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Convergent complex uncertain sequences defined by Orlicz function

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Abstract. In this paper we introduce the notion of convergent sequences of complex uncertain variables with respect to measure, mean, distribution etc. defined by an Orlicz function. We have investigated some of the properties of these classes of sequences. We have established some relationships among these notions as well as with other classes of complex uncertain variables.

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1. Introduction

Uncertainty is an extremely important feature of the real world. How do we understand uncertainty? How do we model uncertainty? In order to answer those questions, the notion of uncertainty theory was introduced by Liu [2]. Nowadays uncertainty theory has become a branch of mathematics for modelling human uncertainty.
In this section, we procure some fundamental concepts and theorems in uncertainty theory are introduced, which will be used throughout the paper.

Definition 1.1. (Liu [2]) Let L be a $\sigma-$algebra on a nonempty set Γ. A set function M is called an uncertain measure if it satisfies the following axioms:
Axiom 1. (Normality Axiom) $M\{\Gamma\} = 1$;
Axiom 2. (Duality Axiom) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any $\Lambda \in L$;
Axiom 3. (Subadditivity Axiom) For every countable sequence of $\{\lambda_j\} \in L$, we have

$$M\left\{\bigcup_{j=1}^{\infty} \lambda_j\right\} \leq \sum_{j=1}^{\infty} M\{\lambda_j\}.$$

The triplet (Γ, L, M) is called an uncertainty space, and each element Λ in L is called an event.
In order to obtain an uncertain measure of compound event, a product uncertain measure is define by Liu ([2]) as follows:
Axiom 4. (Product Axiom) Let (Γ_k, L_k, M_k) be uncertainty space for $k = 1, 2, 3, ...$ The product uncertain measure M is an measure satisfying

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, ...,$ respectively.

Definition 1.2. (Liu [2]) An uncertain variable ξ is a measurable function from an uncertainty space (Γ, L, M) to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma : \xi(\gamma) \in B\}$$

is an event.

Definition 1.3. (Liu [2]) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = M\{\xi \leq x\}, \ \forall x \in \mathbb{R}.$$

Definition 1.4. (Liu [2]) The uncertain variables $\xi_1, \xi_2, ..., \xi_n$ are said to be independent if

$$M\left\{\bigcap_{j=1}^{n}(\xi_j \in B_j)\right\} = \bigwedge_{j=1}^{n} M\{\xi_j \in B_j\}$$

for any Borel sets $B_1, B_2, ..., B_n$ of real numbers.

Definition 1.5. (Liu [2]) Let ξ be an uncertain variable. The *expected value* of ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq r\}dr - \int_{-\infty}^{0} M\{\xi \leq r\}dr$$

provided that at least one of the above two integrals is finite.

Considering the importance of the role of convergence of sequence in mathematics, some concepts of convergence for uncertain sequences were introduced by Liu (See for instance [2]) as follows:

Definition 1.6. The uncertain sequence $\{\xi_n\}$ is said to be *convergent almost surely(a.s.) to* ξ if there exists an event Λ with $M\{\Lambda\} = 1$ such that

$$\lim_{n\to\infty} |\xi_n(\gamma) - \xi(\gamma)| = 0,$$

for every $\gamma \in \Lambda$. In that case we write $\xi_n \to \xi$, a.s. as $n \to \infty$

Definition 1.7. The uncertain sequence $\{\xi_n\}$ is said to be *convergent in measure to* ξ if

$$\lim_{n\to\infty} M\{|\xi_n - \xi| \geq \varepsilon\} = 0,$$

for every $\varepsilon > 0$.

Definition 1.8. The uncertain sequence $\{\xi_n\}$ is said to be *convergent in mean to* ξ if

$$\lim_{n\to\infty} E[|\xi_n - \xi|] = 0.$$

Definition 1.9. Let $\Phi, \Phi_1, \Phi_2, ...$ be the uncertainty distributions of uncertain variables $\xi, \xi_1, \xi_2, ...,$ respectively. We say the uncertain sequence $\{\xi_n\}$ *converges in distribution to* ξ if

$$\lim_{n\to\infty} \Phi_n(x) = \Phi(x)$$

for all x at which $\Phi(x)$ is continuous.

Definition 1.10. The uncertain sequence $\{\xi_n\}$ is said to be *convergent uniformly almost surely(a.s.) to* ξ if there exists an sequence of events $\{E_k\}$, $M\{E_k\} \to 0$ such that $\{\xi_n\}$ converges uniformly to ξ in $\Gamma - E_k$, for any fixed k.

Tripathy and Nath [13] have introduced the notion of statistical convergence of sequence of complex uncertain variables and investigated some of their properties. Uncertainty theory has also been studied by Liu ([3], [1], [5]) from different aspects. It is studied from the concept of sequence spaces by You [16] and Chen et.al. [22], from finance point of view by Chen [21] and many others.

2. Complex Uncertain Variable

In this section, we procure some definitions, concepts and results on complex uncertain variables those can be found in Peng [23].
As a complex function on uncertainty space, complex uncertain variable is mainly used to model a complex uncertain quantity.

Definition 2.1. A complex uncertain variable is a measurable function ζ from an uncertainty space (Γ, L, M) to the set of complex numbers, i.e., for any Borel set B of complex numbers, the set

$$\{\zeta \in B\} = \{\gamma \in \Gamma : \zeta(\gamma) \in B\}$$

is an event.

Theorem 2.1. *A variable ζ from an uncertainty space (Γ, L, M) to the set of complex numbers is a complex uncertain variable if and only if Reζ and Imζ are uncertain variables where Reζ and Imζ represent the real and the imaginary part of ζ, respectively.*

Definition 2.2. The complex uncertainty distribution $\Phi(x)$ of a complex uncertain variable ζ is a function from $\mathbb{C}$ to $[0, 1]$ defined by

$$\Phi(c) = M\{Re(\zeta) \leq Re(c), Im(\zeta) \leq Im(c)\}$$

for any complex c.

Theorem 2.2. *A function $\Phi : \mathbb{C} \to [0, 1]$ is a complex uncertainty distribution if and only if it is increasing with respect to the real part Re(c) and imaginary part Im(c) such that*
(i) $\lim_{x\to-\infty} \Phi(x + ib) \neq 1$, $\lim_{y\to-\infty} \Phi(a + iy) \neq 1$, *for any* $a, b \in \mathbb{R}$*;*
(ii) $\lim_{x\to+\infty, y\to+\infty} \Phi(x + iy) \neq 0$,
where $i = \sqrt{-1}$ *is the imaginary unit.*

Definition 2.3. An Orlicz function is a function $\mathcal{M} : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $\mathcal{M}(0) = 0$, $\mathcal{M}(x) > 0$ for $x > 0$ and $\mathcal{M}(x) \to \infty$ as $x \to \infty$.

If convexity of Orlicz function $\mathcal{M}$ is replaced by

$$\mathcal{M}(x + y) \leq \mathcal{M}(x) + \mathcal{M}(y)$$

then this function is called Modulus function, defined and discussed by Ruckle [20]Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to construct the sequence

space
$$\ell_{\mathcal{M}} = \left\{ x \in \omega : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$
The space $\ell_{\mathcal{M}}$ with the norm
$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$
becomes a Banach space Which is called an Orlicz sequence space. Lindenstrauss and Tzafriri [17]proved that every Orlicz sequence space $\ell_{\mathcal{M}}$ contains a subspace isomorphic to c_0 or some ℓ_p, positively for a class of spaces.

The space $\ell_{\mathcal{M}}$ is closely related to the space ℓ_p which is an Orlicz sequence space with $\mathcal{M}(x) = x^p; 1 \leq p \leq \infty$.

Applying the concept of Orlicz function, different classes of sequences have been introduced by Lindenstrauss [18], Tripathy and Borgogain ([14], [15]), Tripathy and Dutta ([7]), Tripathy and Dutta ([11], [12]), Tripathy and Hazarika [8], Tripathy and Sarma [10], Thorpe [6] and investigated their algebraic and topological properties have been investigated. Now we introduce the notion of different types of convergent sequences of complex uncertain sequences defined by an Orlicz function.

Definition 2.4. The sequence spaces given by Orlicz function for the complex uncertain sequences $\{\zeta_n\}$ which are *convergent almost surely(a.s.) to* ζ is
$$c(\mathcal{M}; a.s) = \left\{ \{\zeta_n\} : \lim_{n\to\infty} \|\zeta_n(\gamma) - \zeta(\gamma)\| = 0, \ \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

Definition 2.5. The sequence spaces given by Orlicz function for the complex uncertain sequences $\{\zeta_n\}$ which are *convergent in measure to* ζ is
$$c(\mathcal{M}; m) = \left\{ \{\zeta_n\} : \lim_{n\to\infty} M\{\|\zeta_n - \zeta\| \geq \varepsilon\} = 0, \ \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

Definition 2.6. The sequence spaces given by Orlicz function for the complex uncertain sequences $\{\zeta_n\}$ which are *convergent in mean to* ζ is
$$c(\mathcal{M}; mean) = \left\{ \{\zeta_n\} : \lim_{n\to\infty} E[\|\zeta_n - \zeta\|] = 0, \ \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

Definition 2.7. Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$, respectively. Then the sequence spaces given by Orlicz function for the complex uncertain sequences $\{\zeta_n\}$ which are *converges in distribution to* ζ is
$$c(\mathcal{M}; Dis) = \left\{ \{\zeta_n\} : \lim_{n\to\infty} \Phi_n(c) = \Phi(c) \text{ and } \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

Definition 2.8. The sequence spaces given by Orlicz function for the complex uncertain sequences $\{\zeta_n\}$ which are *convergent uniformly almost surely(u.a.s.) to* ζ is
$$c(\mathcal{M}; u.a.s) = \left\{ \{\zeta_n\} : \zeta_n \to^{u.a.s} \zeta \text{ and } \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

3. Relationship between the Sequence Spaces Introduced

In this section we establish some relationship between the sequence spaces introduced in this article.

Theorem 3.1. *$c(\mathcal{M}; mean) \subset c(\mathcal{M}; m)$ and the inclusion is strict.*

Proof. Let $\zeta_n \in c(\mathcal{M}; mean)$. Then by definition there exists $\zeta \in c(\mathcal{M}; mean)$ such that

$$\lim_{n\to\infty} E[\|\zeta_n - \zeta\|] = 0 \text{ and } \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty \text{ for some } \rho > 0.$$

It follows from the Markov inequality that for any given $\varepsilon > 0$, we have

$$M\{\|\zeta_k - \zeta\| \geq \varepsilon\} \leq \frac{E[\|\zeta_k - \zeta\|]}{\varepsilon} \to 0.$$

Thus $\{\zeta_n\}$ converges in measure to ζ. Hence we get

$$\lim_{n\to\infty} M\{\|\zeta_n - \zeta\| \geq \varepsilon\} = 0 \text{ and } \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty \text{for some } \rho > 0.$$

□

The inclusion is strict follows from the following example.

Example 3.1. Consider the uncertainty space (Γ, L, M) to be $\gamma_1, \gamma_2, ...$ with

$$M\{\Lambda\} = \begin{cases} \sup_{\gamma_n\in\Lambda} \frac{1}{(n+1)}, & \text{if } \sup_{\gamma_n\in\Lambda} \frac{1}{(n+1)} < 0.5 \\ 1 - \sup_{\gamma_n\in\Lambda^c} \frac{1}{(n+1)}, & \text{if } \sup_{\gamma_n\in\Lambda^c} \frac{1}{(n+1)} < 0.5 \\ 0.5, & \text{otherwise} \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & \text{if } \gamma = \gamma_n \\ 0, & \text{otherwise} \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. For some small number $\varepsilon > 0$ and $n \geq 2$, we have

$$M\{\|\zeta_k - \zeta\| \geq \varepsilon\} = M\{\gamma : \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} = M\{\gamma_n\} = \frac{1}{n+1} \to 0$$

as $n \to \infty$. So the sequence $\{\zeta_n\}$ converges in measure to ζ. However, for each $n \geq 2$, we have the uncertainty distribution of uncertain variable $\|\zeta_n - \zeta\| = \|\zeta_n\|$ is

$$\Phi_n(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - \frac{1}{n+1}, & \text{if } 0 \leq x < n+1 \\ 1, & x \geq n+1. \end{cases}$$

So for each $n \geq 2$, we have

$$E[\|\zeta_n - \zeta\|] = \int_0^{n+1} 1 - (1 - \frac{1}{n+1})dx = 1.$$

That is, the sequence $\{\zeta_n\}$ does not converge in mean to ζ.
Hence the result follows.

Theorem 3.2. *$c(\mathcal{M}; m) \subset c(\mathcal{M}; Dis)$ and the inclusion is strict.*

Proof. Let $c = a + ib$ be a given continuity point of the complex uncertainty distribution Φ. On the one hand, for any $\alpha > a, \beta > b$, we have

$$\begin{aligned}\{\xi_n \le a, \eta_n \le b\} =& \{\xi_n \le a, \eta_n \le b, \xi \le \alpha, \eta \le \beta\} \cup \{\xi_n \le a, \eta_n \le b, \xi > \alpha, \eta > \beta\} \\ & \cup \{\xi_n \le a, \eta_n \le b, \xi \le \alpha, \eta > \beta\} \cup \{\xi_n \le a, \eta_n \le b, \xi > \alpha, \eta \le \beta\} \\ & \subset \{\xi \le \alpha, \eta \le \beta\} \cup \{|\xi_n - \xi| \ge \alpha - a\} \cup \{|\eta_n - \eta| \ge \beta - b\}.\end{aligned}$$

It follows from the subadditivity axiom that

$$\Phi_n(c) = \Phi_n(a + ib) \le \Phi(\alpha + i\beta) + M\{|\xi_n - \xi| \ge \alpha - a\} + M\{|\eta_n - \eta| \ge \beta - b\}.$$

Since $\{\xi_n\}$ and $\{\eta_n\}$ converges in measure to ξ and η, respectively, we have $\lim_{n\to\infty} M\{|\xi_k - \xi\| \ge \alpha - a\} = 0$ and $\lim_{n\to\infty} M\{\|\xi_k - \xi\| \ge \beta - b\} = 0$.
Thus we obtain $\limsup_{n\to\infty} \Phi_n(c) \le \Phi(\alpha + i\beta)$ for any $\alpha > a, \beta > b$. Letting $\alpha + i\beta \to a + ib$, we get

$$\limsup_{n\to\infty} \Phi_n(c) \le \Phi(c). \tag{1}$$

On the other hand, for any $x < a, y < b$ we have

$$\begin{aligned}\{\xi \le x, \eta \le y\} =& \{\xi_n \le a, \eta_n \le b, \xi \le x, \eta \le y\} \cup \{\xi_n \le a, \eta_n \le b, \xi \le x, \eta \le y\} \\ & \cup \{\xi_n > a, \eta_n \le b, \xi \le x, \eta \le y\} \cup \{\xi_n > a, \eta_n > b, \xi \le x, \eta \le y\} \\ & \subset \{\xi_n \le a, \eta_n \le b\} \cup \{|\xi_n - \xi| \ge a - x\} \cup \{|\eta_n - \eta| \ge b - y\}.\end{aligned}$$

Which implies

$$\Phi(x + iy) \le \Phi_n(a + ib) + M\{|\xi_n - \xi| \ge a - x\} + M\{|\eta_n - \eta| \ge b - y\}.$$

Since $\lim_{n\to\infty} M\{\|\xi_k - \xi\| \ge a - x\} = 0$ and $\lim_{n\to\infty} M\{\|\xi_k - \xi\| \ge b - y\} = 0$, we obtain $\Phi(x+iy) \le \liminf_{n\to\infty} \Phi_n(a+ib)$ for any $x < a, y < b$. Taking $x+iy \to a+ib$, we get

$$\Phi(c) \le \liminf_{n\to\infty} \Phi_n(c). \tag{2}$$

It follows from (1) and (2) that $\Phi_n(c) \to \Phi(c)$ as $n \to \infty$. That is the complex uncertain sequence $\{\zeta_n\}$ is convergent in distribution to $\zeta = \xi + i\eta$. Hence the result follows. □

The inclusion is strict follows from the following example.

Example 3.2. Consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2\}$ with $M\{\gamma_1\} = M\{\gamma_2\} = \frac{1}{2}$. We define a complex uncertain variable as

$$\zeta(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1 \\ -i, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\zeta_n = -\zeta$ for $n = 1, 2, \ldots$. Then ζ_n and ζ have the same distribution

$$\Phi_n(c) = \Phi_n(a + ib) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty \\ 0, & \text{if } a \ge 0, b < -1 \\ \frac{1}{2}, & \text{if } a \ge 0, -1 \le b < 1 \\ 1, & \text{if } a \ge 0, b \ge 1. \end{cases}$$

Then $\{\zeta_n\}$ convergence in distribution to ζ. However, for a given $\varepsilon > 0$, we have

$$M\{\|\zeta_k - \zeta\| \ge \varepsilon\} = M\{\gamma : \|\zeta_k(\gamma) - \zeta(\gamma)\| \ge \varepsilon\} = 1.$$

That is, the sequence $\{\zeta_n\}$ does not converge in measure to ζ. By Theorem 3.2, the real part and imaginary part of $\{\zeta_n\}$ also do not convergent in measure.

In addition, since $\zeta_n = -\zeta$ for $n = 1, 2, ...,$ the sequence $\{\zeta_n\}$ does not converge a.s to ζ.

Proposition 3.3. *$c(\mathcal{M}; a.s)$ does not imply $c(\mathcal{M}; m)$.*

Proof. The result follows from the following example. □

Example 3.3. Consider the uncertainty space (Γ, L, M) to be $\gamma_1, \gamma_2, ...$ with

$$M\{\Lambda\} = \begin{cases} \sup_{\gamma_n \in \Lambda} \frac{n}{(2n+1)}, & \text{if } \sup_{\gamma_n \in \Lambda} \frac{n}{(2n+1)} < 0.5 \\ 1 - \sup_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)}, & \text{if } \sup_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Then we define a complex uncertain variables as

$$\zeta_n(\gamma) = \begin{cases} in, & \text{if } \gamma = \gamma_n \\ 0, & \text{otherwise} \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. Then the sequence $\{\zeta_n\}$ convergence a.s to ζ. However for some small number $\varepsilon > 0$, we have

$$M\{\|\zeta_k - \zeta\| \geq \varepsilon\} = M\{\gamma : \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} = M\{\gamma_n\} = \frac{n}{2n+1} \to \frac{1}{2}$$

as $n \to \infty$.That is, the sequence $\{\zeta_n\}$ does not converge in measure to ζ.
In addition the complex uncertainty distributions of $\|\zeta_n\|$ are

$$\Phi_n(c) = \phi_n(a + ib) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty \\ 0, & \text{if } a \geq 0, b < 0 \\ 1 - \frac{n}{2n+1}, & \text{if } a \geq 0, 0 \leq b < n \\ 1, & a \geq 0, b \geq n. \end{cases}$$

for $n = 1, 2, ...,$ respectively. And the complex uncertainty distribution of ζ is

$$\Phi(c) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty \\ 0, & \text{if } a \geq 0, b < 0 \\ 1, & a \geq 0, b \geq 0. \end{cases}$$

Clearly $\Phi_n(c)$ does not converge to $\Phi(c)$ at $a \geq 0, b \geq 0$. That is, the sequence $\{\zeta_n\}$ does not converge to ζ in distribution.

Proposition 3.4. *$c(\mathcal{M}; m)$ also does not imply $c(\mathcal{M}; a.s)$.*

Proof. The result follows from the following example. □

Example 3.4. Consider the uncertainty space (Γ, L, M) to be $[0, 1]$ with Borel algebra and Lebesgue measure. For any positive integer n, there is an integer m such that $n = 2^m + k$ where k is an integer between 0 and $2^m - 1$. Then we define a complex uncertain variable by

$$\zeta_n(\gamma) = \begin{cases} i, & \text{if } \frac{k}{2^m} \leq \gamma \leq \frac{(k+1)}{2^m} \\ 0, & \text{otherwise} \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. For some small number $\varepsilon > 0$, we have

$$M\{\|\zeta_k - \zeta\| \geq \varepsilon\} = M\{\gamma : \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} = \frac{1}{2^m} \to 0$$

as $n \to \infty$. So the sequence $\{\zeta_n\}$ converges in measure to ζ. In addition, we have

$$E[\|\zeta_k - \zeta\|] = \frac{1}{2^m} \to 0$$

as $n \to \infty$. Thus the sequence $\{\zeta_n\}$ also converges in mean to ζ.
However, for any $\gamma \in [0, 1]$, there is an infinite number of intervals of the form $[\frac{k}{2^m}, \frac{k+1}{2^m}]$ containing γ. Thus $\zeta_n(\gamma)$ does not converge to 0. In other words, the sequence $\{\zeta_n\}$ does not converge a.s to ζ.

Proposition 3.5. *$c(\mathcal{M}; a.s)$ does not imply $c(\mathcal{M}; mean)$.*

Example 3.5. Consider the uncertainty space (Γ, L, M) to be $\gamma_1, \gamma_2, ...$ with

$$M\{\Lambda\} = \sum_{\gamma_n \in \Lambda} \frac{1}{2^n}.$$

The complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} i2^n, & \text{if } \gamma = \gamma_n \\ 0, & \text{otherwise} \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. Then the sequence $\{\zeta_n\}$ convergence a.s to ζ. However, the uncertainty distributions of $\|\zeta_n\|$ are

$$\Phi_n(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - \frac{1}{2^n}, & \text{if } 0 \le x < 2^n \\ 1, & x \ge 2^n. \end{cases}$$

for $n = 1, 2, ...$, respectively. Then we have

$$E[\|\zeta_k - \zeta\|] = \int_0^{2^n} 1 - (1 - \frac{1}{2^n})dx = 1.$$

So the sequence $\{\zeta_n\}$ does not converge in mean to ζ.
From Example 6, we can obtain $c(\mathcal{M}; mean)$ does not imply $c(\mathcal{M}; a.s)$.

Proposition 3.6. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then $\{\zeta_n\}$ converges a.s to ζ if and only if for any $\varepsilon > 0$, we have*

$$M\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{\|\zeta_n - \zeta\| \ge \varepsilon\}\right) = 0.$$

Proof. By the definition of convergence a.s., we have that there exists an event Λ with $M\{\Lambda\} = 1$ such that $\lim_{n\to\infty} \|\zeta_n - \zeta\| = 0$. Then for any $\varepsilon > 0$, there exists k such that $\|\zeta_n - \zeta\| < \varepsilon$ where $n > k$ and for any $\gamma \in \Lambda$, that is equivalent to

$$M\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{\|\zeta_n - \zeta\| < \varepsilon\}\right) = 1.$$

It follows from the duality axiom of uncertain measure that

$$M\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{\|\zeta_n - \zeta\| \ge \varepsilon\}\right) = 0.$$

□

Proposition 3.7. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then $\{\zeta_n\}$ converges uniformly a.s to ζ if and only if for any $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} M\left(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right) = 0.$$

Proof. If $\{\zeta_n\}$ converges uniformly a.s to ζ, then for any $\delta > 0$ there exists B such that $M\{B\} < \delta$ and $\{\zeta_n\}$ uniformly converges to ζ on $\Gamma - B$. Thus, for any $\varepsilon > 0$, there exists $k > 0$ such that $\|\zeta_n - \zeta\| < \varepsilon$ where $n \geq k$ and $\gamma \in \Gamma - B$. That is

$$\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\} \subset B.$$

It follows from the subadditivity axiom that

$$M\left(\bigcup_{n=k}^{\infty}\{\|\zeta_k - \zeta\| \geq \varepsilon\}\right) \leq M\{B\}) < \delta.$$

Then

$$\lim_{n\to\infty} M\left(\bigcup_{n=k}^{\infty}\{\|\zeta_k - \zeta\| \geq \varepsilon\}\right) = 0.$$

On the contrary, if $\lim_{n\to\infty} M(\cup_{n=k}^{\infty}\{\|\zeta_k - \zeta\| \geq \varepsilon\}) = 0.$ for any $\varepsilon > 0$, then for given $\delta > 0$ and $m \geq 1$, there exists m_k such that

$$M\left(\bigcup_{n=m_k}^{\infty}\{\|\zeta_n - \zeta\| \geq \frac{1}{m}\}\right) < \frac{\delta}{2^m}.$$

Let $B = \cup_{m=1}^{\infty} \cup_{n=m_k}^{\infty} \{\|\zeta_n - \zeta\| \geq \frac{1}{m}\}$. Then

$$M\{B\} \leq \sum_{m=1}^{\infty} M(\bigcup_{n=m_k}^{\infty}\{\|\zeta_n - \zeta\| \geq \frac{1}{m}\}) \leq \sum_{m=1}^{\infty}\frac{\delta}{2^m} = \delta.$$

Furthermore, we have

$$\sup_{\gamma\in\Gamma-B} \|\zeta_n - \zeta\| < \frac{1}{m}$$

for any $m = 1, 2, ...$ and $n > m_k$. The proposition is thus proved. □

Theorem 3.8. *If $\{\zeta_n\} \in c(\mathcal{M}; u.a.s)$, then $\{\zeta_n\} \in c(\mathcal{M}; a.s)$.*

Proof. It follows from above Proposition that if $\{\zeta_n\}$ converges uniformly a.s to ζ, then

$$\lim_{n\to\infty} M\left(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right) = 0.$$

Since

$$M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right) \leq M\left(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right),$$

taking the limit as $n \to \infty$ on both side of above inequality, we obtain

$$M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right) = 0.$$

By Proposition 3.4, $\{\zeta_n\}$ converges a.s to ζ. □

Theorem 3.9. *If a complex uncertain sequence* $\{\zeta_n\} \in c(\mathcal{M}; u.a.s)$*, then* $\{\zeta_n\} \in c(\mathcal{M}; m)$*.*

Proof. If $\{\zeta_n\}$ converges uniformly a.s. to ζ, then from Proposition above we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \lim_{n\to\infty} M\left(\bigcup_{n=k}^{\infty} \|\zeta_k - \zeta\| \ge \varepsilon\right) \ge \delta\}| = 0.$$

And

$$\delta\left(M\{\|\zeta_n - \zeta\| \ge \varepsilon\}\right) \le \left(M\left(\bigcup_{n=k}^{\infty} \{\|\zeta_n - \zeta\| \ge \varepsilon\}\right)\right).$$

Letting $n \to \infty$, we can obtain $\{\zeta_n\}$ converges in measure to ζ. □

As in seen from Example 3.4, $\{\zeta_n\}$ converges in measure to ζ. However, it does not converges a.s. to $zeta$. It follows from above Theorem that $\{\zeta_n\}$ does not converges uniformly a.s. to ζ.

4. Conclusion

In this paper we have defined different types of convergent sequences of complex uncertain variables defined by an Orlicz function. We have studied their different properties and established the relationship between them. The idea can be applied for introducing some other classes of sequences of complex uncertain variables and study their properties.

References

[1] B. Liu, Uncertainty Theory: A Branch of Mathematics for Modelling Human Uncertainty, *Springer-Verlag*, Berlin, 2010.

[2] B. Liu, Uncertainity Theory, *Springer-Verlag*, 5th Ed., Berlin, 2016.

[3] B. Liu, Some research problems in uncertainty theory, *Journal of Uncertain Systems* **3** (2009), no. 1, 3–10.

[4] B. Liu, Uncertainty risk analysis and uncertain reliabilityanalysis, *Journal of Uncertain Systems* **4**(2010), no.3, 163–170.

[5] B. Liu, Why is there a need for Uncertainty Theory?, *Journal of Uncertain Systems* **6** (2012), no. 1, 3–10.

[6] B. Thorpe, Sequence spaces defined by Orlicz functions, *Bulletin of the London Mathematical Society* **13** (1981), 301–302.

[7] B.C. Tripathy, A.J. Dutta, Statistically pre-Cauchy fuzzy real-valued sequences defined by Orlicz function, *Proyecciones Journal of Mathematics* **33** (2014), no. 3, 235–243.

[8] B.C. Tripathy, B. Hazarika, I-convergent sequences spaces defined by Orlicz function, *Acta Mathematica Applicatae Sinica* **27** (2011), no. 1, 149–154.

[9] B.C. Tripathy and B. Sarma, Vector valued double sequence spaces defined by Orlicz function, *Mathematica Slovaca*, **59** (2009), no. 6, 767–776.

[10] B.C. Tripathy and B. Sarma, Double sequence spaces of fuzzy numbers defined by Orlicz function, *Acta Mathematica Scientia* **31** (2011), no. 1, 134–140.

[11] B.C. Tripathy, H. Dutta, On some new paranormed difference sequence spaces defined by Orlicz functions, *Kyungpook Mathematical Journal* **50** (2010), 59–69.

[12] B.C. Tripathy, H. Dutta, On some lacunary difference sequence spaces defined by a sequence of Orlicz functions and q-lacunaryΔ_m^n-statistical convergence, *Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica* **20** (2012), no. 1, 417–430.

[13] B.C. Tripathy, P.K. Nath, Statistical Convergent of complex uncertain sequences, *New Mathematics and Natural Computing* **13** (2017), no. 3, 359-374.

[14] B.C. Tripathy, S. Borgogain, Some classes of differencesequence spaces of fuzzy real numbers defined by Orlicz function, *Advances in Fuzzy Systems* **(2011)**, Article ID216414, 6 pages.
[15] B.C. Tripathy, S. Borgogain, Sequence space $m(M, \phi)^F$ of fuzzy real numbers defined by Orlicz functions with fuzzy metric, *Kyungpook Mathematical Journal* **53** (2013), no. 3, 319–332.
[16] C.L. You, On the convergence of uncertain sequences, *Mathematical and Computer Modelling* **49** (2009), 482–487.
[17] J. Lindenstrauss, L. Tzafriri, On Orlicz sequence spaces, *Isreal Journal of Mathematics* **10** (1971), 379–390.
[18] J. Lindenstrauss, Some sequence spaces defined by Orlicz functions, *Advances in Mathematics* **5** (1970), 159–180.
[19] M.A. Krasnoselskii, Y.B. Rutitsky, *Convex Function and Orlicz Spaces*, Groningen, Metherlands, 1961.
[20] W.H. Ruckle, FK spaces in which the sequence of coordinate vectors in bounded, *Candian Journal of Mathematics* **25** (1973), 973–978.
[21] X.W. Chen, American option pricing formula for uncertain financial market, *International Journal of Operations Research*, **8** (2011), no. 2, 32–37.
[22] X. Chen, Y. Ning, X. Wang, Convergence of Complex Uncertain Sequences, *Journal of Intelligent and Fuzzy Systems*, **30** (2016), no. 6, 3357–3366.
[23] Z.X. Peng, Complex Uncertain Variable, *Doctoral Dissertation*, Tsinghua University (2012).

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