Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 45(2), 2018, Pages 232–242
ISSN: 1223-6934

On a Hardy's inequality for a fractional integral operator

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Abstract. In the present work we find inequalities that generalize some results found by Iqbal in [4, 5] regarding Riemann-Liouville fractional integral, by means of a Hardy inequality, using fractional integral operators defined by R.K. Raina in [9].

2010 Mathematics Subject Classification. Primary 35A23; Secondary 26A33.
Key words and phrases. Hardy's Inequality, Fractional Integral Operator.

1. Introduction

On the inequalities found by H.G. Hardy, one of them has been the subject of new research, among which are those that generalize them. From these, we have taken as starting point those corresponding to Iqbal S., Krulić K. and Pečarić [4, 5].

Once the fractional integral of Riemann Liouville was defined, Hardy demonstrates that the mentioned integral is bounded in $L_p([a,b]), (1 \le p \le \infty)$, as an operator acting from $L_p([a,b])$ to $L_p([a,b])$ (See [3]).

Fractional calculus originally appeared in the letter between L'Hospital and Leibniz. Since 1695, Riemann, Liouville, Caputo, Hadamard and other famous mathematicians paid attention to study such a branch of mathematical analysis. Meanwhile, fractional calculus have been widely applied to the fields of electricity, biology, economics and signal and image processing [6, 7, 8].

In recent years research articles have been published and contain generalizations of others that have been relevant in the area of Fractional Calculus. An example of this are those published by R.K. Raina in [9] and R.P. Agarwal, M.J. Luo and R.K. Raina in [1].

Motivated by the results of S. Iqbal [4] and R.P. Agarwal et.al. [1], this work shows a generalization of those found in the aforementioned articles, regarding fractional integral operators using the method and results found by Iqbal [4].

2. Preliminaries

Below we present some basic notions for the development of this work.

In [9], R.K. Raina introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \tag{1}$$

Received June 29, 2017. Accepted December 10, 2018.
The presented work has been conducted by CDCHT from Universidad Centroccidental Lisandro Alvarado.

where $\rho, \lambda > 0, |x| < \mathbb{R}$,($\mathbb{R}$ is the set of real numbers) and $\sigma = (\sigma(1), .., \sigma(k), ..)$ is a bounded sequence of positive real numbers.

Making use of (1), and differentiating term-wise the right-side (which is permissible provided the series converges uniformly in any compact set of $\mathbb{C}$), we readily obtain

$$\left(\frac{d}{dx}\right)^n x^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wx^{\rho}) = x^{\lambda-n-1}\mathcal{F}^{\sigma}_{\rho,\lambda-n}(wx^{\rho})$$

where $\rho, \lambda, w \in \mathbb{C}$ $(\mathrm{Re}\,(\rho) > 0, \mathrm{Re}\,(\lambda) > 0)$; $n \in \mathbb{N}$.

Similarly, we can obtain the following result:

$$\underbrace{\int_0^x ... \int_0^x}_{n\ times} t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})\,(dt)^n = x^{\lambda+n-1}\mathcal{F}^{\sigma}_{\rho,\lambda+n}(wx^{\rho})$$

where $\rho, \lambda, w \in \mathbb{C}$ $(\mathrm{Re}\,(\rho) > 0, \mathrm{Re}\,(\lambda) > 0)$; $n \in \mathbb{N}$.

Using (1), R.P. Agarwal, M.J. Luo and R.K. Raina [1] defined the following left-sided and right-sided fractional integral operators respectively, as follows

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-t)^{\rho}\right]\varphi(t)dt, \quad (x > a) \tag{2}$$

and

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(t-x)^{\rho}\right]\varphi(t)dt, \quad (x < b)\,, \tag{3}$$

where $\lambda, \rho > 0$, $w \in R$ and φ is such that the integral on the right side exits.

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann-Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} de order $\alpha > 0$:

$$\left(I^{\alpha}_{a+}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt, \qquad (x > a, \alpha > 0)$$

and

$$\left(I^{\alpha}_{b-}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt, \qquad (x < b, \alpha > 0)$$

follow from (2) and (3) setting $\lambda = \alpha, \sigma(0) = 1$ and $w = 0$. These fractional integrals, viewed as operator in $L_p\left([a,b]\right)$, are bounded, i.e.

$$\left\|I^{\alpha}_{a+}\varphi\right\|_p \leq K\left\|\varphi\right\|_p \qquad \text{and} \qquad \left\|I^{\alpha}_{b-}\varphi\right\|_p \leq K\left\|\varphi\right\|_p \tag{4}$$

where

$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

Inequalities (4) , which involve the left-sided and right-sided of Riemann-Liouville fractional integral, was proved by H.G. Hardy in one of his first papers, see [3]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive $\sigma-$finite measures, and let $k : \Omega_1 \times \Omega_2 \to R$ be a non-negative function, and

$$K(x) = \int_{\Omega_2} k(x,y)d\mu_2,\ x \in \Omega_1 \tag{5}$$

Throughout this paper, we suppose that $K(x) > 0$ a.e. on Ω_1, and by a weight function shortly: a weight , we mean a nonnegative measurable function on the actual set. Let $\mathcal{U}(k)$ denote the class of functions $g : \Omega_1 \to R$ with the representation

$$g(x) = \int_{\Omega_2} k(x,y) f(y) d\mu_2$$

where $f : \Omega_2 \to R$ is a mesurable function.

One result of interest for our work is the following, found by S. Iqbal, K. Krulić, and J. Pečarić in [4].

Theorem 2.1. *Let u be a weight function on Ω_1, k a nonnegative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (5) . Assume that the function $x \to u(x)k(x)/K(x)$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by*

$$v(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)} d\mu_1(x).$$

If $\phi : (0,\infty) \to R$ is convex and inreasing function, then the inequality

$$\int_{\Omega_1} u(x) \phi\left(\left| \frac{g(x)}{K(x)} \right| \right) d\mu_1(x) \leq \int_{\Omega_2} v(y) \phi(|f(y)|) d\mu_2(y)$$

holds for all measurable functions $f : \Omega_2 \to R$ and for all functions $g \in \mathcal{U}(k)$.

Next results are important for this work. The lemma is found in [2].

Lemma 2.2. *For $s \in R$, let function $\phi_s : (0,\infty) \to R$ be defined by*

$$\phi_s(x) = \begin{cases} \frac{x^s}{s(s-1)} & if \;\; s \neq 1, 0 \\ \log x & if \;\; s = 0 \\ x \log x & if \;\; s = 1. \end{cases} \tag{6}$$

Then $\phi_s''(x) = x^{s-2}$, that is ϕ_s is a convex function.

Theorem 2.3. [2] *Let the conditions of Theorem 2.1 be satisfied and ϕ_s be defined by (6). Let f be a positive function. Then the function $\xi : R \to [0,\infty)$ defined by*

$$\xi(s) = \int_{\Omega_2} v(y) \phi_s(f(y)) d\mu_2(y) - \int_{\Omega_1} u(x) \phi_s\left(\frac{g(x)}{K(x)} \right) d\mu_1(x)$$

is exponentially convex.

This theorem is found in [2].

Theorem 2.4. *Let $(\Omega_1, \sum_1, \mu_1), (\Omega_2, \sum_2, \mu_2)$ be measure spaces with $\sigma-$finite measures and $u : \Omega_1 \to R$ be a weight function. Let I be a compact interval of R, $h \in C^2(I)$, and $f : \Omega_2 \to R$ a measurable function such that $Imf \subseteq I$. Then there exists $\eta \in I$ such that*

$$\int_{\Omega_2} v(y) h(f(y)) d\mu_2(y) - \int_{\Omega_1} u(x) h\left(\frac{g(x)}{K(x)} \right) d\mu_1(x)$$
$$= \frac{h''(\eta)}{2} \left[\int_{\Omega_2} v(y) f^2(y) d\mu_2(y) - \int_{\Omega_1} u(x) \left(\frac{g(x)}{K(x)} \right)^2 d\mu_1(x \right]$$

3. Main Results

Theorem 3.1. *Let u be a weight function on $[a,b]\,, a,b\in\mathbb{R}, a<b$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f$ denotes the fractional integral operator defined by (2). Define v on $[a,b]$ by*

$$v(y):=\int_y^b u(x)\frac{(x-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-y)^{\rho}\right]}{(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right]}dx$$

Then we have

$$\int_a^b u(x)\phi\left(\frac{\left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(x)\right|}{(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right]}\right)dx\leq\int_a^b v(y)\phi(|f(y)|)dy.$$

Proof. Applying Theorem 2.1 with $\Omega_1=\Omega_2=(a,b)\,, d\mu_1(x)=dx, d\mu_2(y)=dy$ and $k:\Omega_1\times\Omega_2\to\mathbb{R}$ be a non-negative function defined by

$$k(x,y)=\begin{cases}(x-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-y)^{\rho}\right] & \text{if } a\leq y\leq x\\ 0 & \text{if } x<y\leq b\end{cases},$$

$$g(x)=\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(x),\ \text{for } x\in[a,b]\,,$$

and

$$\begin{aligned}K(x)&=\int_a^b k(x,y)dy\\&=\int_a^x (x-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-y)^{\rho}\right]dy\\&=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda)}w^k\int_a^x (x-y)^{\lambda-1}(x-y)^{k\rho}\,dy\\&=(x-a)^{\lambda}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)}\left[w\,(x-a)\right]^{k\rho}\\&=(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right].\end{aligned}$$

Then we have

$$\int_a^b u(x)\phi\left(\frac{\left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(x)\right|}{(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right]}\right)dx\leq\int_a^b v(y)\phi(|f(y)|)dy.$$

The proof is complete. □

Remark 3.1. If in Theorem 3.1 we put $\lambda=\alpha, w=0$ and $\sigma=(1,0,0,...)$ then we get

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha+1)\left|\left(I^{\alpha}_{a+}f\right)(x)\right|}{(x-a)^{\alpha}}\right)dx\leq\int_a^b v(y)\phi(|f(y)|)dy,$$

making coincidence with Corollary 2.2 in [4]. Also, if $u(x)=(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right]$ and

$$v(y)=\int_y^b (x-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-y)^{\rho}\right]dx=(b-y)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-y)^{\rho}\right],$$

then we can get

$$\int_a^b (x-a)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(x-a)^\rho\right]\phi\left(\frac{\left|\left(\mathcal{J}^\sigma_{\rho,\lambda,a+;w}f\right)(x)\right|}{(x-a)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(x-a)^\rho\right]}\right)dx$$
$$\leq \int_a^b (b-y)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(b-y)^\rho\right]\phi(|f(y)|)dy, \tag{7}$$

therefore, if $\lambda = \alpha, w = 0$ and $\sigma = (1,0,0,...)$ then we get

$$\int_a^b (x-a)^\alpha \phi\left(\frac{\Gamma(\alpha+1)\left|\left(I^\alpha_{a+}f\right)(x)\right|}{(x-a)^\alpha}\right)dx \leq \int_a^b (b-y)^\alpha \phi(|f(y)|)dy,$$

making coincidence with Remark 2.3.in [4]. If in addition $\alpha = 1$, we have

$$\int_a^b (x-a)\,\phi\left(\frac{1}{(x-a)}\left|\int_a^x f(t)dt\right|\right)dx \leq \int_a^b (b-y)\,\phi(|f(y)|)dy.$$

for the Riemann integral.

Remark 3.2. Let $q > 1$ and $\phi(x) = x^q$. Using (7) we see that

$$\int_a^b (x-a)^{\lambda(1-q)}\left(\mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(x-a)^\rho\right]\right)^{1-q}\left|\left(\mathcal{J}^\sigma_{\rho,\lambda,a+;w}f\right)(x)\right|^q dx$$
$$\leq \int_a^b (b-y)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(b-y)^\rho\right]|f(y)|^q dy.$$

Note that $\lambda(1-q) < 0$, in consequence we can write

$$\int_a^b (x-a)^{\lambda(1-q)}\left(\mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(x-a)^\rho\right]\right)^{1-q}\left|\left(\mathcal{J}^\sigma_{\rho,\lambda,a+;w}f\right)(x)\right|^q dx$$
$$\geq (b-a)^{\lambda(1-q)}\left(\mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(b-a)^\rho\right]\right)^{1-q}\int_a^b \left|\left(\mathcal{J}^\sigma_{\rho,\lambda,a+;w}f\right)(x)\right|^q dx$$

and also

$$\int_a^b (b-y)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(b-y)^\rho\right]|f(y)|^q dy$$
$$\leq (b-a)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(b-a)^\rho\right]\int_a^b |f(x)|^q dx$$

therefore

$$\int_a^b \left|\left(\mathcal{J}^\sigma_{\rho,\lambda,a+;w}f\right)(x)\right|^q dx \leq (b-a)^{\lambda q}\left(\mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(b-a)^\rho\right]\right)^q \int_a^b |f(x)|^q dx$$

i.e.,

$$\left\|\left(\mathcal{J}^\sigma_{\rho,\lambda,a+;w}f\right)\right\|_q \leq (b-a)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(b-a)^\rho\right]\|f\|_q .$$

In addition, if $\lambda = \alpha, w = 0$ and $\sigma = (1,0,0,...)$ we have

$$\left\|I^\alpha_{a+}f\right\|_q \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\|f\|_q .$$

This last expression coincides with inequality (4) proved by H.G. Hardy.

Corollary 3.2. *Let u be a weight function on $[a,b]$, and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f$ defined by (3) denotes the fractional integral operator defined in* [1]. *Define v on $[a,b]$ by*

$$v(y) := \int_y^b u(x) \frac{(x-y)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-y)^{\rho}\right]}{(x-a)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right]} dx.$$

Then we have

$$\int_a^b u(x)\phi\left(\frac{\left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(x)\right|}{(x-a)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right]}\right) dx \leq \int_a^b v(y)\phi(|f(y)|)dy.$$

Proof. Similar as the proof of Theorem 3.1, taking

$$k(x,y) = \begin{cases} 0 & \text{if } a \leq y \leq x \\ (x-y)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-y)^{\rho}\right] & \text{if } x < y \leq b \end{cases}$$

The proof is complete. □

With this result also we find similar forms, as in the previous remarks, for $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f$.

Remark 3.3. Setting $\lambda = \alpha, \sigma = (1,0,0,...)$ and $w = 0$, we can establish the following for Riemann-Liouville fractional integral

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha+1)\left|\left(\mathcal{I}^{\alpha}_{a+}f\right)(x)\right|}{(x-a)^{\alpha}}\right) dx \leq \int_a^b v(y)\phi\left(|f(y)|\right) dy$$

and

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha+1)\left|\left(\mathcal{I}^{\alpha}_{b-}f\right)(x)\right|}{(b-x)^{\alpha}}\right) dx \leq \int_a^b v(y)\phi\left(|f(y)|\right) dy,$$

also if we take $u(x) = (x-a)^{\alpha}$ in Theorem 3.1 and $u(x) = (b-x)^{\alpha}$ in Corollary 3.2, respectively, we have

$$\int_a^b (x-a)^{\lambda}\phi\left(\frac{\Gamma(\alpha+1)\left|\left(\mathcal{I}^{\alpha}_{a+}f\right)(x)\right|}{(x-a)^{\lambda}}\right) dx \leq \int_a^b (b-y)^{\alpha}\phi\left(|f(y)|\right) dy$$

and

$$\int_a^b (b-x)^{\lambda}\phi\left(\frac{\Gamma(\alpha+1)\left|\left(\mathcal{I}^{\alpha}_{b-}f\right)(x)\right|}{(b-x)^{\lambda}}\right) dx \leq \int_a^b (y-a)^{\alpha}\phi\left(|f(y)|\right) dy,$$

moreover

$$\left\|\mathcal{I}^{\alpha}_{a+}f\right\|_{L_q(a,b)} \leq \frac{(b-a)^{\lambda}}{\Gamma(\alpha+1)} \|f\|_{L_q(a,b)}$$

and

$$\left\|\mathcal{I}^{\alpha}_{b-}f\right\|_{L_q(a,b)} \leq \frac{(b-a)^{\lambda}}{\Gamma(\alpha+1)} \|f\|_{L_q(a,b)}.$$

Theorem 3.3. *Let $s > 1$ and $\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)$ denote the fractional integral operator and defined by (2). Then the following inequality holds*

$$\xi_1(s) \leq H_1(s)$$

where

$$\xi_1(s) = \frac{1}{s(s-1)} \left(\int_a^b (b-y)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-y)^{\rho}\right] |f(y)|^s \, dy \right.$$
$$\left. - \int_a^b (x-a)^{\lambda(1-s)} \left(\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right]\right)^{1-s} \left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} f\right)(x)\right|^s dx \right)$$

and

$$H_1(s) = \frac{(b-a)^{\lambda(1-s)} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{s(s-1)} \left((b-a)^{\lambda s} \int_a^b |f(y)|^s \, dy \right.$$
$$\left. - \left(\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]\right)^{-s} \int_a^b \left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} f\right)(x)\right|^s dx \right)$$

Proof. Let $s > 1$, ϕ_s defined in Lemma 2.2. From Theorem 2.3 and Theorem 3.1, we can get

$$\xi_1(s) = \frac{1}{s(s-1)} \left(\int_a^b (b-y)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-y)^{\rho}\right] |f(y)|^s \, dy \right.$$
$$\left. - \int_a^b (x-a)^{\lambda(1-s)} \left(\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right]\right)^{1-s} \left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} f\right)(x)\right|^s dx \right)$$
$$\leq \frac{(b-a)^{\lambda(1-s)} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{s(s-1)} \left((b-a)^{\lambda s} \left(\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]\right) \int_a^b |f(y)|^s \, dy \right.$$
$$\left. - \left(\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]\right)^{-s} \int_a^b \left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} f\right)(x)\right|^s dx \right) = H_1(s).$$

□

Remark 3.4. If in Theorem we put $\lambda = \alpha, w = 0$ and $\sigma = (1, 0, 0, ...)$ we have

$$\xi_1(s) = \frac{1}{s(s-1)} \left(\int_a^b (b-y)^{\alpha} \frac{1}{\Gamma(\alpha+1)} |f(y)|^s \, dy \right.$$
$$\left. - \int_a^b (x-a)^{\alpha(1-s)} \left(\frac{1}{\Gamma(\alpha+1)}\right)^{1-s} \left|\left(I^{\alpha}_{a+} f\right)(x)\right|^s dx \right)$$
$$\leq \frac{(b-a)^{\alpha(1-s)}}{\Gamma(\alpha+1)s(s-1)} \left((b-a)^{\alpha s} \int_a^b |f(y)|^s \, dy - \Gamma(\alpha+1)^s \int_a^b \left|\left(I^{\alpha}_{a+} f\right)(x)\right|^s dx \right) H_1(s)$$

making coincidence with Theorem 2.3 in [5].

Theorem 3.4. *Let $s > 1$ and $\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w} f\right)$ denote the fractional integral operator and defined by 3. Then the following inequality holds*

$$\xi_2(s) \leq H_2(s)$$

where

$$\xi_2(s) = \frac{1}{s(s-1)}\left(\int_a^b (y-a)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(y-a)^{\rho}\right]|f(y)|^s\,dy - \int_a^b (b-x)^{\lambda(1-s)}\left(\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-x)^{\rho}\right]\right)^{1-s}\left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(x)\right|^s dx\right)$$

and

$$H_2(s) = \frac{(b-a)^{\lambda(1-s)}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{s(s-1)}\left((b-a)^{\lambda s}\int_a^b |f(y)|^s\,dy - \left(\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]\right)^{-s}\int_a^b \left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(x)\right|^s dx\right)$$

Proof. Similar to the proof of Theorem 3.3. □

Remark 3.5. If in Theorem we put $\lambda = \alpha, w = 0$ and $\sigma = (1,0,0,...)$ we have

$$\xi_2(s) = \frac{1}{s(s-1)}\left(\int_a^b (b-y)^{\alpha}\frac{1}{\Gamma(\alpha+1)}|f(y)|^s\,dy - \int_a^b (x-a)^{\alpha(1-s)}\left(\frac{1}{\Gamma(\alpha+1)}\right)^{1-s}\left|\left(I^{\alpha}_{b-}f\right)(x)\right|^s dx\right)$$
$$\leq \frac{(b-a)^{\alpha(1-s)}}{\Gamma(\alpha+1)s(s-1)}\left((b-a)^{\alpha s}\int_a^b |f(y)|^s\,dy - \Gamma(\alpha+1)^s\int_a^b \left|\left(I^{\alpha}_{b-}f\right)(x)\right|^s dx\right) = H_2(s)$$

making coincidence with Theorem 2.4 in [5].

Theorem 3.5. *Let $[a,b]$ be an interval, $u : [a,b] \to R$ be a weight function. Let I be a compact interval of R, $h \in C^2(I)$, and $f : [a,b] \to R$ a measurable function such that $Imf \subseteq I$. Then there exists $\eta \in I$ such that*

$$\int_a^b v(y)h(f(y))dy - \int_a^b u(x)h\left(\frac{g(x)}{K(x)}\right)dx = \frac{h''(\eta)}{2}\left[\int_a^b v(y)f^2(y)d\mu_2(y) - \int_a^b u(x)\left(\frac{g(x)}{K(x)}\right)^2 dx\right]$$

Proof. Making $\Omega_1 = \Omega_2 = (a,b)\,, d\mu_1(x) = dx, d\mu_2(y) = dy$ and $k : \Omega_1 \times \Omega_2 \to R$ in Theorem 2.4

$$\int_a^b v(y)h(f(y))dy - \int_a^b u(x)h\left(\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(x)}{(x-a)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right]}\right)dx$$
$$= \frac{h''(\eta)}{2}\left[\int_a^b v(y)f^2(y)d\mu_2(y) - \int_a^b u(x)\left(\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(x)}{(x-a)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(x-a)^{\rho}\right]}\right)^2 dx\right].$$

The proof is complete. □

Remark 3.6. Making $\lambda = \alpha, w = 0$ and $\sigma = (1, 0, 0, ...)$ in Theorem 3.5, we have

$$\int_a^b v(y)h(f(y))dy - \int_a^b u(x)h\left(\frac{\Gamma(\alpha+1)\left(I_{a+}^{\alpha}f\right)(x)}{(x-a)^{\alpha}}\right)dx$$
$$= \frac{h''(\eta)}{2}\left[\int_a^b v(y)f^2(y)d\mu_2(y) - \int_a^b u(x)\left(\frac{\Gamma(\alpha+1)\left(I_{a+}^{\alpha}f\right)(x)}{(x-a)^{\alpha}}\right)^2 dx\right].$$

In addition if $u(x) = (x-a)^{\lambda} / \mathcal{F}_{\rho,\lambda+1}^{\sigma}\left[w(x-a)^{\rho}\right] = (x-a)^{\alpha} / \Gamma(\alpha+1)$ and

$$v(y) = \int_y^b (x-y)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}\left[w(x-y)^{\rho}\right] dx = (b-y)^{\alpha} / \Gamma(\alpha+1),$$

we have

$$\int_a^b \frac{(b-y)^{\alpha}}{\Gamma(\alpha+1)}h(f(y))dy - \int_a^b \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}h\left(\frac{\Gamma(\alpha+1)\left(I_{a+}^{\alpha}f\right)(x)}{(x-a)^{\alpha}}\right)dx$$
$$= \frac{h''(\eta)}{2}\left[\int_a^b \frac{(b-y)^{\alpha}}{\Gamma(\alpha+1)}f^2(y)d\mu_2(y) - \int_a^b \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\left(\left(I_{a+}^{\alpha}f\right)(x)\right)^2 dx\right].$$

If $\alpha = 1$ the we get

$$\int_a^b (b-y)\,h(f(y))dy - \int_a^b (x-a)\,h\left(\frac{1}{(x-a)^{\alpha}}\int_0^x f(t)dt\right)dx$$
$$= \frac{h''(\eta)}{2}\left[\int_a^b (b-y)\,f^2(y)dy - \int_a^b (x-a)\left(\int_a^x f(t)dt\right)^2 dx\right].$$

Theorem 3.6. *Let $p, q > 1$ such that $p^{-1} + q^{-1} = 1, \lambda \geq 1$ and $\left(\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma}f\right)$ and $\left(\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma}f\right)$ defined by 2 and 3 respectively. Then we have*

$$\int_a^b \left|\left(\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma}f\right)(x)\right|^p dx \leq C\int_a^b |f(y)dy|$$

and

$$\int_a^b \left|\left(\mathcal{J}_{\rho,\lambda,b-;w}^{\sigma}f\right)(x)\right|^q dx \leq C\int_a^b |f(y)dy|$$

where

$$C = \frac{\left(\mathcal{F}_{\rho,\lambda}^{\sigma}\left[w(b-a)^{\rho}\right]\right)^q}{(p(\lambda-1)+1)^{q/p}}\frac{(b-a)^{q\lambda}}{q\lambda}.$$

Proof. Using Hölder inequality we see that

$$\begin{aligned}
\left|\left(\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma}f\right)(x)\right| &= \left|\int_a^x (x-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\left[w(x-t)^{\rho}\right]f(t)dt\right| \\
&\leq \left(\int_a^x \left((x-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\left[w(x-t)^{\rho}\right]\right)^p dt\right)^{1/p}\left(\int_a^x |f(t)|^q dt\right)^{1/q} \\
&\leq \frac{(x-a)^{(\lambda-1)+1/p}}{(p(\lambda-1)+1)^{1/p}}\left(\mathcal{F}_{\rho,\lambda}^{\sigma}\left[w(x-a)^{\rho}\right]\right)\left(\int_a^b |f(t)|^q dt\right)^{1/q},
\end{aligned}$$

and similarly

$$\left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(x)\right| \leq \frac{(b-x)^{(\lambda-1)+1/p}}{(p(\lambda-1)+1)^{1/p}}\left(\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-x)^{\rho}\right]\right)\left(\int_a^b |f(t)|^q\,dt\right)^{1/q},$$

consequently

$$\left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(x)\right|^q \leq \frac{(x-a)^{q(\lambda-1)+q/p}}{(p(\lambda-1)+1)^{q/p}}\left(\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-a)^{\rho}\right]\right)^q\left(\int_a^b |f(t)|^q\,dt\right)$$

and

$$\left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(x)\right|^q \leq \frac{(b-x)^{q(\lambda-1)+q/p}}{(p(\lambda-1)+1)^{q/p}}\left(\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-x)^{\rho}\right]\right)^q\left(\int_a^b |f(t)|^q\,dt\right).$$

Integrating over $x \in [a,b]$, we have

$$\begin{aligned}
&\int_a^b \left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(x)\right|^q dx \\
&\quad\leq \left(\int_a^b (x-a)^{q(\lambda-1)+q/p}\left(\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-a)^{\rho}\right]\right)^q dx\right)\left(\int_a^b |f(t)|^q\,dt\right) \\
&\quad\leq \frac{\left(\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}\right]\right)^q}{(p(\lambda-1)+1)^{q/p}}\frac{(b-a)^{q(\lambda-1)+q/p+1}}{q(\lambda-1)+q/p+1}\left(\int_a^b |f(t)|^q\,dt\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(x)\right|^q dx \\
&\quad\leq \left(\int_a^b (b-x)^{q(\lambda-1)+q/p}\left(\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-x)^{\rho}\right]\right)^q dx\right)\left(\int_a^b |f(t)|^q\,dt\right) \\
&\quad\leq \frac{\left(\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}\right]\right)^q}{(p(\lambda-1)+1)^{q/p}}\frac{(b-a)^{q(\lambda-1)+q/p+1}}{(q(\lambda-1)+q/p+1)}\left(\int_a^b |f(t)|^q\,dt\right),
\end{aligned}$$

and since $p^{-1}+q^{-1}=1$, we get

$$C = \frac{\left(\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}\right]\right)^q}{(p(\lambda-1)+1)^{q/p}}\frac{(b-a)^{q\lambda}}{q\lambda}$$

we get the desired inequality. □

Remark 3.7. Putting $\lambda=\alpha, w=0$ and $\sigma=(1,0,0,...)$ in Theorem 3.6, we have

$$\int_a^b \left|\left(I^{\alpha}_{a+}f\right)(x)\right|^q dx \leq C\int_a^b |f(t)|^q\,dt$$

and

$$\int_a^b \left|\left(I^{\alpha}_{b-}f\right)(x)\right|^q dx \leq C int_a^b |f(t)|^q\,dt$$

where

$$C=\frac{(b-a)^{q\alpha}}{(\Gamma(\alpha))^{q}\,q\alpha\,(p(\alpha-1)+1)^{q/p}}$$

making coincidence with Theorem 2.6 and Remark 2.7 in [4].

4. Conclusion

As a generalization, using the fractional integral operator $\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)$ defined by R.K. Raina in [9], this work contains some new expressions of others results, exposed in previous investigations [4, 5], about the Hardy's inequality and others improvements. Also, as a contribution to the development of the theory of inequalities, this work is expected to give a motivation to future research on this topic.

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