Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(2), 2017, Pages 278–282
ISSN: 1223-6934

Properties of integrable punctual convex functions

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Abstract. The paper refers to the *convexity at a point* as defined in [1]. We present some properties of the class of integrable functions which are convex at a fixed point. In this way, we extend some known properties.

2010 Mathematics Subject Classification. 26A51, 26B25, 26D10.
Key words and phrases. convex functions; convexity at a point.

1. Introduction

The "weak" convexity is extensively studied in the literature. In this note, we refer to a particular type of weak convexity, meaning the punctual convexity. A version of this concept was introduced and discussed in [1]. This kind of punctual convexity finds some interesting applications (see [1]). Let us recall the definition of *the convexity at a point*.

Definition 1.1. Let I be an open real interval. We say that a function $f : I \to \mathbb{R}$ is convex at the point $c \in I$, denoted by $f \in Conv_c(I)$, if

$$f(c) + f(x + y - c) \leq f(x) + f(y), \tag{1}$$

for all $x, y \in I$, such that $x < c < y$.

Note that a convex function on the interval I is convex at each point $c \in I$. The class of functions $\mathrm{Conv}_c(I)$ does not enjoy many properties. For example, the continuity (which is satisfied by any convex function on I) is not a specific property of a punctual convex function. Instead, the continuous functions of the class $\mathrm{Conv}_c(I)$ have interesting properties. In addition, the differentiable functions of the set $\mathrm{Conv}_c(I)$ can be accurately characterized (see [1]).

We focus here on Riemann integrable functions of the set $\mathrm{Conv}_c(I)$. Thus, we extend Jensen's inequality for punctual convex functions from the class of continuous functions to the class of Riemann integrable functions. We also obtain a specific integral inequality.

2. Main results

Throughout this paper, we assume that $I \subset \mathbb{R}$ is an open interval and $c \in I$. We prove that the "punctual" version of Jensen's inequality holds for locally integrable punctual convex functions. This result extends Lemma 3 in [1].

Received May 23, 2017.

Theorem 2.1. *Let $f \in Conv_c(I)$ be a Riemann locally integrable function on I and let $n \geq 2$ an integer number. For all positive real numbers $\lambda_1, \cdots, \lambda_n$, with $\sum_{i=1}^{n} \lambda_i = 1$, and for all $x_1, \cdots, x_n \in I$, such that $\sum_{i=1}^{n} \lambda_i x_i = c$, we have*

$$f(c) \leq \sum_{i=1}^{n} \lambda_i f(x_i). \tag{2}$$

Proof. Let us consider $a, b \in I$, with $a < c < b$. From the definition of the punctual convexity at c, we have

$$f(c) + f(a + t - c) \leq f(a) + f(t), \ \forall \ t \in [c, b].$$

By integrating these inequalities on the interval $[c, b]$ we obtain

$$f(c)(b - c) + \int_c^b f(a + t - c) \, dt \leq f(a)(b - c) + \int_c^b f(t) dt.$$

Similarly, we find

$$f(c)(c - a) + \int_a^c f(b + t - c) \, dt \leq f(b)(c - a) + \int_a^c f(t) dt.$$

By summing the above relations, we get

$$f(c)(b - a) + \int_c^b f(a + t - c) \, dt + \int_a^c f(b + t - c) \, dt$$

$$\leq f(a)(b - c) + f(b)(c - a) + \int_a^b f(t) dt.$$

But

$$\int_c^b f(a + t - c) \, dt + \int_a^c f(b + t - c) \, dt = \int_a^{a+b-c} f(t) dt + \int_{a+b-c}^b f(t) dt = \int_b^b f(t) dt.$$

Hence $f(c)(b - a) \leq f(a)(b - c) + f(b)(c - a)$, or

$$\frac{f(c) - f(a)}{c - a} \leq \frac{f(b) - f(c)}{b - c}. \tag{3}$$

Denote $s = \sup_{a \in I, \, a < c} \frac{f(c) - f(a)}{c - a}$ and $d = \inf_{b \in I, \, b > c} \frac{f(b) - f(c)}{b - c}$. From (3) we obtain $s, d \in \mathbb{R}$, with $s \leq d$. Let us consider $m \in [s, d]$. Therefore,

$$f(x) - f(c) \geq m(x - c), \ \forall \ x \in I. \tag{4}$$

For $n \geq 2$, assume now $x_1, \cdots, x_n \in I$ and $\lambda_1, \cdots, \lambda_n > 0$, with $\sum_{i=1}^{n} \lambda_i = 1$, such that $\sum_{i=1}^{n} \lambda_i x_i = c$. From (4), we obtain

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i f(x_i) - f(c) &= \sum_{i=1}^{n} \lambda_i \left[f(x_i) - f(c) \right] \\ &\geq \sum_{i=1}^{n} \lambda_i m(x_i - c) = m \left[\sum_{i=1}^{n} \lambda_i x_i - c \right] = 0. \end{aligned}$$

Thus, the inequality (2) is proved. □

The above theorem states that c a is a *point of convexity of* f (see, for example, [3]). Note that we can apply Lemma 3.1 of [3] for the last part of the proof. Also remark that our result shows that Theorem 1 of [1] holds for the locally integrable functions of the class $\mathrm{Conv}_c(I)$. In addition, we can highlight another specific integral inequality.

Theorem 2.2. *For a locally integrable function $f \in Conv_c(I)$, we will denote*

$$s = \sup_{x\in I,\, x<c} \frac{f(x)-f(c)}{x-c} \text{ and } d = \inf_{x\in I,\, x>c} \frac{f(x)-f(c)}{x-c}.$$

Then, for all $a, b \in I$, such that $a < c < b$, the following inequality holds

$$\int_a^b f(x)dx \geq f(c)(b-a) + \frac{d(b-c)^2 - s(a-c)^2}{2}. \tag{5}$$

In particular, if f is differentiable on I, then the inequality (5) becomes

$$\int_a^b f(x)dx \geq (b-a)\left[f(c) + f'(c)\left(\frac{a+b}{2} - c\right)\right].$$

Proof. Following the arguments of the proof of Theorem 1, we find

$$f(x) \geq f(c) + s(x-c),\ \forall\, x \in [a,c]$$

and

$$f(x) \geq f(c) + d(x-c),\ \forall\, x \in [c,b].$$

Hence

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \geq f(c)(b-a) + \frac{d(b-c)^2 - s(a-c)^2}{2}.$$

If f is differentiable on I, then from Theorem 2 of [1] and the Mean Value Theorem, we easily obtain $s = d = f'(c)$. So we get the conclusion. □

Let us characterize now the locally integrable functions having a finite number of points of convexity. We assume in the following that a locally integrable function $f : I \to \mathbb{R}$ is convex at the points $c_1 < c_2 < \cdots < c_n$ of the open inteval I. We will denote $f \in \mathrm{Conv}_{c_1,\cdots,c_n}(I)$. For each point of convexity c_i, we define $s_i = \sup_{x\in I,\, x<c_i} \frac{f(c_i)-f(x)}{c_i - x}$ and $d_i = \inf_{x\in I,\, x>c_i} \frac{f(x)-f(c_i)}{x-c_i}$. From (3) we have $s_i \leq d_i$, for $i = 1, \cdots, n$. Since $c_i < c_{i+1}$, we also obtain

$$d_i \leq \frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i} \leq s_{i+1},\ i = 1, \cdots, n-1. \tag{6}$$

We consider now the linear polynomial functions $g_i, h_i : \mathbb{R} \to \mathbb{R}$, defined by

$$g_i(x) = s_i(x - c_i) + f(c_i) \text{ and } h_i(x) = d_i(x-c_i) + f(c_i),\ i = 1, \cdots, n.$$

In order to study the behavior of the function f we highlight a set of $n-1$ "intermediate" points z_i, defined below

$$z_i = \begin{cases} c_i, & \text{if } d_i = s_{i+1} = \frac{f(c_{i+1})-f(c_i)}{c_{i+1}-c_i} \\ \frac{s_{i+1}c_{i+1} - d_i c_i - [f(c_{i+1}) - f(c_i)]}{s_{i+1} - d_i}, & \text{if } d_i < s_{i+1} \end{cases},\ i = 1, \cdots, (n-1). \tag{7}$$

If $d_i < s_{i+1}$, then, from the above definition and the inequalities (6), we have

$$z_i = c_i + \frac{c_{i+1} - c_i}{s_{i+1} - d_i}\left(s_{i+1} - \frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i}\right) \geq c_i$$

and

$$z_i = c_{i+1} - \frac{c_{i+1} - c_i}{s_{i+1} - d_i}\left(\frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i} - d_i\right) \leq c_{i+1}.$$

So we have $z_i \in [c_i, c_{i+1}]$, for $i = 1, \cdots, n-1$.

The following theorem offers a characterization of the functions with a finite numbers of points of convexity.

Theorem 2.3. *Let $f \in Conv_{c_1,\cdots,c_n}(I)$ be a function which is convex at the points $c_1 < c_2 < \cdots < c_n$ of the open interval I. By using the above notations, let us define the function $\underline{f} : I \to \mathbb{R}$,*

$$\underline{f}(x) = \begin{cases} g_1(x), & x < c_1 \\ h_i(x), & x \in [c_i, z_i],\ i = 1, \cdots, n-1 \\ g_{i+1}(x), & x \in (z_i, c_{i+1}),\ i = 1, \cdots, n-1 \\ h_n(x), & x \geq c_n \end{cases}.$$

The following statements hold:

(1) *the function $\underline{f}$ is convex on I;*

(2) $f(x) \geq \underline{f}(x),\ \forall\, x \in I$;

(3) $$\int_a^b f(x)dx \geq (c_1 - a)f(c_1) + f(c_n)(b - c_n) + \frac{d_n(b - c_n)^2 - s_1(c_1 - a)^2}{2}$$
$$+ \sum_{i=1}^{n-1}\left[(z_i - c_i)f(c_i) + (c_{i+1} - z_i)f(c_{i+1}) + \frac{d_i(z_i - c_i)^2 - s_{i+1}(c_{i+1} - z_i)^2}{2}\right],$$
for all $a, b \in I$, such that $a \leq c_1$ and $b \geq c_n$.

Proof. (1) The function $\underline{f}$ is continuous and linear on the intervals $I \cap (-\infty, c_1]$, $I \cap [c_n, \infty)$ and the intervals $[c_i, z_i]$ and $[z_i, c_{i+1}]$, for $i = 1, \cdots, n-1$. Note that $\underline{f}(c_i) = g_i(c_i) = h_i(c_i) = f(c_i)$, for $i = 1, \cdots, n$, and

$$\underline{f}(z_i) = h_i(z_i) = g_{i+1}(z_i) = \frac{d_i s_{i+1}(c_{i+1} - c_i) + s_{i+1}f(c_i) - d_i f(c_{i+1})}{s_{i+1} - d_i},$$

for all $i \in \{1, \cdots, n-1\}$ such that $d_i < s_{i+1}$. The convexity of the function $\underline{f}$ is due to the increasing sequence $s_1 \leq d_1 \leq s_2 \leq d_2 \leq \cdots \leq s_{n-1} \leq d_{n-1} \leq s_n \leq d_n$ of the slopes of its consecutive linear portions.

(2) Let $x \in I$.

If $x < c_1$, then $\dfrac{f(c_1) - f(x)}{c_1 - x} \leq s_1$. Hence $f(x) \geq s_1(x - c_1) + f(c_1) = g_1(x) = \underline{f}(x)$. Assume $i \in \{1, \cdots, n-1\}$. If $d_i = s_{i+1}$ (so $z_i = c_i$) and $x \in (c_i, c_{i+1})$, then we have $\dfrac{f(c_{i+1}) - f(x)}{c_{i+1} - x} \leq s_{i+1}$, and therefore $f(x) \geq s_{i+1}(x - c_{i+1}) + f(c_{i+1}) = g_{i+1}(x) = \underline{f}(x)$. If $d_i < s_{i+1}$, then we obtain $f(x) \geq d_i(x - c_i) + f(c_i) = h_i(x) = \overline{f}(x)$, for $x \in [c_i, z_i]$, and $f(x) \geq s_{i+1}(x - c_{i+1}) + f(c_{i+1}) = g_{i+1}(x) = \overline{f}(x)$, for $x \in (z_i, c_{i+1}]$. Finally, if $x \geq c_n$ then $f(x) \geq d_n(x - c_n) + f(c_n) = h_n(x) = \overline{f}(x)$. As a result, we get

$f \geq \underline{f}$ on I.
(3) Let $a, b \in I$ such that $a \leq c_1 < c_n \leq b$. Since $f(x) \geq \underline{f}(x),\ \forall\, x \in I$, we have

$$\int_a^b f(x)dx \geq \int_a^b \underline{f}(x)dx.$$

On the other hand,

$$\int_a^b \underline{f}(x)dx = \int_a^{c_1} g_1(x)dx + \int_{c_n}^b h_n(x)dx + \sum_{i=1}^{n-1}\left(\int_{c_i}^{z_i} h_i(x)dx + \int_{z_i}^{c_{i+1}} g_{i+1}(x)dx\right)$$
$$= (c_1 - a)f(c_1) + f(c_n)(b - c_n) + \frac{d_n(b-c_n)^2 - s_1(c_1-a)^2}{2}$$
$$+ \sum_{i=1}^{n-1}\left[(z_i - c_i)f(c_i) + (c_{i+1} - z_i)f(c_{i+1}) + \frac{d_i(z_i-c_i)^2 - s_{i+1}(c_{i+1}-z_i)^2}{2}\right].$$

□

References

[1] A. Florea, E. Păltănea, *On a class of punctual convex functions*, Mathematical Inequalities & Applications **17** (2014), no. 1, 389–399.

[2] C. P. Niculescu, L.-E. Persson, *Convex Functions and their Applications: A Contemporary Approach* (*CMS Books in Mathematics*), Springer-Verlag New York Inc., New York, 2006.

[3] C. P. Niculescu, I. Rovenţa, *Hardy-Littlewood-Pólya theorem of majorization in the framework of generalized convexity*, CARPATHIAN J. MATH. **33** (2017), no. 1, online version.

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