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Ideals with linear resolution in Segre products

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Abstract. We consider a homogeneous graded algebra on a field K, which is the Segre product of a $K-$polynomial ring in m variables and the second squarefree Veronese subalgebra of a $K-$polynomial ring in n variables, generated over K by elements of degree 1. We describe a class of graded ideals of the Segre product with a linear resolution, provided that the minimal system of generators satisfies a suitable condition of combinatorial kind.

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1. Introduction

Let A and B be two homogeneous graded algebras and let $A * B$ be their Segre product $K[u_1, \ldots, u_n]$, where all generators have degree 1. In [14] the notion of strongly Koszul algebra is introduced and the main consequence is that the maximal graded ideal has linear quotients, hence a linear resolution. In particular if $A = K[x_1, \ldots, x_n]$ and $B = K[y_1, \ldots, y_m]$ are polynomial rings, the graded maximal ideal $(x_1y_1, \ldots, x_ny_m)$ of $A * B$ has linear quotients and a linear resolution. For the significant applications in combinatorics, the case where A and B are monomial algebras received a lot of attention from algebrists. In this case, note that the generators $u_1, \ldots, u_n$ are monomials and the subtended affine semigroup reflects properties of the algebra. The problem to yield monomial ideals with linear quotients and having linear resolution is particularly interesting for homogeneous semigroup rings. The aim of this paper is to investigate if the class of monomial ideals of the semigroup ring studied in [12], and with linear quotients, has a linear resolution.
More precisely, in Section 1, we consider two polynomials rings $A = K[x_1, \ldots, x_n]$ and $B = K[y_1, \ldots, y_m]$ with the standard graduation and the Segre product $B * A^{(2)}$ between B and the second squarefree Veronese ring $A^{(2)}$ generated over K by all squarefree monomials of degree 2 of A. We recall in particular the property P considered in [12], on ordered subsets of the generators of C, that has an interpretation in algebraic combinatorics. In Section 2, we focus our attention to monomial ideals of $B * A^{(2)}$, that admit quotient ideals linearly generated and, as a consequence, they have a linear resolution, being linear modules, following the definition given in [3]. We examine a class of ideals, generated by a suitable subset of the set of the generators of the K-algebra $B * A^{(2)}$, studied in [12] and with linear quotients. The main point is to require that the set of generators satisfies a property able to guarantee that a family of colon ideals of the ideal has linear quotients.

2. Preliminaries and known results

Let $A = K[x_1, \ldots, x_n]$ and $B = K[y_1, \ldots, y_m]$ be two polynomial rings in n and m variables respectively with coefficients in any field K. Let $A^{(2)} \subset A$ be the 2nd squarefree Veronese algebra of A and let $C = B * A^{(2)}$ be the Segre product of B and $A^{(2)}$. We consider C as a standard $K-$algebra generated in degree 1 by the monomials $y_\alpha x_i x_j$, with $1 \leq \alpha \leq m$, $1 \leq i < j \leq n$. For convenience, we will indicate such a monomial by αij.

In [12] we computed all quotient ideals of principal ideals of C, generated by generators of the graded maximal ideal m^* of C in order to obtain the intersection degree of this algebra [13], [14]. The description of the generators of the colon ideals will be used in the following.

Theorem 2.1. [12, Theorem 1.1] *Let $C = B * A^{(2)}$ be the Segre product and let $m^* = (u_1, \ldots, u_N)$, $N = m\binom{n}{2}$ the maximal ideal of C. Let $(u_r) : (u_s)$, $1 \leq r, s \leq N, r \neq s$, a colon ideal of generators of m^*, in the lexicographic order. Then we have:*

1. $(\alpha\, ij_1) : (\alpha\, ij_2) = (\beta kj_1, k \neq j_1, j_2,\, \beta \in \{1, \ldots, m\})$
2. $(\alpha_1 ij_1) : (\alpha_2 ij_2) = (\alpha_1 kj_1,\, k \neq j_1, j_2)$
3. $(\alpha\, i_1 j) : (\alpha\, i_2 j) = (\beta i_1 k, k \neq i_1, i_2,\, \beta \in \{1, \ldots, m\})$
4. $(\alpha_1 i_1 j) : (\alpha_2 i_2 j) = (\alpha_1 i_1 k,\, k \neq i_1, i_2)$
5. $(\alpha\, i_1 j) : (\alpha\, jj_2) = (\beta i_1 k,\, k \neq i_1, j_2,\, \beta \in \{1, \ldots, m\})$
6. $(\alpha\, ij_1) : (\alpha\, i_2 i) = (\beta kj_1,\, k \neq j_1, i_2,\, \beta \in \{1, \ldots, m\})$
7. $(\alpha_1 i_1 j) : (\alpha_2 jj_2) = (\alpha_1 i_1 k,\, k \neq i_1, j_2)$
8. $(\alpha_1 ij_1) : (\alpha_2 i_2 i) = (\alpha_1 kj_1,\, k \neq i_2, j_1)$
9. $(\alpha_1 i_1 j_1) : (\alpha_2 i_2 j_2) = (\alpha_1 i_1 j_1, (\alpha_1 i_1 s)(\beta j_1 s), \beta \in \{1, \ldots, m\}, s \neq i_1, j_1, i_2, j_2)$

10 $(\alpha_1 ij) : (\alpha_2 ij) = (\alpha_1 kl, k \neq l)$

Corollary 2.2. [12, Corollary 1.2] *Let $B * A^{(2)}$ be the Segre product as in Theorem 2.1, where all generators are of degree 1. Then the intersection degree of the monomial algebra $B * A^{(2)}$ is equal to 3 for $n > 4$.*

The fact that there are colon ideals not generated in degree 1 can to not be a problem for special classes of monomial ideals. In particular the strong condition that we consider monomial ideals generated by subsets of generators that verify the property P implies that a family of associated quotients ideals are generated in degree 1, provided a suitable order on the generators.

For this end, we introduce in the set of monomials of $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ the lexicographic order with the order on the variables $y_1 > \ldots > y_m > x_1 > \ldots > x_n$. Moreover, following [12], we call "bad pair" a pair of monomials ij, kl in $A^{(2)}$ or $\alpha\, ij, \beta\, kl$ in C, with $i \neq k$ and $j \neq l$.

Definition 2.1. Let $(u_1, \ldots, u_t)$ be an ideal of $C = B * A^{(2)}$ generated by a sequence $\mathcal{L} = \{\alpha_1 i_1 j_1, \ldots, \alpha_t i_t j_t\}$ of generators of C, with $u_1 > \ldots > u_t$. Fixed $\alpha kl \in \mathcal{L}$, let $\mathcal{L}_{\alpha kl} = \{\beta rs \in \mathcal{L}/\beta rs > \alpha kl$ and $rs > kl\}$ and $\mathcal{L}'_{\alpha kl} = \{\beta rs \in \mathcal{L}/\beta rs < \alpha kl$ and $rs > kl\}$ be. We say that the sequence $\mathcal{L}$ satisfies the property P if:

(1) for each bad pair $\alpha ij > \alpha kl$ in $\mathcal{L}$, $\alpha ik \in \mathcal{L}_{\alpha kl}$ or $\alpha il \in \mathcal{L}_{\alpha kl}$ or $\alpha kl \in \mathcal{L}_{\alpha jk}$ or $\alpha jl \in \mathcal{L}_{\alpha kl}$

(2) for each bad pair $\alpha ij > \beta kl$ in $\mathcal{L}$, with $ij > kl$, $\alpha ik \in \mathcal{L}_{\beta kl}$ or $\alpha il \in \mathcal{L}_{\beta kl}$ or $\alpha jk \in \mathcal{L}_{\beta kl}$ or $\alpha jl \in \mathcal{L}_{\beta kl}$

(3) for each bad pair $\alpha ij > \beta kl$ in $\mathcal{L}$, with $ij < kl$, or $\beta ki \in \mathcal{L}^{'}_{\alpha ij}$ or $\beta kj \in \mathcal{L}^{'}_{\alpha ij}$ or $\beta il \in \mathcal{L}^{'}_{\alpha ij}$ or $\beta jl \in \mathcal{L}^{'}_{\alpha ij}$.

By using this definition, we have:

Theorem 2.3. *Let $(u_1, \dots, u_t)$ be the ideal of $B * A^{(2)}$ generated by a sequence $\mathcal{L} = \{\alpha_1 i_1 j_1, \dots, \alpha_t i_t j_t\}$ of generators of M, with $u_1 > \dots > u_t$. Fixed $\alpha kl \in \mathcal{L}$, let $\mathcal{L}_{\alpha kl} = \{\beta rs \in \mathcal{L} / \beta rs > \alpha kl$ and $rs > kl\}$ and $\mathcal{L}^{'}_{\alpha kl} = \{\beta rs \in \mathcal{L} / \beta rs < \alpha kl$ and $rs > kl\}$ be. Suppose that the sequence $\mathcal{L}$ satisfies the property P. Then $(u_1, \dots, u_t)$ has linear quotients.*

Proof. See [12, Theorem 2.3]. □

Example 2.1. For $n = 2$, $m = 5$, consider $C = K[y_1, y_2] * K[x_1, x_2, x_3, x_4, x_5]$. The sequences $\mathcal{L}_1 = \{112, 113, 114, \dots, 145, 212, 213, 214, \dots, 245\}$ and $\mathcal{L}_2 = \{112, 113, 123, 125, 135, 212, 213, 223, 225, 235\}$ satisfy the property P. For $\mathcal{L}_1$ the result is obvious, since $\mathcal{L}_1$ is the generating sequence of the maximal irrelevant ideal of C. For $\mathcal{L}_2$, we observe that it comes from the colon ideal $(112, 113) : (114) = (112, 113, 123, 125, 135, 212, 213, 223, 225, 235)$. Consider the bad pair $112 > 135$, with $12 > 35$. Then $\mathcal{L}_{135} = \{112, 113, 123, 125\}$. We have $113 \in \mathcal{L}_{135}, 123 \in \mathcal{L}_{135}, 125 \in \mathcal{L}_{135}, 115 \notin \mathcal{L}_{135}$. Consider the bad pair $125 > 213$, with $25 < 13$, $\mathcal{L}^{'}_{125} = \{212, 213, 223\}$. We have $212 \in \mathcal{L}^{'}, 215 \notin \mathcal{L}^{'}, 223 \in \mathcal{L}^{'}, 235 \notin \mathcal{L}^{'}$. Then the property P is satisfied.

3. Monomial Ideals with linear quotients

The aim of this section is to prove that the class of monomial ideals of the Segre product $C = B * A^{(2)}$ described in [12], having linear quotients, has a linear resolution on C. For this we need the following;

Theorem 3.1. *Let $(u_1, \dots u_t)$ be the ideal of $C = B * A^{(2)}$ generated by the sequence $\mathcal{L}$ as in Theorem 2.1 and $I_{q-1} = (\alpha_1 i_1 j_1, \dots, \alpha_{q-1} i_{q-1} j_{q-1}), q \leq t-1$. Then the colon ideal $I : \alpha_{q-1} i_q j_q$ satisfies condition P.*

Proof. Note that the monomial ideal $I : \alpha_q i_q j_q$ is generated by all colon ideals $\alpha_p i_p j_p : \alpha_q i_q j_q$ such that each pair $\alpha_p i_p j_p, \alpha_q i_q j_q$ is not a bad pair for $p < q$ (see [12, Theorem 2.3]). Set $i = i_p$ and $j = j_p$. Assume $i < j$. Consider a bad pair $a, b \in I : \alpha_q ij$:

I case: $a \in \alpha\, i_s j_s : \alpha\, ij$, $i_s j_s, ij$ is not a bad pair
$b \in \alpha\, i_t j_t : \alpha\, ij$, $i_t j_t, ij$ is not a bad pair

II case: $a \in \alpha\, i_s j_s : \beta\, ij$, $i_s j_s, ij$ is not a bad pair
$b \in \beta\, i_s j_s : \beta\, ij$, $i_s j_s, ij$ is not a bad pair

III case: $a \in \alpha\, i_s j_s : \beta\, ij$, $i_s j_s, ij$ is not a bad pair
$b \in \gamma\, i_s j_s : \beta\, ij, \gamma \neq \alpha$, $i_s j_s, ij$ is not a bad pair

IV case: $a \in \alpha\, i_s j_s : \alpha\, ij$, $i_s j_s, ij$ is not a bad pair
$b \in \beta\, i_t j_t : \alpha\, ij$, $i_t j_t, ij$ is not a bad pair

I case: Note that $i_sj_s > ij, i_tj_t > ij$ and $i_s < j_s, i < j, i_t < j_t$. Write $i = i_t$ and $j = j_s$, the colon ideals to be considered are $\alpha i_sj : \alpha ij$ and $\alpha ij_t : \alpha ij$. Let $a \in \alpha\, i_sj : \alpha\, ij$ and $b \in \alpha\, ij_t : \alpha\, ij$ be. Suppose $a = \alpha\, i_sk, k \neq i_s, j$ and $b = \alpha\, lj_t, l \neq j_t, j$.

We look to the following cases:

i) $i_s < k,\ l < j_t$. If $\alpha\, i_sk > \alpha\, lj_t$ (that is $a > b$) then $i_sk > lj_t$, $i_s < l < j_t$, hence $i_s < j_t$. Since $i < j_t$, it follows that $\alpha\, i_sj_t$ is a generator of the colon ideal $I : \alpha ij$ (since $j_t \neq i_s, i$). It follows that $\alpha\, i_sj_t > \alpha\, lj_t$, that is $\alpha\, i_sj_t \in \mathcal{L}_{\alpha\, lj_t}$. If $a < b$, $i_sk < lj_t, l < i_s$, $\alpha i_sk < \alpha li_s$ and $i_sk < li_s$. It follows $\alpha li_s \in \mathcal{L}_{\alpha\, i_sk}$ (that is the property P).

ii) $i_s < k,\ l > j_t,\ a = \alpha\, i_sk,\ b = \alpha j_tl$. If $a > b$, $\alpha\, i_sk > \alpha\, j_tl$ and $i_s < j_t < l$. Since $i < j_t$, it follows that αi_sj_t is a generator of the colon ideal $I : \alpha\, ij$ (since $j_t \neq i_s, i$) hence $\alpha\, i_sj_t > \alpha j_tl, \alpha\, i_sj_t \in \mathcal{L}_{\alpha\, j_tl}$(that is the property P). If $a < b$, $i_sk < j_tl$. Then $j_t < i_s < k$, hence $j_ti_s > i_sk$ and $\alpha j_ti_s \in \mathcal{L}_{\alpha i_sk}$.

iii) $i_s > k, l < j_t$. If $a > b$, $ki_s > lj_t$, then $k < l$ and so $k < l < j_t$. Since $k < j$, the element αkj_t is a generator of $I : \alpha ij$ and $\alpha kj_t > \alpha lj_t$, so $\alpha kj_t \in \mathcal{L}_{\alpha lj_t}$. If $ki_s < lj_t$, $l < k < i_s$ and, since $l < i$, it follows that αli_s is a generator of $I : \alpha ij$ and $\alpha li_s > \alpha ki_s$, $\alpha li_s \in \mathcal{L}_{\alpha\, ki_s}$ (that is the property P).

iv) $i_s > k, l > j_t$. If $a > b$, write $a = \alpha ki_s$, $b = \alpha j_tl$. If $ki_s > j_tl$, $k < j_t < l$ and $kj_t > j_tl$. Since $k < i_s < i < j$, αkj_t is a generator of $I : \alpha ij$. It follows $\alpha kj_t \in \mathcal{L}_{\alpha\, j_tl}$ (that is the property P). If $ki_s < j_tl$, $j_t < k < i_s$ and $j_t < i_s < i$. Hence αj_ti_s is a generator of $I : \alpha ij$, $\alpha j_ti_s > \alpha ki_s$, and $\alpha j_ti_s \in \mathcal{L}_{\alpha\, ki_s}$.

Indeed, we have to achieve the property P for the remaining cases. In synthesis, we can suppose:

$$a = \beta\, i_sk,\ k \neq i_s, i \text{ and } \beta \neq \alpha, \beta > \alpha$$

$$b = \gamma\, lj_t,\ l \neq j_t, j,\ \beta\, i_sk > \gamma lj_t \ \text{ then } \beta > \gamma.$$

We can have:

a) $i_sk > lj_t$

b) $i_sk < lj_t$.

We look to the following cases:

i') $i_s < k$, $l < j_t$, ii') $i_s < k$, $l > j_t$, iii') $i_s > k, l < j_t$, iv') $i_s > k, l > j_t$:

i') For a), $i_s < k$ and $l < j_t$, hence $i_s < l < j_t$ and $j_t \neq i_s$. Since $j_t > i$, $j_t \neq i$, it follows that βi_sj_t is an element of $I : \alpha ij$ and $\beta i_sj_t > \gamma lj_t$ (that is the property P). For b), $i_s > l$. Since $j_t > i, j_t \neq i$. It follows that the monomial βi_sj_t is an element of $I : \alpha ij$ and $\beta i_sj_t > \gamma lj_t$, $\gamma lj_t \in \mathcal{L}'_{\beta i_sj_t}$ (that is the property P).

ii') For a), write $i_sk > j_tl$. Then $i_s < j_t < l$. Since $i < j_t$, it follows that βi_sj_t is a generator of the colon ideal $I : \alpha\, ij$ (since $j_t \neq i_s, i$) hence $\beta\, i_sj_t > \gamma j_tl, \beta\, i_sj_t \in \mathcal{L}_{\gamma j_tl}$(that is the property P). For b), $i_sk < j_tl$. Then $j_t < i_s < k$, hence $j_ti_s > i_sk$,$\beta j_ti_s > \gamma i_sk$ and $\beta j_ti_s \in \mathcal{L}_{\gamma i_sk}$.

iii') For a) $ki_s > lj_t$, then $k < l$ and so $k < l < j_t$. Since $k < j$, the element βkj_t is a generator of $I : \alpha ij$ and $\beta kj_t > \gamma lj_t$, so $\beta kj_t \in \mathcal{L}_{\gamma lj_t}$. For b), $l < k < i_s$ and, since $l < i$, it follows that $\gamma li_s < \beta ki_s$, $\gamma li_s \in \mathcal{L}'_{\beta ki_s}$.

iv') For a), write $ki_s > j_tl$, $k < j_t < l$ and $kj_t > j_tl$. Since $k < i_s < i < j$, γkj_t is a generator of $I : \alpha ij$, and $\gamma kj_t < \beta j_tl$, it follows $\gamma kj_t \in \mathcal{L}'_{\beta j_tl}$, that is the property P. For b), write $ki_s < j_tl$, $j_t < k < i_s$ and $j_t < i_s < i$. Hence βj_ti_s is a generator of $I : \alpha ij$, $\beta j_ti_s > \gamma ki_s$, and $\beta j_ti_s \in \mathcal{L}_{\gamma ki_s}$.

The proof of cases II, III, IV is analogous. □

Now we recall the definition of linear module, as found in [3].

Definition 3.1. Let $R = K[u_1, \dots, u_n]$ be a homogeneous $K-$algebra, K a field, finitely generated over K by elements of degree 1, and let M a graded $R-$module. M is said to be linear if it has a system of generators $m_1, \dots, m_t$ all of the same degree, such that for $j = 1, \dots, t$ the colon ideals:

$$(Rm_1 + \dots Rm_{j-1}) : m_j$$

is generated by a subset of $\{u_1, \dots, u_n\}$.

Proposition 3.2. [14, Theorem 1.2] *Suppose R a strongly Koszul K-algebra. Let $I \subset R$ be a homogeneous ideal generated by a subset of generators of the maximal irrelevant ideal of R. Then I has linear quotients and a linear resolution on R.*

Proposition 3.3. *Let C be the monomial algebra $B * A^{(2)}$ and let I be a monomial ideal $(u_1, \dots, u_t)$ generated by a sequence $\mathcal{L}$ of generators of the algebra that satisfies the property P. Then I has a linear resolution.*

Proof. By Definition 3.1, I is a linear module. Hence the statement will be true if we show that I has linear relations and its first syzygy module is again a linear module. For the first assertion, if $a_1u_1 + \dots + a_ru_r, 1 \le r \le t$, is a homogeneous generating relation of I, let a_j be the last non zero coefficient of that relation, then a_j is a generator of the colon ideal $(u_1, \dots, u_{j-1}) : u_j$. Hence a_j is a generator of the algebra of degree 1, and the relation is linear. Let $Syz_1(I)$ be the first syzygy module of I. We will prove that $Syz_1(I)$ is a linear module by induction on the number of generators. If the ideal I is principal, then $Syz_1(I) = \{0\}$. Suppose $Syz_1(I)$ generated by r elements $g_1, \dots, g_s, s > 1$,such that with respect to them $Syz_1(I)$ is a linear module. Consider the submodule $D = Cg_1 + \dots + Cg_{s-1}$ that is linear by induction and so its $Syz_1(D)$ module, with respect to a system of minimal generators $l_1, \dots, l_u$. By the exact sequence

$$0 \to Syz_1(D) \to Syz_1(Syz_1(I)) \to Syz_1(Syz_1(I)/Syz_1(D)) \to 0,$$

the module $Syz_1(I)/Syz_1(D)$ is cyclic with annihilator ideal $Cg_1 + \dots + Cg_{s-1} : Cg_s$,then $Syz_1(Syz_1(I)/Syz_1(D)) \cong (u_{i_1}, \dots, u_{i_v})$, $1 \le i_1 < \dots < i_v \le t$, that verifies the Property P by induction and then it is a linear module. Now we can complete the set $l_1, \dots, l_u$ in $Syz_1(D)$, hence in $Syz_1(Syz_1(I))$, choosing homogeneous elements $h_1, h_2, \dots, h_v$ of $Syz_1(Syz_1(I))$, such that they can be mapped onto in the set $u_{i_1}, \dots, u_{i_v}$. We claim that the module $Syz_1(Syz_1(I))$, generated by the set $l_1, \dots, l_u$, $h_1, h_2, \dots, h_v$ is a linear module with respect to these generators. In fact the quotient ideals $Cl_1 + \dots + Cl_{j-1} : Cl_j$, $1 \le j \le s$,are generated by a subset of generators. By induction, each colon ideal $Cl_{i_j} : Ch_{j_k} = (0)$, $1 \le i_j \le u$, $1 \le i_k \le v$, and $Ch_1 + Ch_{k-1} : Ch_k, 1 \le k \le v$, are generated by a subset of variables. For this, let m be a monomial generator, then $mh_k = b_1h_1 + \dots + b_{k-1}h_{k-1}$ and mapping onto in

$Syz_1(Syz_1/Syz_1(D))$, we obtain the relation $mu_{i_k} = b_1u_{i_1} + b_k - 1u_{i_{k-1}}$ in C. So m is a generator of the quotient ideal $(u_{i_1}, \dots, u_{i_{k-1}}) : u_{i_k}$, hence of degree 1. □

Corollary 3.4. *Let $I = (u_1, \dots, u_t)$ be an ideal of $B * A^{(2)}$ as in Theorem 2.1. Let I_r be any colon ideal $(u_1, \dots, u_r) : (u_{r+1})$ of I, $r = 1, \dots, t-1$. Then we have:*
(1) *I_r has linear quotients*
(2) *I_r has a linear resolution.*

Proof. (1) By Theorem 3.1 and (2) by Proposition 3.3. □

Remark 3.1. We proved in Theorem 3.1 that any colon ideal I_r of I verifies the property P. In the same way any colon ideal of I_r verifies P and so on. The previous condition characterizes the sequentially Koszul algebras, as defined in [1].

Remark 3.2. For $n = 4$, $A^{(2)}$ is a strongly Koszul algebra and consequently the Segre product $B * A^{(2)}$ [14]. As a consequence any ideal generated by a subset of generators has a linear resolution.

Remark 3.3. For homogeneous semigroup rings arising from Grassmann varieties, Hankel varieties of $\mathbb{P}^n$ and their subvarieties [7], [8], [9], [10], [15], the problem is more difficult. For $G(1,3) = H(1,3)$ its toric ring is strongly Koszul, being a quotient of the polynomial ring $K[[12],[13],[14],[23],[24],[34]]$ for the ideal generated by the binomial relation $[14][23] - [13][24]$, where $[i,j]$ is the variable corresponding to the minor with columns i, j, $i < j$, of a 2×4 generic matrix. The semigroup ring of $\mathbb{G}(1,4)$ is a subring of $K[t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}]$, t_{ij} the generic entry of a 2×5- matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} \end{pmatrix}$$

and it is generated by the diagonal initial terms of ten 2×2 minors of the matrix. The semigroup of $H(1,4)$ is a subring of $K[t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}]$, generated by the diagonal initial terms of ten 2×2 minors of the Hankel matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \end{pmatrix}.$$

Both rings have a toric ideal generated by a Gröbner basis of degree 2 [15], [8] and they are Koszul. The problem to find monomial ideals generated by subsets of generators of the semigroup ring with linear resolution is open, for $n > 4$.

Remark 3.4. Segre products between polynomial rings on any field K and squarefree Veronese rings have been employed for algebraic models in statistic, in graphs theory, in transportation problems [4], [5], [6]. In particular, if I_r and J_s are respectively the rth squarefree Veronese ideal of $K[x_1, \dots, x_n]$ and the sth squarefree Veronese ideal of $K[y_1, \dots, y_m]$, we can consider the sum $I_r + J_s$ or the product I_rJ_s in the ring $K[x_1, \dots, x_n; y_1, \dots, y_m]$ that describe particular simple graphs and the semigroup rings $K[I_r]$, $K[I_r, J_s]$, $K[I_rJ_s]$, respectively subrings of $K[x_1, \dots, x_n]$, $K[x_1, \dots, x_m; y_1, \dots, y_n]$ generated by the minimal system of generators of I_r,$I_r + J_s$ and I_rJ_s. Observe that we have that $C = K[J_1I_2]$. Since the sorted Gröbner basis of the defining ideals of the previous semigroup rings is quadratic [15], initial simplicial complexes with respect the a total order received a lot of attention in several articles. Indeed the subtended affine semigroup presents easy triangulations [11],[15]. Alternately, one studied classify the simplicial complexes defined by the squarefree monomial ideals $I_r + J_s$ and $I_r J_s$ to obtain combinatorial statements [16].

In this paper we referred to the excellent books whose in [2], [17].

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