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A note on η-Ricci solitons in 3-dimensional trans-Sasakian manifolds

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Abstract. In this paper we study η-Ricci soliton on 3-dimensional trans-Sasakian manifold. First we obtain the existence of η-Einstein soliton on 3-dimensional trans-Sasakian manifold. Next we establish some results on 3-dimensional trans-Sasakian manifold satisfying an η-Ricci soliton when the manifold is Ricci-symmetric, has Codazzi or cyclic η-recurrent Ricci curvature tensor. Later we observe η-Ricci Soliton on 3-dimensional trans-Sasakian manifold satisfying the conditions $\tau \cdot S = 0$, $S \cdot \tau = 0$, $\mathcal{M} \cdot S = 0$ and $S \cdot \mathcal{M} = 0$. Also we construct an example of almost-η-Ricci soliton on 3-dimensional trans-Sasakian manifold.

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1. Introduction

In 1982, Hamilton introduced the concept of the Ricci flow in [7] to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation on a smooth manifold M with Riemannian metric $g(t)$ given by

$$\frac{\partial}{\partial t} g(t) = -2S.$$

Ricci solitons appear as self-similar solutions to Hamiltons's Ricci flow and often arise as limits of dilations of singularities in the Ricci flow [8]. Ricci solitons and η-Ricci solitons are natural generalizations of Einstein metrics. A Ricci soliton is defined on a Riemannian manifold (M, g) by

$$S + \frac{1}{2}\mathcal{L}_Y g = \lambda g$$

where $\mathcal{L}_Y g$ is the Lie derivative along the vector field Y, S is the Ricci tensor of (M, g) and λ is a real constant. If $Y = \nabla f$ for some function f on M, the Ricci soliton alters to a gradient Ricci soliton. A soliton becomes shrinking, steady and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

The concept of η-Ricci soliton was introduced by J.C. Cho and M. Kimura [6] in 2009. They established that in a non-flat complex space form, a real hypersurface considering an η-Ricci soliton becomes a Hopf-hypersurface. An η-Ricci soliton is defined on a Riemannian manifold (M, g) by the following equation

$$2S + \mathcal{L}_\xi g + 2\lambda g + 2\mu \eta \otimes \eta = 0, \tag{1}$$

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where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor of (M,g) and λ, μ are real constants. When λ, μ are smooth functions, η-Ricci soliton becomes almost η-Ricci soliton [13]. If $\mu = 0$, then η-Ricci soliton becomes Ricci soliton.

In [4], A. M. Blaga introduced η-Einstein soliton that is generalization of η-Ricci soliton is defined by the following equation

$$2S + \mathcal{L}_\xi g + (2\lambda - r)g + 2\mu\eta \otimes \eta = 0, \tag{2}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S, r are the Ricci tensor and scalar curvature, respectively of the metric, and λ, μ are real constants.

In the last few years, many geometers have studied various types of Ricci soliton and their generalizations in different Contact metric manfolds in [1], [2], [9] etc. In 2014, B. Y. Chen and S. Deshmukh [5] proved the characterizations of compact shrinking trivial Ricci solitons. A.M. Blada worked on η-Ricci soliton on para-kenmotsu manifold in [3]. D. G. Prakasha, B. S. Hadimani [15] studied the non-existence of certain geometric characteristics of para-Sasakian η-Ricci solitons in 2016. In [12], S. Pahan, T. Dutta, and A. Bhattacharyya worked on various types of curvature tensors on Generalized Sasakian space form admitting Ricci soliton and η-Ricci soliton. They also studied conformal Killing vector field, torse forming vector field on Generalized Sasakian space form.

In this paper we study the existence of η-Einstein soliton on 3-dimensional trans-Sasakian manifold. Next we observe some results on 3-dimensional trans-Sasakian manifold satisfying an η-Ricci soliton when the manifold becomes Ricci-symmetric, has Codazzi or cyclic η-recurrent Rici curvature tensor. Next we give an example of an almost η-Ricci soliton on 3-dimensional trans-Sasaian manifold. Later we obtain some different types of curvature tensors and their properties under certain conditions.

2. Preliminaries

The product $\bar{M} = M \times R$ has a natural almost complex structure J with the product metric G being Hermitian metric. The geometry of the almost Hermitian manifold $(\bar{M}, J, G)$ gives the geometry of the almost contact metric manifold (M, ϕ, ξ, η, g). Sixteen different types of structures on M like Sasakian manifold, Kenmotsu manifold etc are given by the almost Hermitian manifold $(\bar{M}, J, G)$. Oubina [11] introduced the idea of trans-Sasakian manifolds in 1985. Then J. C. Marrero [10] have obtained the local structure of trans-Sasakian manifolds. In general a trans-Sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type (α, β). An n $(= 2m+1)$ dimensional Riemannian manifold (M, g) is called an almost contact manifold if there exists a (1,1) tensor field ϕ, a vector field ξ and a 1-form η on M such that

$$\phi^2(X) = -X + \eta(X)\xi, \tag{3}$$

$$\eta(\xi) = 1, \eta(\phi X) = 0, \tag{4}$$

$$\phi\xi = 0, \tag{5}$$

$$\eta(X) = g(X, \xi), \tag{6}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{7}$$

$$g(X, \phi Y) + g(Y, \phi X) = 0, \tag{8}$$

for any vector fields X, Y on M.

A 3-dimensional almost contact metric manifold M is called a trans-Sasakian manifold if it satisfies the following condition

$$(\nabla_X\phi)(Y) = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\}, \tag{9}$$

for some smooth functions α, β on M and we say that the trans-Sasakian structure is of type (α,β). For 3-dimensional trans-Sasakian manifold, from (9) we have,

$$\nabla_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{10}$$

$$(\nabla_X\eta)(Y) = -\alpha g(\phi X,Y) + \beta g(\phi X,\phi Y). \tag{11}$$

In a 3-dimensional trans-Sasakian manifold, we have

$$\begin{aligned}
R(X,Y)Z = &\left[\frac{r}{2} - 2(\alpha^2 - \beta^2 - \xi\beta)\right][g(Y,Z)X - g(X,Z)Y]\\
&- \left[\frac{r}{2} - 3(\alpha^2-\beta^2) + \xi\beta\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi\\
&+ [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)][\phi\ grad\ \alpha -\ grad\ \beta]\\
&- \left[\frac{r}{2} - 3(\alpha^2-\beta^2) + \xi\beta\right]\eta(Z)[\eta(Y)X - \eta(X)Y]\\
&- [Z\beta + (\phi Z)\alpha]\eta(Z)[\eta(Y)X - \eta(X)Y]\\
&- [X\beta + (\phi X)\alpha][g(Y,Z)\xi - \eta(Z)Y] - [Y\beta + (\phi Y)\alpha][g(X,Z)\xi - \eta(Z)X],
\end{aligned}$$

$$\begin{aligned}
S(X,Y) = &\left[\frac{r}{2} - (\alpha^2-\beta^2-\xi\beta)\right]g(X,Y) - \left[\frac{r}{2} - 3(\alpha^2-\beta^2) + \xi\beta\right]\eta(X)\eta(Y)\\
&- [Y\beta + (\phi Y)\alpha]\eta(X) - [X\beta + (\phi X)\alpha]\eta(Y).
\end{aligned}$$

When α and β are constants the above equations reduce to,

$$R(\xi,X)\xi = (\alpha^2-\beta^2)(\eta(X)\xi - X), \tag{12}$$

$$S(X,\xi) = 2(\alpha^2-\beta^2)\eta(X), \tag{13}$$

$$R(\xi,X)Y = (\alpha^2-\beta^2)(g(X,Y)\xi - \eta(Y)X). \tag{14}$$

$$R(X,Y)\xi = (\alpha^2-\beta^2)(\eta(Y)X - \eta(X)Y). \tag{15}$$

Definition 2.1. A trans-Sasakian manifold M^3 is said to be η-Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions.

3. η-Einstein solitons on trans-Sasakian manifolds

To study the existence conditions of η-Einstein solitons on 3-dimensional trans-Sasakian manifolds, first we consider a symmetric $(0,2)$-tensor field L which is parallel with respect to the Levi-Civita connection ($\nabla L = 0$). Then it follows that

$$L(R(X,Y)Z,W) + L(Z,R(X,Y)W) = 0, \tag{16}$$

for an arbitrary vector field $W,\ X,\ Y,\ Z$ on M. Put $X = Z = W = \xi$ we get

$$L(R(X,Y)\xi,\xi) = 0, \tag{17}$$

for any $X,\ Y\ \in\ \chi(M)$ By using the equation (15)

$$L(Y,\xi) = g(Y,\xi)L(\xi,\xi), \tag{18}$$

for any $Y \in \chi(M)$. Differentiating the equation (18) covariantly with respect to the vector field $X \in \chi(M)$ we have

$$L(\nabla_X Y,\xi)+L(Y,\nabla_X\xi)=g(\nabla_X Y,\xi)L(\xi,\xi)+g(Y,\nabla_X\xi)L(\xi,\xi), \tag{19}$$

Using the equation (10) we have

$$\beta L(X,Y)-\alpha L(\phi X,Y)=-\alpha g(\phi X,Y)L(\xi,\xi)+\beta L(\xi,\xi)g(X,Y). \tag{20}$$

Interchanging X by Y we have

$$\beta L(X,Y)-\alpha L(X,\phi Y)=-\alpha g(X,\phi Y)L(\xi,\xi)+\beta L(\xi,\xi)g(X,Y). \tag{21}$$

Then adding the above two equations we get

$$\beta L(X,Y)-\frac{\alpha}{2}[L(\phi X,Y)+L(X,\phi Y)]=\beta L(\xi,\xi)g(X,Y). \tag{22}$$

We see that $\beta L(X,Y)-\frac{\alpha}{2}[L(\phi X,Y)+L(X,\phi Y)]$ is a symmetric tensor of type $(0,2)$. Let $\beta L(X,Y)-\frac{\alpha}{2}[L(\phi X,Y)+L(X,\phi Y)]=\mathcal{L}_\xi g(X,Y)+2S(X,Y)+2\mu\eta(X)\eta(Y)-rg(X,Y)$.
Then we compute

$$\beta L(\xi,\xi)g(X,Y)=\mathcal{L}_\xi g(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)-rg(X,Y).$$

As L is parallel so, $L(\xi,\xi)$ is constant. Hence, we can write $L(\xi,\xi)=-\frac{2}{\beta}\lambda$ where β is constant and $\beta\neq 0$.
Therefore $\mathcal{L}_\xi g(X,Y)+2S(X,Y)+2\mu\eta(X)\eta(Y)-rg(X,Y)=-2\lambda g(X,Y)$ and so (g,ξ,μ) becomes an η-Einstein soliton. Hence we have the following theorem.

Theorem 3.1. *Let $(M,g,\phi,\eta,\xi,\alpha,\beta)$ be a 3-dimensional trans-Sasakian manifold with α, β constants $(\beta\neq 0)$. If the symmetric $(0,2)$ tensor field L satisfying the condition $\beta L(X,Y)-\frac{\alpha}{2}[L(\phi X,Y)+L(X,\phi Y)]=\mathcal{L}_\xi g(X,Y)+2S(X,Y)+2\mu\eta(X)\eta(Y)-rg(X,Y)$ is parallel with respect to the Levi-Civita connection associated to g. Then (g,ξ,μ) becomes an η-Einstein soliton.*

Corollary 3.2. *Let $(M,g,\phi,\eta,\xi,\alpha,\beta)$ be a 3-dimensional trans-Sasakian manifold with α, β constants $(\beta\neq 0)$. If the symmetric $(0,2)$ tensor field L satisfying the condition $\beta L(X,Y)-\frac{\alpha}{2}[L(\phi X,Y)+L(X,\phi Y)]=\mathcal{L}_\xi g(X,Y)+2S(X,Y)+2\mu\eta(X)\eta(Y)$ is parallel with respect to the Levi-Civita connection associated to g. Then (g,ξ,μ) becomes an η-Ricci soliton.*

Next we obtain some results on 3-dimensional trans-Sasakian manifold satisfying an η-Ricci soliton when the manifold is Ricci-symmetric, has Codazzi or cyclic η-recurrent Ricci curvature tensor.

Theorem 3.3. *Let $(M,g,\phi,\eta,\xi,\alpha,\beta)$ be a 3-dimensional trans-Sasakian manifold with α, β constants $(\beta\neq 0)$ satisfying η-Ricci soliton.*
(i) If the manifold (M,g) is Ricci symmetric (i.e. $\nabla S=0$), then $\mu=\beta$.
(ii) If the Ricci tensor is η-recurrent (i.e. $\nabla S=\eta\otimes S$), then $\mu=2\beta-\frac{\alpha^2}{\beta}$.
(iii) If the Ricci tensor is Codazzi (i.e. $(\nabla_X S)(Y,Z)=(\nabla_Y S)(X,Z)$, for all vector fields X,Y,Z), then $\mu=\beta$.

Proof. From the equation (1) we get

$$2S(X,Y)=-g(\nabla_X\xi,Y)-g(X,\nabla_Y\xi)-2\lambda g(X,Y)-2\mu\eta(X)\eta(Y). \tag{23}$$

By using the equation (10) we get

$$S(X,Y) = -(\beta+\lambda)g(X,Y) + (\beta-\mu)\eta(X)\eta(Y) \tag{24}$$

and

$$S(X,\xi) = -(\lambda+\mu)\eta(X). \tag{25}$$

Also from (25) we have

$$\lambda+\mu = 2(\beta^2-\alpha^2). \tag{26}$$

The Ricci operator Q is defined by $g(QX,Y) = S(X,Y)$. Then we get

$$QX = (\mu-\beta+2(\alpha^2-\beta^2))X + (\beta-\mu)\eta(X)\xi. \tag{27}$$

(i) We consider that the manifold (M, g) is Ricci symmetric i.e.

$$\nabla S = 0. \tag{28}$$

Now we have

$$\nabla_X S(Y,Z) = XS(Y,Z) - S(\nabla_X Y,Z) - S(\nabla_X Z,Y).$$

Using the equations (24) and (28), we obtain

$$(\beta-\mu)[-\alpha(g(\phi X,Y)+g(\phi X,Z))+\beta(g(X,Y)\eta(Z)-g(X,Z)\eta(Y))-2\beta\eta(X)\eta(Y)\eta(Z)] = 0.$$

Putting $Y = Z = \xi$, the above equation becomes $\mu = \beta$.
(ii) We assume that the manifold (M, g) is η-recurrent i.e.

$$\nabla S = \eta \otimes S. \tag{29}$$

Now we have

$$\nabla_X S(Y,Z) = \eta(X)S(Y,Z), \tag{30}$$

for all vector fields X,Y,Z. Using the equations (24) and (30), we obtain $\mu = 2\beta - \frac{\alpha^2}{\beta}$.
(iii) If the Ricci tensor is Codazzi i.e. $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$, for all vector fields X,Y,Z, then we have

$$XS(Y,Z) - S(\nabla_X Y,Z) - S(\nabla_X Z,Y) = YS(X,Z) - S(\nabla_Y X,Z) - S(\nabla_Y Z,X).$$

Using the equation (24) and then putting $Y = Z = \xi$, we observe $\mu = \beta$. □

4. Example of almost η-Ricci solitons on 3-dimensional trans-Sasakian manifolds

We consider the three dimensional manifold $M = \{(x,y,z) \in R^3 : x \neq 0\}$ where (x,y,z) are the standard coordinates in R^3. The vector fields

$$e_1 = \frac{\partial}{\partial x}, e_2 = x^2\frac{\partial}{\partial y}, e_3 = x^2\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_1)$ for any $Z \in \chi(M^3)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = 0, \phi(e_2) = -e_3, \phi(e_3) = e_2$. Then using the linearity property of ϕ and g we have

$$\eta(e_1) = 1,\ \phi^2(Z) = -Z + \eta(Z)e_1,\ g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_1 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M. Now, after some calculation we have,

$$[e_1, e_3] = \frac{2}{x}e_3, [e_2, e_3] = 0, [e_1, e_2] = \frac{2}{x}e_2.$$

The Riemannian connection ∇ of the metric is given by the Koszul's formula which is

$$2g(\nabla_X Y, Z) = Xg(Y, Z)+Y g(Z, X)-Zg(X, Y)-g(X, [Y, Z])-g(Y, [X, Z])+g(Z, [X, Y]).$$

By Koszul's formula we get,

$$\nabla_{e_1}e_1 = 0, \nabla_{e_2}e_1 = -\frac{2}{x}e_2, \nabla_{e_3}e_1 = -\frac{2}{x}e_3, \nabla_{e_1}e_2 = 0, \nabla_{e_2}e_2 = \frac{2}{x}e_1,$$

$$\nabla_{e_3}e_2 = 0, \nabla_{e_1}e_3 = 0, \nabla_{e_2}e_3 = 0, \nabla_{e_3}e_3 = \frac{2}{x}e_1.$$

From the above it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold of type $(0, -2)$.
Here

$$R(e_1, e_2)e_2 = -\frac{6}{x^2}e_1, R(e_2, e_3)e_2 = \frac{4}{x^2}e_3, R(e_1, e_3)e_3 = -\frac{6}{x^2}e_1, R(e_2, e_3)e_3 = -\frac{4}{x^2}e_2,$$

$$R(e_1, e_3)e_1 = \frac{6}{x^2}e_3, R(e_1, e_2)e_1 = \frac{6}{x^2}e_2.$$

So, we have

$$S(e_1, e_1) = -\frac{12}{x^2}, S(e_2, e_2) = S(e_3, e_3) = -\frac{10}{x^2}.$$

From the equation (1) we get $\lambda = \frac{2}{x}$ and $\mu = \frac{12-2x}{x^2}$. Therefore, (g, ξ, λ, μ) is an almost η-Ricci soliton on $M^3(\phi, \xi, \eta, g)$.

5. η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $\tau(\xi, X) \cdot S = 0$

M.M. Tripathi and P. Gupta introduced a new curvature tensor named as the τ-curvature tensor of semi-Riemannian manifold M is defined by [16]

$$\begin{aligned}\tau(X, Y)Z =& a_0R(X, Y)Z + a_1S(Y, Z)X + a_2S(X, Z)Y + a_3S(X, Y)Z + a_4g(Y, Z)QX \\ &+ a_5g(X, Z)QY + a_6g(X, Y)QZ + a_7r(g(Y, Z)X - g(X, Z)Y), \qquad (31)\end{aligned}$$

where a_i, $i = 1, 2, ..., 7$ are some smooth functions on M and R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator of type $(1, 1)$ and the scalar curvature respectively.
(i) τ-curvature tensor becomes Ricci curvature tensor R if $a_0 = 1,$, $a_i = 0$ for $i = 1, 2, ..., 7$.
(ii) τ-curvature tensor becomes concircular curvature tensor K if $a_0 = 1$, $a_7 = -\frac{1}{6}$ $a_i = 0$ for $i = 1, 2, ..., 6$.

First we suppose that 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfy the condition

$$\tau(\xi, X) \cdot S = 0.$$

Then we have

$$S(\tau(\xi, X)Y, Z) + S(Y, \tau(\xi, X)Z) = 0$$

for any $X, Y, Z \in \chi(M)$.

Using the equations (14), (24), (25), (26) we get

$$\begin{aligned}
&g(X,Y)\eta(Z)[a_0(\beta^2-\alpha^2)(\lambda+\mu)+a_1(\beta+\lambda)(\lambda+\mu)+2a_4(\beta^2-\alpha^2)(\lambda+\mu)-a_7r(\lambda+\mu)\\
&-a_0(\beta^2-\alpha^2)(\lambda+\beta)+a_2(\beta+\lambda)(\lambda+\mu)-a_5(\mu-\beta-2(\beta^2-\alpha^2))(\lambda+\beta)+a_7r(\lambda+\beta)]\\
&+g(X,Z)\eta(Y)[a_0(\beta^2-\alpha^2)(\lambda+\mu)+a_1(\beta+\lambda)(\lambda+\mu)+2a_4(\beta^2-\alpha^2)(\lambda+\mu)\\
&-a_7r(\lambda+\mu)-a_0(\beta^2-\alpha^2)(\lambda+\beta)+a_2(\beta+\lambda)(\lambda+\mu)-a_5(\mu-\beta-2(\beta^2-\alpha^2))(\lambda+\beta)\\
&+a_7r(\lambda+\beta)]+g(Y,Z)\eta(X)[2a_3(\beta+\lambda)(\lambda+\mu)-a_6(\mu-\beta-2(\beta^2-\alpha^2))(\lambda+\beta)]\\
&+2\eta(X)\eta(Y)\eta(Z)[(a_0-a_7r)(\beta-\mu)-(a_1+a_3)(\lambda+\mu)(\beta-\mu)\\
&+(a_5+a_6)](\beta-\mu)(2(\alpha^2-\beta^2)-\lambda-\beta)]=0. \qquad (32)
\end{aligned}$$

Put $Z = \xi$ we have

$$\begin{aligned}
&g(X,Y)[a_0(\beta^2-\alpha^2)(\lambda+\mu)+a_1(\beta+\lambda)(\lambda+\mu)+2a_4(\beta^2-\alpha^2)(\lambda+\mu)-a_7r(\lambda+\mu)\\
&-a_0(\beta^2-\alpha^2)(\lambda+\beta)+a_2(\beta+\lambda)(\lambda+\mu)-a_5(\mu-\beta-2(\beta^2-\alpha^2))(\lambda+\beta)+a_7r(\lambda+\beta)]\\
&+g(X,\xi)\eta(Y)[a_0(\beta^2-\alpha^2)(\lambda+\mu)+a_1(\beta+\lambda)(\lambda+\mu)+2a_4(\beta^2-\alpha^2)(\lambda+\mu)\\
&-a_7r(\lambda+\mu)-a_0(\beta^2-\alpha^2)(\lambda+\beta)+a_2(\beta+\lambda)(\lambda+\mu)-a_5(\mu-\beta-2(\beta^2-\alpha^2))(\lambda+\beta)\\
&+a_7r(\lambda+\beta)]+g(Y,\xi)\eta(X)[2a_3(\beta+\lambda)(\lambda+\mu)-a_6(\mu-\beta-2(\beta^2-\alpha^2))(\lambda+\beta)]\\
&+2\eta(X)\eta(Y)[(a_0-a_7r)(\beta-\mu)-(a_1+a_3)(\lambda+\mu)(\beta-\mu)\\
&+(a_5+a_6)](\beta-\mu)(2(\alpha^2-\beta^2)-\lambda-\beta)]=0.
\end{aligned}$$

Setting $X = \phi X$ and $Y = \phi Y$ in the above equation we get

$$\begin{aligned}
&g(\phi X,\phi Y)[a_0(\beta^2-\alpha^2)(\lambda+\mu)+a_1(\beta+\lambda)(\lambda+\mu)+2a_4(\beta^2-\alpha^2)(\lambda+\mu)-a_7r(\lambda+\mu)\\
&-a_0(\beta^2-\alpha^2)(\lambda+\beta)+a_2(\beta+\lambda)(\lambda+\mu)-a_5(\mu-\beta-2(\beta^2-\alpha^2))(\lambda+\beta)\\
&+a_7r(\lambda+\beta)]=0. \qquad (33)
\end{aligned}$$

If $a_1 + a_2 = -2k$, $a_4 = k$ and $a_5 = k$ with $k(\neq 0) \in \mathbb{R}$ then we get

$$(\mu-\beta)[2k(\mu-\beta)+a_0(\lambda+\mu)-2ra_7]=0,$$

Again using the equation (26) we have

$$\mu=\beta,\ \lambda=2(\beta^2-\alpha^2)-\beta$$

or

$$\mu=\beta+\frac{a_7r-2a_0(\beta^2-\alpha^2)}{k},\ \lambda=\beta+\frac{a_7r+(a_0+2k)(\alpha^2-\beta^2)}{k}$$

Also we can easily see that M is an Einstein manifold. So we have the following theorem.

Theorem 5.1. *If a 3-dimensional trans-Sasakian manifold* $(M, g, \phi, \eta, \xi, \alpha, \beta)$ *with* α, β *constants admitting an* η*-Ricci soliton satisfies the condition* $\tau(\xi, X) \cdot S = 0$ *and* $a_1 + a_2 = -2k$, $a_4 = k$ *and* $a_5 = k$ *with* $k(\neq 0) \in \mathbb{R}$ *then* $\mu = \beta$, $\lambda = 2(\beta^2 - \alpha^2) - \beta$ *or* $\mu = \beta + \frac{a_7r-2a_0(\beta^2-\alpha^2)}{k}$, $\lambda = \beta + \frac{a_7r+(a_0+2k)(\alpha^2-\beta^2)}{k}$ *and* M *is an Einstein manifold.*

Corollary 5.2. *A 3-dimensional trans-Sasakian manifold with α, β constants satisfies the condition $\tau(\xi, X) \cdot S = 0$, there is no Ricci soliton with the potential vector field ξ.*

Theorem 5.3. *If a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $R(\xi, X) \cdot S = 0$ i.e. $a_0 = 1$, $a_i = 0$ for $i = 1, 2, ..., 7$ then $\mu = \beta$, $\lambda = 2(\beta^2 - \alpha^2) - \beta$ and M is an Einstein manifold.*

Theorem 5.4. *If a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $K(\xi, X) \cdot S = 0$ i.e. $a_0 = 1$,, $a_7 = -\frac{1}{6}$ $a_i = 0$ for $i = 1, 2, ..., 6$ then $\mu = \beta$, $\lambda = 2(\beta^2 - \alpha^2) - \beta$ or $r = 6(\alpha^2 - \beta^2)$ and M is an Einstein manifold.*

6. η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $S(\xi, X) \cdot \tau = 0$

We consider 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfying the condition

$$S(\xi, X) \cdot \tau = 0.$$

So we have

$$S(X, \tau(Y, Z)W)\xi - S(\xi, \tau(Y, Z)W)X + S(X, Y)\tau(\xi, Z)W - S(\xi, Y)\tau(X, Z)W$$
$$+S(X, Z)\tau(Y, \xi)W - S(\xi, Z)\tau(Y, X)W + S(X, W)\tau(Y, Z)\xi - S(\xi, W)\tau(Y, Z)X = 0.$$

Taking inner product with ξ then the above equation becomes

$$S(X, \tau(Y, Z)W) - S(\xi, \tau(Y, Z)W)\eta(X) + S(X, Y)\eta(\tau(\xi, Z)W)$$
$$-S(\xi, Y)\eta(\tau(X, Z)W) + S(X, Z)\eta(\tau(Y, \xi)W) - S(\xi, Z)\eta(\tau(Y, X)W)$$
$$+S(X, W)\eta(\tau(Y, Z)\xi) - S(\xi, W)\eta(\tau(Y, Z)X) = 0. \tag{34}$$

Put $W = \xi$ and using the equations (12), (14), (24), (25), (26) we get

$$g(Y, Z)\eta(X)[-2(\alpha^2 - \beta^2)a_3(\beta + \lambda) + 4a_6(\alpha^2 - \beta^2)^2] + \eta(X)\eta(Y)\eta(Z)[a_3(\beta - \mu)^2]$$
$$+ g(X, Y)\eta(Z)[-(\beta + \lambda)\{a_0(\alpha^2 - \beta^2) + a_7 r + 2a_1(\alpha^2 - \beta^2) + a_4(\mu - \beta + 2(\alpha^2 - \beta^2))\}]$$
$$+ g(X, Z)\eta(Y)[-(\beta + \lambda)\{-a_0(\alpha^2 - \beta^2) - a_7 r + 2a_2(\alpha^2 - \beta^2) + a_5(\mu - \beta + 2(\alpha^2 - \beta^2))\}]$$
$$+ \eta(X)\eta(Y)\eta(Z)[-(\beta + \lambda)(\beta - \mu)\sum_{i=3}^{5} a_i + 2(\alpha^2 - \beta^2)(\beta - \mu)(a_1 + a_2 + a_4 + a_5)]$$
$$+ (\lambda + \mu)g(Y, Z)\eta(X)[-a_3(\beta + \lambda) + 2a_6(\alpha^2 - \beta^2)] + (\lambda + \mu)\eta(X)\eta(Y)\eta(Z)[-(\lambda + \mu)\sum_{i=1}^{2} a_i$$
$$+ (\beta - \mu)\sum_{i=3}^{5} a_i + \{\mu - \beta + 2(\alpha^2 - \beta^2)\}\sum_{i=4}^{5} a_i] - 2(\alpha^2 - \beta^2)\eta(X)\eta(Y)\eta(Z)(\beta - \mu)\sum_{i=1}^{3} a_i$$

$$+ 2(\alpha^2 - \beta^2)\sum_{i=1}^{6} a_i[-(\beta+\lambda)g(X,Y)\eta(Z) + (\beta-\mu)]\eta(X)\eta(Y)\eta(Z)$$
$$+ 2(\alpha^2 - \beta^2)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y) - g(X,Y)\eta(Z))(-a_3(\beta+\lambda) + 2a_6(\alpha^2-\beta^2))$$
$$- 2(\alpha^2 - \beta^2)\eta(X)\eta(Y)\eta(Z)[-(\lambda+\mu)\sum_{i=1}^{2} a_i + (\beta-\mu)\sum_{i=3}^{5} a_i + \{\mu - \beta + 2(\alpha^2-\beta^2)\}\sum_{i=4}^{5} a_i]$$
$$- 2(\alpha^2 - \beta^2)g(X,Z)\eta(Y)[a_0(\alpha^2-\beta^2) + a_7 r + 2a_4(\alpha^2-\beta^2) - a_1(\beta+\lambda)]$$
$$- 2(\alpha^2 - \beta^2)g(X,Y)\eta(Z)[-a_0(\alpha^2-\beta^2) - a_7 r + 2a_5(\alpha^2-\beta^2) - a_2(\beta+\lambda)]$$
$$- 2(\alpha^2 - \beta^2)g(Y,Z)\eta(X)[2a_6(\alpha^2-\beta^2)^2 - a_3(\beta+\lambda)] = 0.$$

Putting $Z = \xi$ and setting $X = \phi X$ and $Y = \phi Y$ in the above equation we get

$$g(\phi X, \phi Y)[-(\beta+\lambda)\{a_0(\alpha^2-\beta^2) + a_7 r + 2a_1(\alpha^2-\beta^2) + a_4(\mu - \beta + 2(\alpha^2-\beta^2))\}]$$
$$- (\beta+\lambda)2(\alpha^2-\beta^2)\sum_{i=1}^{6} a_i + 2a_3(\beta+\lambda)(\alpha^2-\beta^2) - 4a_6(\alpha^2-\beta^2)^2 + 2a_0(\alpha^2-\beta^2)^2$$
$$+ 2a_7 r(\alpha^2-\beta^2) + 2(\alpha^2-\beta^2)a_2(\beta+\lambda) - 4a_5(\alpha^2-\beta^2)^2] = 0. \quad (35)$$

i.e.

$$g(\phi X, \phi Y)[-(\beta+\lambda)\{a_0(\alpha^2-\beta^2) + a_7 r + 2(a_0 - a_2 - a_3)(\alpha^2-\beta^2) - a_4(\beta+\lambda)$$
$$+ 2(\alpha^2-\beta^2)\sum_{i=1}^{6} a_i\} + 2(\alpha^2-\beta^2)\{(\alpha^2-\beta^2)(a_0 - 2a_5 - 2a_6) + a_7 r\}] = 0.$$

If $r = \frac{(\alpha^2-\beta^2)}{a_7}[2a_5 + 2a_6 - a_0]$ with $a_7 \neq 0$ then we obtain $\lambda = -\beta$, $\mu = \beta - 2(\alpha^2-\beta^2)$. So we have the following theorem.

Theorem 6.1. *If a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $S(\xi, X)\cdot\tau = 0$ and $r = \frac{(\alpha^2-\beta^2)}{a_7}[2a_5 + 2a_6 - a_0]$ with $a_7 \neq 0$ then $\lambda = -\beta$, $\mu = \beta - 2(\alpha^2-\beta^2)$.*

Corollary 6.2. *A 3-dimensional trans-Sasakian manifold with α, β constants satisfies the condition $S(\xi, X)\cdot\tau = 0$, there is no Ricci soliton with the potential vector field ξ.*

Theorem 6.3. *If a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $S(\xi, X)\cdot R = 0$ i.e. $a_0 = 1$, $a_i = 0$ for $i = 1, 2, ..., 7$ then $\mu = \beta + 4(\beta^2 - \alpha^2)$, $\lambda = -[2(\beta^2-\alpha^2) + \beta]$.*

Theorem 6.4. *If a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $S(\xi, X)\cdot K = 0$ i.e. $a_0 = 1$, $a_7 = -\frac{1}{6}$ $a_i = 0$ for $i = 1, 2, ..., 6$ then $\mu = \beta + 4(\beta^2-\alpha^2)$, $\lambda = -[2(\beta^2-\alpha^2) + \beta]$, or $r = 6(\alpha^2-\beta^2)$.*

7. η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $\mathcal{M}(\xi, X)\cdot S = 0$

Definition 7.1. Let M be 3-dimensional trans-Sasakian manifold. The $\mathcal{M}$-projective curvature tensor of M is defined by [14]

$$\mathcal{M}(X,Y)Z = R(X,Y)Z - \frac{1}{4}(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY). \quad (36)$$

We assume 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfying the condition

$$\mathcal{M}(\xi, X) \cdot S = 0.$$

Then we have

$$S(\mathcal{M}(\xi, X)Y, Z) + S(Y, \mathcal{M}(\xi, X)Z) = 0$$

for any $X, Y, Z \in \chi(M)$.

Using the equations (14), (24), (25), (26), (36) we get

$$\begin{aligned}&[-\frac{1}{2}(\alpha^2-\beta^2)(\lambda+\mu) - \frac{1}{4}(\beta+\lambda) + (\alpha^2-\beta^2)(\lambda+\beta)\\ &+\frac{1}{4}(\lambda+\beta)(\mu+\lambda) - \frac{(\lambda+\beta)}{4}(\mu-\beta+2(\alpha^2-\beta^2))](g(X,Y)\eta(Z)+g(X,Z)\eta(Y))\\ &+[-(\alpha^2-\beta^2)(\beta-\mu)+\frac{(\beta-\mu)}{4}(2(\alpha^2-\beta^2)-(\beta+\lambda))]\eta(X)\eta(Y)\eta(Z) = 0.\end{aligned}$$

Put $Z = \xi$ in the above equation we get

$$\begin{aligned}&[-\frac{1}{2}(\alpha^2-\beta^2)(\lambda+\mu) - \frac{1}{4}(\beta+\lambda) + (\alpha^2-\beta^2)(\lambda+\beta)\\ &+\frac{1}{4}(\lambda+\beta)(\mu+\lambda) - \frac{(\lambda+\beta)}{4}(\mu-\beta+2(\alpha^2-\beta^2))](g(X,Y)+g(X,\xi)\eta(Y))\\ &+[-(\alpha^2-\beta^2)(\beta-\mu)+\frac{(\beta-\mu)}{4}(2(\alpha^2-\beta^2)-(\beta+\lambda))]\eta(X)\eta(Y) = 0.\end{aligned}$$

Setting $X = \phi X$ and $Y = \phi Y$ in the above equation we get

$$\begin{aligned}&[-\frac{1}{2}(\alpha^2-\beta^2)(\lambda+\mu) - \frac{1}{4}(\beta+\lambda) + (\alpha^2-\beta^2)(\lambda+\beta)\\ &+\frac{1}{4}(\lambda+\beta)(\mu+\lambda) - \frac{(\lambda+\beta)}{4}(\mu-\beta+2(\alpha^2-\beta^2))]g(\phi X,\phi Y) = 0.\end{aligned} \tag{37}$$

Again using the equation (26) we have

$$\mu = \beta,\ \lambda = 2(\beta^2 - \alpha^2) - \beta.$$

So we have the following theorem.

Theorem 7.1. *If a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $\mathcal{M}(\xi, X) \cdot S = 0$ then $\mu = \beta,\ \lambda = 2(\beta^2 - \alpha^2) - \beta$.*

Corollary 7.2. *A 3-dimensional trans-Sasakian manifold with α, β constants satisfies the condition $\mathcal{M}(\xi, X) \cdot S = 0$, there is no Ricci soliton with the potential vector field ξ.*

8. η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $S(\xi, X) \cdot \mathcal{M} = 0$

Suppose that 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfy the condition

$$S(\xi, X) \cdot \mathcal{M} = 0.$$

So we have

$$\begin{aligned}&S(X,\mathcal{M}(Y,Z)V)\xi - S(\xi,\mathcal{M}(Y,Z)V)X + S(X,Y)\mathcal{M}(\xi,Z)V - S(\xi,Y)\mathcal{M}(X,Z)V\\ &+S(X,Z)\mathcal{M}(Y,\xi)V - S(\xi,Z)\mathcal{M}(Y,X)V + S(X,V)\mathcal{M}(Y,Z)\xi - S(\xi,V)\mathcal{M}(Y,Z)X = 0.\end{aligned}$$

Taking inner product with ξ then the above equation becomes

$$\begin{aligned} &S(X,\mathcal{M}(Y,Z)V)-S(\xi,\mathcal{M}(Y,Z)V)\eta(X)+S(X,Y)\eta(\mathcal{M}(\xi,Z)V)\\ &-S(\xi,Y)\eta(\mathcal{M}(X,Z)V)+S(X,Z)\eta(\mathcal{M}(Y,\xi)V)-S(\xi,Z)\eta(\mathcal{M}(Y,X)V)\\ &+S(X,V)\eta(\mathcal{M}(Y,Z)\xi)-S(\xi,V)\eta(\mathcal{M}(Y,Z)X)=0. \end{aligned} \tag{38}$$

Put $V=\xi$ and using the equations (10), (14), (24), (25), (26), (36) the equation (38) becomes

$$\begin{aligned} &[(2\lambda+\mu+\beta)(\alpha^2-\beta^2)+\frac{(2\lambda+\mu+\beta)^2}{4}+(2\lambda+\mu+\beta)\{(\alpha^2-\beta^2)\\ &\qquad+\frac{(2\lambda+\mu+\beta)}{4}\}](g(X,Z)\eta(Y)-g(X,Y)\eta(Z))=0. \end{aligned} \tag{39}$$

Using the equation (27) we have

$$\mu=\beta,\ \lambda=2(\beta^2-\alpha^2)-\beta$$

or

$$\lambda=2(\alpha^2-\beta^2)-\beta,\ \mu=-4(\alpha^2-\beta^2)+\beta.$$

So we have the following theorem.

Theorem 8.1. *If Let a 3-dimensional trans-Sasakian manifold $(M,g,\phi,\eta,\xi,\alpha,\beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $S(\xi,X)\cdot\mathcal{M}=0$ then*

$$\mu=\beta,\ \lambda=2(\beta^2-\alpha^2)-\beta$$

or $\lambda=2(\alpha^2-\beta^2)-\beta,\ \mu=-4(\alpha^2-\beta^2)+\beta$.

Corollary 8.2. *A 3-dimensional trans-Sasakian manifold with α, β constants satisfies the condition $S(\xi,X)\cdot\mathcal{M}=0$, there is no Ricci soliton with the potential vector field ξ.*

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