |g(t)|^2\,dt\right)^{1/2} \tag{48}$$
$$-\left|\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)\overline{g(t)}dt\right|$$
$$\geq\left|\frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b\overline{g(t)}dt\right.$$
$$\left.-\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)f(t)\,dt\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)\overline{g(t)}dt\right|$$

for any $f,g\in L^2[a,b]$.

On making use of Theorem 2.7 one can state similar discrete and integral inequalities. However the details are not presented here.

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