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Instantaneous shrinking of compact support of solutions of semi-linear parabolic equations with singular absorption

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Abstract. We prove an existence of weak solutions of semi-linear parabolic equations with a strong singular absorption term. Moreover, we study the qualitative behavior of solutions such as the quenching phenomenon, the finite speed of propagation and the instantaneous shrinking of compact support.

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1. Introduction

In this paper, we are interested in nonnegative solutions of the following equation:

$$\begin{cases} \partial_t u - \Delta u + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\beta \in (0,1)$, and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$, i.e:

$$\chi_{\{u>0\}} = \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{if } u \leq 0. \end{cases}$$

Note that the absorption term $u^{-\beta}\chi_{\{u>0\}}$ becomes singular when u is near to 0, and we impose tactically $u^{-\beta}\chi_{\{u>0\}} = 0$ whenever $u = 0$. Through this paper, $f : [0,\infty) \longrightarrow \mathbb{R}$ is a nondecreasing continuous function such that $f(0) = 0$.

Problem (1) can be considered as a limit of mathematical models describing enzymatic kinetics (see [1]), or the Langmuir-Hinshelwood model of the heterogeneous chemical catalyst (see, e.g. [20] p. 68, [11], [18]). This problem has been studied by the authors in [18], [14], [15], [17], [10], [7], [21], and references therein. These authors have considered the existence and uniqueness, and the qualitative behavior of these solutions. For example, when $f = 0$, D. Phillips [18] proved the existence of solution for the Cauchy problem associating to equation (1). A partial uniqueness of

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solution of equation (1) was proved by J. Davila and M. Montenegro, [10] for a class of solutions with initial data u_0 satisfying

$$u_0(x) \geq Cdist(x, \partial\Omega)^{\mu}, \quad \text{for } \mu \in (1, \frac{2}{1+\beta}),$$

see also [9] the uniqueness in a different class of solutions. Moreover, M. Winkler, [21] proved that the uniqueness of solution fails in general. One of the most striking phenomenon of solutions of equation (1) is the extinction that any solution vanishes after a finite time even beginning with a positive initial data, see [18], [14] (see also [7] for a quasilinear equation of this type). It is known that this phenomenon occurs according to the presence of the nonlinear singular absorption $u^{-\beta}\chi_{\{u>0\}}$. The same situation happens for the nonlinear absorption u^{β}, for $\beta \in (0,1)$, see [2] and references therein. Furthermore, equation (1) with source term $f(u)$ satisfying the sublinear condition, i.e: $f(u) \leq C(u+1)$, was considered by J. Davila and M. Montenegro, [10]. The authors proved the existence of solution and showed that the measure of the set $\{(x,t) \in \Omega \times (0,\infty) : u(x,t) = 0\}$ is positive (see also a more general statement in [12]). In other words, the solution may exhibit the quenching behavior.

To prove the existence of solutions of equation (1), we must prove the following gradient estimate:

$$|\nabla u(x,t)|^2 \leq Cu^{1-\beta}(x,t), \quad \text{for } (x,t) \in \Omega \times (0,T),$$

where the constant C depends on the f', f, see [10]. Thus, it requires the nonlinear $f \in \mathcal{C}^1([0,\infty))$. In this paper, we show that if f is a nondecreasing function then constant C above is independent of f', so we can remove the regularity $f \in \mathcal{C}^1([0,\infty))$.

Before establishing the existence of solutions of equation (1), it is necessary to introduce a notion of weak solution.

Definition 1.1. Let $u_0 \in L^{\infty}(\Omega)$. A nonnegative function $u(x,t)$ is called a weak solution of equation (1) if $u^{-\beta}\chi_{\{u>0\}} \in L^1(\Omega \times (0,T))$, and $u \in L^2(0,T; W_0^{1,2}(\Omega)) \cap L^{\infty}(\Omega \times (0,T))$ satisfies equation (1) in the sense of distributions $\mathcal{D}'(\Omega \times (0,T))$, i.e:

$$\int_0^T \int_{\Omega} \left(-u\phi_t + \nabla u.\nabla \phi + u^{-\beta}\chi_{\{u>0\}}\phi + f(u)\phi\right) dxdt = 0, \quad \forall \phi \in \mathcal{C}_c^{\infty}(\Omega \times (0,T)). \tag{2}$$

Then, our existence result is as follows:

Theorem 1.1. *Let $u_0 \in L^{\infty}(\Omega)$, and $\beta \in (0,1)$. Then, equation (1) has a maximal weak solution u satisfying*

$$|\nabla u(x,t)|^2 \leq Cu^{1-\beta}(x,t)\left(t^{-1}+1\right), \quad \textit{for a.e } (x,t) \in \Omega \times (0,\infty), \tag{3}$$

where constant $C = C(f, \|u_0\|_{\infty}) > 0$.
Furthermore, if $\nabla(u_0^{\frac{1}{\gamma}}) \in L^{\infty}(\Omega)$, then there is a constant $C = C(f, \|u_0\|_{\infty}, \|\nabla(u_0^{\frac{1}{\gamma}})\|_{\infty}) > 0$ such that

$$|\nabla u(x,t)|^2 \leq Cu^{1-\beta}(x,t), \quad \textit{for a.e } (x,t) \in \Omega \times (0,\infty). \tag{4}$$

Besides, we also study behaviors of solutions of the Cauchy problem associating to equation (1):

$$\begin{cases} \partial_t u - \Delta u + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{5}$$

In [18], Phillips showed that the quenching phenomenon, and the finite speed of propagation hold for the solutions of the Cauchy problem. In this paper, we show that if initial data u_0 satisfies a certain growth condition at infinity, then any weak solution has the instantaneous shrinking of compact support (in short ISS), namely, if initial data u_0 goes to 0 uniformly as $|x| \to \infty$, then the support of any weak solution is bounded for any $t > 0$. This property was first proved in the literature in the study of variational inequalities by Brezis and Friedman, see [5]. After that this phenomenon has been considered for quasilinear parabolic equations, see [4], [13], and references therein. Then, our main result of the Cauchy problem is as follows:

Theorem 1.2. *Let $0 \le u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and $\beta \in (0,1)$. Then, there exists a weak bounded solution $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}^N)) \cap L^2(0,T; W^{1,2}(\mathbb{R}^N))$, satisfying equation (5) in $\mathcal{D}'(\mathbb{R}^N \times (0,\infty))$.*
In addition, if $u_0(x) \to 0$ uniformly as $|x| \to \infty$, then such a weak solution of problem (5) has ISS property.

The paper is organized as follows: In the next section, we prove some gradient estimates for the approximating solutions. In Section 3, we shall prove Theorem 1.1. The last section is devoted to study the Cauchy problem (5) and the instantaneous shrinking of compact support.

Several notations which will be used through this paper are the following: we denote by C a general positive constant, possibly varying from line to line. Furthermore, the constants which depend on parameters will be emphasized by using parentheses. For example, $C = C(\beta, f)$ means that C depends on β, f.

2. Gradient estimate for the approximate solutions

In this section, we consider a regularized equation of (1):

$$(P_{\varepsilon,\eta}) \begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) + f(u_\varepsilon) = 0 & \text{in } \Omega \times (0,\infty), \\ u_\varepsilon = \eta & \text{on } \partial\Omega \times (0,\infty), \\ u_\varepsilon(0) = u_0 + \eta & \text{on } \Omega \end{cases}$$

for any $0 < \eta < \varepsilon$, with $g_\varepsilon(s) = \psi_\varepsilon(s)s^{-\beta}$, $\psi_\varepsilon(s) = \psi(\frac{s}{\varepsilon})$, and $\psi \in \mathcal{C}^\infty(\mathbb{R})$ is a non-decreasing function on $\mathbb{R}$ such that $\psi(s) = 0$ for $s \le 1$, and $\psi(s) = 1$ for $s \ge 2$. Note that g_ε is a globally Lipschitz function for any $\varepsilon > 0$. We will show that solution $u_{\varepsilon,\eta}$ of equation $(P_{\varepsilon,\eta})$ tends to a solution of equation (1) as $\eta, \varepsilon \to 0$. In passing to the limit, we need to derive some gradient estimates for solution $u_{\varepsilon,\eta}$. Then, we have the following result:

Lemma 2.1. *Let $0 \le u_0 \in \mathcal{C}_c^\infty(\Omega)$, $u_0 \ne 0$. There exists a classical unique solution $u_{\varepsilon,\eta}$ of $(P_{\varepsilon,\eta})$ in $\Omega \times (0,\infty)$.*
i) Moreover, there is a constant $C > 0$ only depending on $\beta, f, \|u_0\|_\infty$ such that

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \le C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\big(\tau^{-1} + 1\big), \quad \textit{for any } (x,\tau) \in \Omega \times (0,\infty). \tag{6}$$

ii) If $\nabla(u_0^{\frac{1}{\gamma}}) \in L^\infty(\Omega)$, *then we get*

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \leq C u_{\varepsilon,\eta}^{1-\beta}(x,\tau), \quad \text{for any } (x,\tau) \in \Omega \times (0,\infty), \tag{7}$$

with $C > 0$ *merely depends on* $\beta, f, \|u_0\|_\infty, \|\nabla(u_0^{\frac{1}{\gamma}})\|_\infty$.

Proof. We first prove $i)$.
Fixed $\varepsilon \in (0, \|u_0\|_\infty)$. For any $\eta \in (0,\varepsilon)$, there exists a unique classical solution $u_{\varepsilon,\eta}$ of problem $(P_{\varepsilon,\eta})$ (see [16]). We denote by $u = u_{\varepsilon,\eta}$ for short. It follows from the comparison principle that

$$\eta \leq u(x,t) \leq \|u_0\|_\infty + \eta, \quad \forall (x,t) \in \Omega \times (0,\infty).$$

We can assume $f \in \mathcal{C}^1([0,\infty))$ if not we regularize f by a standard sequence f_n. Note that since f is nondecreasing so is f_n.
Put $u = \phi(v) = v^\gamma$, with $\gamma = 2/(1+\beta)$. Then,

$$v_t - \Delta v = \frac{\phi''}{\phi'}|\nabla v|^2 - \frac{1}{\phi'}\big(g_\varepsilon(\phi(v)) + f(\phi(v))\big). \tag{8}$$

For any $\tau \in (0,T)$, let us consider a cut-off function $\xi(t) \in \mathcal{C}^1(0,\infty)$, $0 \leq \xi(t) \leq 1$, such that

$$\xi(t) = \begin{cases} 1, & \text{on } [\tau, T], \\ 0, & \text{outside } (\frac{\tau}{2}, T + \frac{\tau}{2}), \end{cases}$$

and $|\xi_t| \leq \dfrac{1}{\tau}$.
Next, we set $w = \xi(t)|\nabla v|^2$.
If $\max\limits_{\Omega\times[0,T]} w = 0$, then $\nabla v(\tau) = 0$, so estimate (6) is trivial.
If not, there is a point $(x_0,t_0) \in \Omega \times (\tau/2, T+\tau/2)$ such that $\max\limits_{\Omega\times[0,T]} w = w(x_0,t_0)$.
Thus, we have at (x_0,t_0)

$$w_t = \nabla w = 0, \quad \Delta w \leq 0. \tag{9}$$

This implies

$$0 \leq w_t - \Delta w = \xi_t|\nabla v|^2 + 2\xi(t)\big(\nabla v.\nabla v_t - \nabla v.\nabla(\Delta v)\big) - 2\xi(t)|D^2 v|^2.$$

Or,

$$0 \leq \xi_t|\nabla v|^2 + 2\xi(t)\nabla v.\nabla(v_t - \Delta v). \tag{10}$$

A combination of (8) and (10) provides us

$$0 \leq \xi_t|\nabla v|^2 + 2\xi(t)\nabla v.\nabla\Big(\frac{\phi''}{\phi'}|\nabla v|^2 - \frac{g_\varepsilon(\phi(v)) + f(\phi(v))}{\phi'}\Big).$$

Since $\xi(t_0) > 0$, we get

$$0 \leq \frac{1}{2}\xi^{-1}\xi_t|\nabla v|^2 + \nabla v.\nabla\Big(\frac{\phi''}{\phi'}|\nabla v|^2 - \frac{g_\varepsilon(\phi(v)) + f(\phi(v))}{\phi'}\Big). \tag{11}$$

At the moment, we estimate the terms on the right hand side of (11). First of all, we have from (9) that $\nabla(|\nabla v(x_0,t_0)|^2) = 0$, so

$$\nabla v.\nabla\Big(\frac{\phi''}{\phi'}|\nabla v|^2\Big) = \nabla v.\nabla\Big(\frac{\phi''}{\phi'}\Big)|\nabla v|^2 = (\gamma-1)(2\gamma-3)v^{-2}|\nabla v|^4. \tag{12}$$

Next, we have

$$\nabla v.\nabla\left(\frac{f(\phi)}{\phi'}\right)=f'(\phi)|\nabla v|^2-f(\phi)\frac{\phi''}{\phi'^2}|\nabla v|^2$$
$$=f'(\phi)|\nabla v|^2-(\frac{\gamma-1}{\gamma})f(\phi)v^{-\gamma}|\nabla v|^2. \tag{13}$$

Since $f,f'\geq 0$, and $\gamma>1$, it follows from (13) that

$$-\nabla v.\nabla\left(\frac{f(\phi)}{\phi'}\right)\leq(\frac{\gamma-1}{\gamma})f(\phi)v^{-\gamma}|\nabla v|^2. \tag{14}$$

After that, we have

$$\nabla v.\nabla\left(\frac{g_\varepsilon(\phi)}{\phi'}\right)=(g_\varepsilon'-g_\varepsilon\frac{\phi''}{\phi'^2})|\nabla v|^2=\left(\psi_\varepsilon'(\phi)v^{-\beta}-(\beta+\frac{\gamma-1}{\gamma})\psi_\varepsilon(\phi)v^{-(1+\beta)\gamma}\right)|\nabla v|^2.$$

Since $\psi_\varepsilon'\geq 0$, and $0\leq\psi_\varepsilon\leq 1$, we obtain

$$-\nabla v.\nabla\left(\frac{g(\phi)}{\phi'}\right)\leq(\beta+\frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2. \tag{15}$$

By inserting (12), (14) and (15) into (11), we obtain

$$(\gamma-1)(2\gamma-3)v^{-2}|\nabla v|^4\leq\frac{1}{2}\xi^{-1}\xi_t|\nabla v|^2+(\beta+1-\frac{1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2$$
$$+(\frac{\gamma-1}{\gamma})f(\phi)v^{-\gamma}|\nabla v|^2. \tag{16}$$

Now, we multiply both sides of (16) with v^2 to get

$$(\gamma-1)(2\gamma-3)|\nabla v|^4\leq\frac{1}{2}\xi^{-1}|\xi_t|v^2|\nabla v|^2+(\beta+1-\frac{1}{\gamma})|\nabla v|^2+(\frac{\gamma-1}{\gamma})f(\phi)v^{2-\gamma}|\nabla v|^2. \tag{17}$$

by noting that $(1+\beta)\gamma=2$.
By simplifying the term $|\nabla v|^2$ both sides of the last inequality, we obtain

$$(\gamma-1)(2\gamma-3)|\nabla v|^2\leq\frac{1}{2}\xi^{-1}|\xi_t|v^2+(\beta+1-\frac{1}{\gamma})+(\frac{\gamma-1}{\gamma})f(\phi)v^{2-\gamma}.$$

Multiplying both sides of the last inequality with $\xi(t_0)$ yields

$$(\gamma-1)(2\gamma-3)\xi(t_0)|\nabla v|^2\leq\frac{1}{2}|\xi_t|v^2+\xi(t_0)\left((\beta+1-\frac{1}{\gamma})+(\frac{\gamma-1}{\gamma})f(\phi)v^{2-\gamma}\right). \tag{18}$$

Note that $w(x_0,t_0)=\xi(t_0)|\nabla v(x_0,t_0)|^2$, $0\leq\xi(t)\leq 1$, and $|\xi_t|\leq\tau^{-1}$. It follows from (18) that there is a constant $C=C(\beta)>0$ such that

$$w(x_0,t_0)\leq C\big(\tau^{-1}v^2+f(\phi)v^{2-\gamma}+1\big).$$

Since $w(x_0,t_0)\geq w(x,\tau)=|\nabla v(x,\tau)|^2$, we obtain

$$|\nabla v(x,\tau)|^2\leq C\big(\tau^{-1}v^2+f(\phi)v^{2-\gamma}+1\big).$$

Moreover, we have

$$v^\gamma(x,t)=u(x,t)\leq 2\|u_0\|_\infty,\quad\text{for any }(x,t)\in\Omega\times(0,\infty).$$

Then,

$$|\nabla v(x,\tau)|^2\leq C\left(\tau^{-1}\|u_0\|_\infty^{1+\beta}+\|u_0\|_\infty^{\beta}M_f+1\right),$$

with $M_f = \max_{0\le s\le \|u_0\|_\infty} \{|f(s)|\}$.
Thus,

$$|\nabla u(x,\tau)|^2 \le C_1 u^{1-\beta}\left(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \|u_0\|_\infty^{\beta} M_f + 1\right). \tag{19}$$

This completes the proof of $i)$.

Now, we prove $ii)$.
The proof of estimate (7) is similar to the one of estimate (6). We just make a slight change by considering a cut-off function, still denoted by $\xi(t) \in \mathcal{C}^1(\mathbb{R})$ such that $0 \le \xi(t) \le 1$, $\xi_t(t) \le 0$, and $\xi(t) = \begin{cases} 1, & \text{if } t \le T, \\ 0, & \text{if } t \ge 2T. \end{cases}$

Then, either $w(x,t)$ attains its maximum at the initial data, i.e:

$$\max_{(x,t)\in I\times[0,2T]} w(x,t) = w(x_0,0) = \overline{\xi}(0)|\nabla v(x_0,0)|^2 \le \|\nabla(u_0^{\frac{1}{\gamma}})\|_\infty^2, \quad \text{for some } x_0 \in \Omega,$$

which implies

$$|\nabla u(x,\tau)|^2 \le \gamma^2 \|\nabla(u_0^{\frac{1}{\gamma}})\|_\infty^2 u^{1-\beta}(x,\tau), \quad \text{for any } x \in \Omega. \tag{20}$$

Thus, we get estimate (7) immediately.

Or there is a point $(x_0,t_0) \in \Omega\times(0,2T)$ such that

$$\max_{(x,t)\in\Omega\times[0,2T]} w(x,t) = w(x_0,t_0)$$

Then, we mimic the proof of $i)$ to get an estimate like estimate (16).

$$(\gamma-1)(2\gamma-3)v^{-2}|\nabla v|^4 \le \frac{1}{2}\xi^{-1}\xi_t|\nabla v|^2 + (\beta+1-\frac{1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2 \\ +(\frac{\gamma-1}{\gamma})f(\phi)v^{-\gamma}|\nabla v|^2.$$

Since $\xi_t(t) \le 0$, we get from the above inequality

$$(\gamma-1)(2\gamma-3)v^{-2}|\nabla v|^4 \le (\beta+1-\frac{1}{\gamma})v^{-(1+\beta)\gamma}|\nabla v|^2 + (\frac{\gamma-1}{\gamma})f(\phi)v^{-\gamma}|\nabla v|^2.$$

By repeating the proof of $i)$ after this inequality, we obtain

$$|\nabla u(x,\tau)|^2 \le C u^{1-\beta}(x,\tau)\left(\|u_0\|_\infty^{\beta} M_f + 1\right), \tag{21}$$

for some constant $C = C(\beta) > 0$.
A combination of (20) and (21) yields estimate (7). Or we complete the proof of Lemma 2.1. □

Remark 2.1. Note that gradient estimates (19) and (21) are independent of f'.

As a consequence of Lemma 2.1, we have the following regularity results.

Proposition 2.2. *Let u be a solution of $(P_{\varepsilon,\eta})$. Then, we have*

$$|u(x,t) - u(y,s)| \le C\left(|x-y| + |t-s|^{\frac{1}{3}}\right), \quad \forall (x,t),(y,s) \in \Omega\times(\tau,\infty), \tag{22}$$

for any $\tau > 0$, where $C > 0$ depends on β, τ, $\|u_0\|_\infty$, f.
Moreover, if $\nabla(u_0^{\frac{1}{\gamma}}) \in L^\infty(\Omega)$, then inequality (22) holds for any $(x,t),(y,s) \in \Omega\times(0,\infty)$, and C depends on $\beta, f, \|u_0\|_\infty, \|\nabla(u_0^{\frac{1}{\gamma}})\|_\infty$.

Proof. We refer the proof to Proposition 14, [7] (see also [18]). □

It is obvious that the estimates in Lemma 2.1 are independent of ε, η. Thus, a classical argument allows us to pass to the limit as $\eta \to 0$ in order to obtain $u_{\varepsilon,\eta} \to u_\varepsilon$ (resp. $\nabla u_{\varepsilon,\eta} \to \nabla u_\varepsilon$) uniformly on $\overline{\Omega} \times (0,\infty)$, in that u_ε is a unique classical solution of the following equation:

$$(P_\varepsilon) \begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) = f(u_\varepsilon) & \text{in } \Omega \times (0,\infty), \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u_\varepsilon(0) = u_0 & \text{on } \Omega \end{cases}$$

Remark 2.2. The above gradient estimates also hold for u_ε.

Next, we will pass $\varepsilon \to 0$ to obtain an existence of solution of equation (1).

3. Proof of Theorem 1.1

Let u_ε be a unique solution of equation (P_ε) in $\Omega \times (0,\infty)$. Then, we show that $\{u_\varepsilon\}_{\varepsilon>0}$ is a non-decreasing sequence. Indeed, we have

$$g_{\varepsilon_1}(s) \geq g_{\varepsilon_2}(s), \quad \text{for any } 0 < \varepsilon_1 < \varepsilon_2.$$

This implies that u_{ε_1} is a sub-solution of the equation satisfied by u_{ε_2}. Therefore, the comparison principle yields

$$u_{\varepsilon_1} \leq u_{\varepsilon_2}, \quad \text{in } \Omega \times (0,\infty), \quad \forall \varepsilon_1 < \varepsilon_2,$$

so the conclusion follows. Consequently, there is a nonnegative function u such that $u_\varepsilon \downarrow u$ as $\varepsilon \to 0^+$.
Integrating equation (P_ε) on $\Omega \times (0,T)$ yields

$$\int_\Omega u_\varepsilon(x,T)dx - \int_0^T \int_{\partial\Omega} \nabla u_\varepsilon.\mathbf{n}\, d\sigma ds + \int_0^T \int_\Omega g_\varepsilon(u_\varepsilon) dxds + \int_0^T \int_\Omega f(u_\varepsilon) dxds = \int_\Omega u_\varepsilon(x,0)dx,$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$.
Since $\nabla u_\varepsilon.\mathbf{n} \leq 0$, we obtain

$$\int_0^T \int_\Omega g_\varepsilon(u_\varepsilon) dxds + \int_0^T \int_\Omega f(u_\varepsilon) dxds \leq \int_\Omega u_0(x)dx.$$

This implies that $\|g_\varepsilon(u_\varepsilon)\|_{L^1(\Omega\times(0,T))}$, and $\|f(u_\varepsilon)\|_{L^1(\Omega\times(0,T))}$ are bounded by a constant not depending on ε.
Thanks to Fatou's lemma, there is a function $\Upsilon \in L^1(\Omega \times (0,T))$ such that

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon) = \Upsilon, \quad \text{in } L^1(\Omega \times (0,T)). \tag{23}$$

Next, the monotonicity of $\{u_\varepsilon\}_{\varepsilon>0}$ deduces

$$g_\varepsilon(u_\varepsilon)(x,t) \geq g_\varepsilon(u_\varepsilon)\chi_{\{u>0\}}(x,t), \quad \forall (x,t) \in \Omega \times (0,T),$$

so

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon)(x,t) = \Upsilon(x,t) \geq u^{-\beta}\chi_{\{u>0\}}(x,t), \quad \text{for } (x,t) \in \Omega \times (0,T), \tag{24}$$

which implies that $u^{-\beta}\chi_{\{u>0\}}$ is integrable on $\Omega \times (0,T)$.
In fact, we shall prove

$$\Upsilon = u^{-\beta}\chi_{\{u>0\}}, \quad \text{in } L^1(\Omega \times (0,T)). \tag{25}$$

On the other hand, we can use a result of gradient convergence of Boccardo et al., [3] in order to obtain

$$\nabla u_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \nabla u, \quad \text{for a.e } (x,t) \in \Omega \times (0,T), \tag{26}$$

see the detail of its proof in [9].
As a result, ∇u fulfills estimate (3) for a.e $(x,t) \in \Omega \times (0,T)$, and then for any $\tau \in (0,T)$,

$$\nabla u_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \nabla u, \quad \text{in } L^r(\Omega \times (\tau,T)), \quad \forall r \in [1,\infty). \tag{27}$$

Now, it suffices to demonstrate that u satisfies equation (1) in the sense of distribution. For any $\eta > 0$ fixed, we use the test function $\psi_\eta(u_\varepsilon)\phi$, for any $\phi \in \mathcal{C}_c^\infty(\Omega \times (0,T))$, to the equation satisfied by u_ε. Then, using integration by parts yields

$$\int_{Supp(\phi)} \Big(-\Psi_\eta(u_\varepsilon)\phi_t + \frac{1}{\eta}|\nabla u_\varepsilon|^2\psi'(\frac{u_\varepsilon}{\eta})\phi + \nabla u.\nabla\phi\psi_\eta(u_\varepsilon) + g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi + \\ f(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi\Big) dxds = 0,$$

with $\Psi_\eta(u) = \displaystyle\int_0^u \psi_\eta(s)ds$.
Note that the role of the function $\psi_\eta(.)$ is to avoid the singularity of the term $u^{-\beta}\chi_{\{u>0\}}$ as u is near 0. Thus, there is no problem of passing to the limit as $\varepsilon \to 0$ in the indicated equation in order to get

$$\int_{Supp(\phi)}\Big(-\Psi_\eta(u)\phi_t+\frac{1}{\eta}|\nabla u|^2\psi'(\frac{u}{\eta})\phi+\nabla u.\nabla\phi\psi_\eta(u)+u^{-\beta}\psi_\eta(u)\phi+f(u)\psi_\eta(u)\phi\Big)dxds = 0.$$

Next, we go to the limit as $\eta \to 0$ in the last equation.
By (26), (27), and the integration of $u^{-\beta}\chi_{\{u>0\}}$ in $\Omega \times (0,T)$, it is not difficult to verify

$$\begin{cases} \lim\limits_{\eta\to 0}\displaystyle\int_{Supp(\phi)} \Psi_\eta(u)\phi_t dxds = \int_{Supp(\phi)} u\phi_t dxds, \\ \lim\limits_{\eta\to 0}\displaystyle\int_{Supp(\phi)} \nabla u.\nabla\phi\psi_\eta(u)dxds = \int_{Supp(\phi)} \nabla u.\nabla\phi dxds, \\ \lim\limits_{\eta\to 0}\displaystyle\int_{Supp(\phi)} u^{-\beta}\psi_\eta(u)\phi dxds = \int_{Supp(\phi)} u^{-\beta}\chi_{\{u>0\}}\phi dxds, \\ \lim\limits_{\eta\to 0}\displaystyle\int_{Supp(\phi)} f(u)\psi_\eta(u)\phi dxds = \int_{Supp(\phi)} f(u)\phi dxds. \end{cases} \tag{28}$$

(Note that the assumption $f(0) = 0$ is used in the final limit of (28)).
Next, we show that

$$\lim_{\eta\to 0}\int_{Supp(\phi)} \frac{1}{\eta}|\nabla u|^2\psi'(\frac{u}{\eta})\phi \; dxds = 0. \tag{29}$$

In fact, since u satisfies estimate (3), we have

$$\frac{1}{\eta}\int_{Supp(\phi)}|\nabla u|^2|\psi'(\frac{u}{\eta})\phi|dxds \le C\frac{1}{\eta}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}}u^{1-\beta}dxds$$
$$\le 2C\int_{Supp(\phi)\cap\{\eta<u<2\eta\}}u^{-\beta}dxds,$$

where $Supp(\phi)$ means the support compact of ϕ, and the constant $C>0$ is independent of η. Since $u^{-\beta}\chi_{\{u>0\}}$ is integrable on $\Omega\times(0,T)$, we obtain

$$\lim_{\eta\to 0}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}}u^{-\beta}dxds = 0,$$

which implies the conclusion (29). A combination of (28) and (29) deduces

$$\int_{Supp(\phi)}\big(-u\phi_t+\nabla u.\nabla\phi+u^{-\beta}\chi_{\{u>0\}}\phi+f(u)\phi\big)dxds = 0. \tag{30}$$

In other words, u satisfies equation (1) in $\mathcal{D}'(\Omega\times(0,T))$.
As mentioned above, we prove (25) now. From equation (P_ε), we have

$$\int_{Supp(\phi)}\big(-u_\varepsilon\phi_t+\nabla u_\varepsilon.\nabla\phi+g_\varepsilon(u_\varepsilon)\phi+f(u_\varepsilon)\phi\big)dxds = 0,$$

for any $\phi\in\mathcal{C}_c^\infty(\Omega\times(0,T))$, $\phi\ge 0$.
Then, letting $\varepsilon\to 0$ deduces

$$\int_{Supp(\phi)}(-u\phi_t+\nabla u.\nabla\phi)\,dxds+\lim_{\varepsilon\to 0}\int_{Supp(\phi)}g_\varepsilon(u_\varepsilon)\phi\ dxds+\int_{Supp(\phi)}f(u)\phi\ dxds = 0. \tag{31}$$

By (30) and (31), we get

$$\lim_{\varepsilon\to 0}\int_{Supp(\phi)}g_\varepsilon(u_\varepsilon)\phi\ dxds = \int_{Supp(\phi)}u^{-\beta}\chi_{\{u>0\}}\phi\ dxds. \tag{32}$$

According to (23), (32) and Fatou's lemma, we obtain

$$\int_{Supp(\phi)}u^{-\beta}\chi_{\{u>0\}}\phi dxds \ge \int_{Supp(\phi)}\Upsilon\phi dxds, \quad \forall\phi\in\mathcal{C}_c^\infty(\Omega\times(0,T)),\phi\ge 0.$$

The last inequality and (24) yield conclusion (25).
The conclusion $u\in\mathcal{C}([0,T];L^1(\Omega))$ is well known, so we skip its proof and refer to the compactness result of J. Simon, [19]. Thus, u is a weak solution of equation (1).

To complete the proof of Theorem 1.1, it remains to show that u is the maximal solution of equation (1).

Proposition 3.1. *Let v be any weak solution of equation* (1) *on* $\Omega\times(0,\infty)$. *Then, we have*

$$v(x,t)\le u(x,t), \quad \textit{for a.e } (x,t)\in\Omega\times(0,\infty).$$

In fact, we observe that

$$g_\varepsilon(v)\le v^{-\beta}\chi_{\{v>0\}}, \quad \forall\varepsilon>0.$$

Thus,

$$\partial_t v-\Delta v+g_\varepsilon(v)+f(v)\le 0, \quad \text{in } \mathcal{D}'(\Omega\times(0,\infty)),$$

which implies that v is a sub-solution of equation (P_ε).
By the comparison principle, we get

$$v(x,t) \le u_\varepsilon(x,t), \quad \text{for a.e } (x,t) \in \Omega \times (0,\infty).$$

Letting $\varepsilon \to 0$ yields the result.

Next, it is known that the quenching phenomenon holds for any weak solution of equation (1), see e.g., [18], [9], [7], [8]. By this fact, we show that the condition $f(0) = 0$ is a necessary condition for the existence of a solution of equation (1).

Theorem 3.2. *Assume that $f(0) > 0$. Then equation* (1) *has no nonnegative solution.*

Proof. We assume a contradiction that there is a weak solution u of equation (1). Then, we have the following result:

Lemma 3.3. *Let $0 \le u_0 \in L^\infty(\Omega)$, and $\beta \in (0,1)$. Then, there is a finite time $T_0 > 0$ such that $u(x,t) = 0$, for any $(x,t) \in \Omega \times (T_0,\infty)$.*

We skip the proof of the above lemma, and refer its proof to [18], [9].

Thanks to this lemma, there is a finite time $T_0 > 0$ such that

$$u(x,t) = 0, \quad \forall (x,t) \in \Omega \times (T_0,\infty).$$

This implies that $f(0) = 0$. Then, we get the above theorem. □

4. The instantaneous shrinking of compact support of solutions of the Cauchy problem

4.1. Existence of a weak solution.

Proof. Let u_r be the maximal solution of the following equation

$$\begin{cases} \partial_t u - \Delta u + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0 & \text{in } B_r \times (0,\infty), \\ u = 0, & \partial B_R \times (0,\infty), \\ u(x,0) = u_0(x), & \text{in } B_r, \end{cases} \tag{33}$$

see Theorem 1.1. Obviously, $\{u_r\}_{r>0}$ is a nondecreasing sequence. Moreover, the strong comparison principle deduces

$$u_r(x,t) \le \|u_0\|_{L^\infty(\mathbb{R}^N)}, \quad \text{for } (x,t) \in B_r \times (0,\infty). \tag{34}$$

Thus, there exists a function u such that $u_r \uparrow u$ as $r \to \infty$. We will show that u is a solution of problem (5).

By integrating both sides of (33), we get

$$\begin{cases} \|u_r(.,t)\|_{L^1(B_r)} \le \|u_0\|_{L^1(\mathbb{R}^N)}, \quad \text{for any } t \in (0,\infty), \\ \|f(u_r)\|_{L^1(B_r\times(0,\infty))},\ \|u_r^{-\beta}\chi_{\{u_r>0\}}\|_{L^1(B_r\times(0,\infty))} \le \|u_0\|_{L^1(\mathbb{R}^N)}. \end{cases} \tag{35}$$

It follows immediately from the Monotone Convergence Theorem that $u_r(t)$ converges to $u(t)$ in $L^1(\mathbb{R})$, and $f(u_r)$ converges to $f(u)$ in $L^1(\mathbb{R}^N \times (0,\infty))$ as $r \to \infty$, likewise

$$\begin{cases} \|u(.,t)\|_{L^1(\mathbb{R}^N)} \le \|u_0\|_{L^1(\mathbb{R}^N)}, \quad \text{for any } t \in (0,\infty), \\ \|f(u)\|_{L^1(\mathbb{R}^N\times(0,\infty))} \le \|u_0\|_{L^1(\mathbb{R}^N)}. \end{cases} \tag{36}$$

On the other hand, we have from Lemma 2.1

$$|\nabla u_r(x,t)|^2 \le C u_r^{1-\beta}(x,t)\left(t^{-1}+1\right), \quad \text{for a.e } (x,t) \in B_r \times (0,\infty), \tag{37}$$

for any $r > 0$. By using again a result of [3] (almost everywhere convergence of the gradients), there is a subsequence of $\{u_r\}_{r>0}$ (still denoted as $\{u_r\}_{r>0}$) such that

$$\nabla u_r \overset{r\to\infty}{\longrightarrow} \nabla u, \quad \text{for a.e } (x,t) \in \mathbb{R}^N \times (0,\infty).$$

Thus,

$$|\nabla u(x,t)|^2 \leq C u^{1-\beta}(x,t)\left(t^{-1}+1\right), \quad \text{for a.e } (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{38}$$

and

$$\nabla u_r \overset{r\to\infty}{\longrightarrow} \nabla u, \quad \text{in } L^q_{loc}(\mathbb{R} \times (0,\infty)), \quad \forall q \geq 1. \tag{39}$$

Now, we show that u satisfies equation (5) in the sense of distribution. Indeed, using the test function $\psi_\eta(u_r)\phi$ for the equation satisfied by u_r gives us

$$\int_{Supp(\phi)} \Big(-\Psi_\eta(u_r)\phi_t + \nabla u_r.\nabla\phi\psi_\eta(u_r) + |\nabla u_r|^2\phi\psi'_\eta(u_r) + u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r)\phi +$$
$$f(u_r)\psi_\eta(u_r)\phi\Big)\, dsdx = 0, \quad \forall\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N \times (0,\infty)).$$

We first take care of the term $u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r)\phi$ in passing $r \to \infty$ and $\eta \to 0$. It is not difficult to see that $u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r) = u_r^{-\beta}\psi_\eta(u_r)$ is bounded by $\eta^{-\beta}$. Then for any $\eta > 0$, the Dominated Convergence Theorem yields $u_r^{-\beta}\psi_\eta(u_r) \overset{r\to\infty}{\longrightarrow} u^{-\beta}\psi_\eta(u)$ in $L^1_{loc}(\mathbb{R}^N \times (0,\infty))$, which implies

$$\|u^{-\beta}\psi_\eta(u)\|_{L^1(\mathbb{R}^N\times(0,\infty))} \overset{(35)}{\leq} \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Next, using the Monotone Convergence Theorem deduces $u^{-\beta}\psi_\eta(u) \uparrow u^{-\beta}\chi_{\{u>0\}}$ in $L^1(\mathbb{R}^N \times (0,\infty))$, as $\eta \to 0$, thereby proves

$$\|u^{-\beta}\chi_{\{u>0\}}\|_{L^1(\mathbb{R}^N\times(0,\infty))} \leq \|u_0\|_{L^1(\mathbb{R}^N)}. \tag{40}$$

Thanks to (39), (35) and (34), there is no problem of passing to the limit as $r \to \infty$ in the indicated variational equation in order to get

$$\int_{Supp(\phi)} \Big(-\Psi_\eta(u)\phi_t + \nabla u.\nabla\phi\psi_\eta(u) + |\nabla u|^2\phi\psi'_\eta(u)$$
$$+ u^{-\beta}\psi_\eta(u)\phi + f(u)\psi_\eta(u)\phi\Big)\, dsdx = 0, \quad \forall\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N \times (0,\infty)).$$

By (36), (38), and (40), we can proceed similarly as in the proof of Theorem 1.1 to obtain after letting $\eta \to 0$

$$\int_{Supp(\phi)} \left(-u\phi_t + \nabla u.\nabla\phi + u^{-\beta}\chi_{\{u>0\}}\phi + f(u)\phi\right) dxds = 0, \quad \forall\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N\times(0,\infty)). \tag{41}$$

Or u satisfies equation (5) in the sense of distribution.
The conclusion $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R})^N)$ is classical, so we leave it to the reader. □

4.2. Instantaneous shrinking of compact support of solutions.

Proof. Let u be a solution of equation (1). Since $f(u) \geq 0$, we have for some $q \in (0,1)$

$$f(u) + u^{-\beta}\chi_{\{u>0\}} \geq c_0 u^q,$$

with $c_0 = \dfrac{1}{\|u_0\|^{\beta+q}_{L^\infty(\mathbb{R}^N)}}$. This implies that u is a sub-solution of the following equation:

$$\begin{cases} \partial_t w - \Delta w + c_0 w^q = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ w(x,0) = u_0(x), & \text{in } \mathbb{R}^N. \end{cases} \tag{42}$$

Since equation (42) has a unique solution w, then the comparison principle yields

$$u(x,t) \leq w(x,t), \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

Thanks to the result of Evans et al. [13], w has an instantaneous shrinking of compact support, so does u.

Thus, we obtain the conclusion. □

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References

[1] H. T. Banks, *Modeling and control in the biomedical sciences. Lecture Notes in Biomathematics*, Vol. 6. Springer-Verlag, Berlin-New York, 1975.

[2] Y. Belaud, J.I. Díaz, Abstract results on the finite extinction time property: application to a singular parabolic equation, *Journal of Convex Analysis* **17** (2010), no. 3-4, 827–860.

[3] L. Boccardo, F. Murat, Almost everywhere convergence of the gradients of solutions to Elliptic and Parabolic equations, *Nonlinear Anal. TMA*, **19** (1992), no. 6, 581-597.

[4] M. Borelli, M. Ughi, The fast diffusion equation with strong absorption: the instantaneous shrinking phenomenon, *Rend. Istit. Mat. Univ. Trieste* **26** (1994), 109–140.

[5] H. Brézis, A. Friedman, Estimates on the support of solutions of parabolic variational inequalities, *Illinois J. Math.* **20** (1976), 82–97.

[6] E. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[7] A.N. Dao, J. I. Díaz, A gradient estimate to a degenerate parabolic equation with a singular absorption term: global and local quenching phenomena, *J. Math. Anal. Appl.* **437** (2016), 445–473.

[8] A.N. Dao, J.I. Díaz, The extinction versus the blow-up: Global and non-global existence of solutions of source types of degenerate parabolic equations with a singular absorption, *Submitted.*

[9] A.N. Dao, J.I. Díaz, P. Sauvy, Quenching phenomenon of singular parabolic problem with L^1 initial data, *Elec. Jour. Diff. Equa.* **2016** (2016), no. 136, 1–16.

[10] J. Dávila, M. Montenegro, Existence and asymptotic behavior for a singular parabolic equation, *Transactions of the AMS* **357** (2004) 1801–1828.

[11] J.I. Díaz, *Nonlinear partial differential equations and free boundaries, Research Notes in Mathematics*, vol. 106, Pitman, London, 1985.

[12] J.I. Díaz, On the free boundary for quenching type parabolic problems via local energy methods, *Communications on Pure and Applied Analysis* **13** (2014), 1799–1814.

[13] L.C. Evans, B.F. Knerr, Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities, *Illinois J. of Math.* **23** (1979), no. 1, 153–166.

[14] H. Kawarada, On solutions of initial-boundary problem for $u_t = u_{xx} + \frac{1}{1-u}$, *Publ. Res. Inst. Math. Sci.* **10** (1974/75), 729–736.

[15] B. Kawohl, Remarks on Quenching, *Doc. Math., J. DMV* **1** (1996), 199–208.

[16] O. A. Ladyzenskaja, V.A. Solonnikov, N.N. Uralceva, *Linear and Quasi-Linear Equations of Parabolic Type*, AMS 23, 1988.
[17] H. A. Levine, Quenching and beyond: a survey of recent results, *GAKUTO Internat. Ser. Math. Sci. Appl.* 2 (1993) , Nonlinear mathematical problems in industry II, Gakkotosho, Tokyo, 501–512.
[18] D. Phillips, Existence of solutions of quenching problems, *Applicable Anal.* **24** (1987), 253–264.
[19] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.* **196** (1987), 65–96.
[20] W. Strieder, R. Aris, *Variational Methods Applied to Problems of Diffusion and Reaction*, Berlin: Springer-Verlag, 1973.
[21] M. Winkler, Nonuniqueness in the quenching problem, *Math. Ann.* **339** (2007), 559–597.

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