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The refinements of Hilbert-type inequalities in discrete case

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Abstract. In this article we obtain a refinement of Hilbert-type inequality in discrete case. We derive a pair of equivalent inequalities, and also establish some applications in particular settings.

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1. Introduction

Some of the recent results concerning Hilbert's inequality include extension to multidimensional case, equipped with conjugate exponents p_i, that is, $\sum_{i=1}^n 1/p_i = 1$, $p_i > 1$, $n \geq 2$ (see papers [3], [5] and [7]). Here we refer to paper [1], which provides a unified treatment of the multidimensional Hilbert-type inequality in the setting with conjugate exponents. Suppose $(\Omega_i, \Sigma_i, \mu_i)$ are σ-finite measure spaces and $K : \prod_{i=1}^n \Omega_i \to \mathbb{R}$, $\phi_{ij} : \Omega_j \to \mathbb{R}$, $f_i : \Omega_i \to \mathbb{R}$, $i, j = 1, 2, \ldots, n$, are non-negative measurable functions. If $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$, then the following inequalities hold and are equivalent

$$\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i) d\mu(\mathbf{x}) \leq \prod_{i=1}^n \|\phi_{ii}\omega_i f_i\|_{p_i}, \tag{1}$$

and

$$\left[\int_{\Omega_n} \left(\frac{1}{(\phi_{nn}\omega_n)(x_n)} \int_{\hat{\mathbf{\Omega}}^n} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d\hat{\mu}^n(\mathbf{x})\right)^P d\mu(x_n)\right]^{1/P} \leq \prod_{i=1}^{n-1} \|\phi_{ii}\omega_i f_i\|_{p_i}, \tag{2}$$

where

$$\omega_i(x_i) = \left[\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^n \phi_{ij}^{p_i}(x_j) d\hat{\mu}^i(\mathbf{x})\right]^{1/p_i}, \tag{3}$$

and

$$1/P = \sum_{i=1}^{n-1} (1/p_i), \quad \mathbf{\Omega} = \prod_{i=1}^n \Omega_i, \quad \hat{\mathbf{\Omega}}^i = \prod_{j=1, j\neq i}^n \Omega_j, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n),$$
$$d\mu(\mathbf{x}) = \prod_{i=1}^n d\mu_i(x_i), \quad d\hat{\mu}^i(\mathbf{x}) = \prod_{j=1, j\neq i}^n d\mu_j(x_j). \tag{4}$$

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The abbreviations as in (3) and (4) will be used throughout the whole paper. Also note that $\|\cdot\|_{p_i}$ denotes the usual norm in $L^{p_i}(\Omega_i)$, that is

$$\|\phi_{ii}\omega_i f_i\|_{p_i} = \left[\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i)d\mu_i(x_i)\right]^{1/p_i}, \ i = 1, 2, \ldots, n.$$

Our results will be based on the mentioned results of Krnić and Vuković from [6]. Here are obtained the refinements of inequalities (1) and (2). These results are given in Theorems A and B.

Theorem A. *Let $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces, and let $K : \mathbf{\Omega} \to \mathbb{R}$, $\phi_{ij} : \Omega_j \to \mathbb{R}$, $f_i : \Omega_i \to \mathbb{R}$, $i, j = 1, 2, \ldots, n$, be non-negative measurable functions. If $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$ and the functions $F_i : \mathbf{\Omega} \to \mathbb{R}$ are defined by*

$$F_i(\mathbf{x}) = K^{\frac{1}{p_i}}(\mathbf{x}) f_i(x_i) \prod_{j=1}^n \phi_{ij}(x_j), \quad i = 1, 2, \ldots, n,$$

then

$$\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i) d\mu(\mathbf{x}) \le n^{\frac{n}{M}} G(F_1, F_2, \ldots, F_n) \prod_{i=1}^n \|\phi_{ii}\omega_i f_i\|_{p_i}^{1-\frac{p_i}{M}}, \tag{5}$$

where $M = \max_{1\le i\le n} p_i$, ω_i is defined by (3), $\phi_{ii}\omega_i f_i \in L^{p_i}(\Omega_i)$, $i = 1, 2, \ldots, n$, and G is defined by

$$G(F_1, F_2, \ldots, F_n) = \int_{\mathbf{\Omega}} \left[\sum_{i=1}^n \frac{F_i^{p_i}(\mathbf{x})}{p_i \|F_i\|_{p_i}^{p_i}}\right] \left[\frac{\prod_{i=1}^n F_i^{\frac{p_i}{n}}(\mathbf{x})}{\sum_{i=1}^n \frac{F_i^{p_i}(\mathbf{x})}{\|F_i\|_{p_i}^{p_i}}}\right]^{\frac{n}{M}} d\mu(\mathbf{x}). \tag{6}$$

Theorem B. *Suppose that the assumptions as in Theorem A are fulfilled. Then,*

$$\left[\int_{\Omega_n} \left(\frac{1}{(\phi_{nn}\omega_n)(x_n)} \int_{\hat{\mathbf{\Omega}}^n} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d\hat{\mu}^n(\mathbf{x})\right)^P d\mu(x_n)\right]^{\frac{1}{P}+\frac{1}{M}}$$
$$\le n^{\frac{n}{M}} G(F_1, F_2, \ldots, F_{n-1}, \widetilde{F}_n) \prod_{i=1}^{n-1} \|\phi_{ii}\omega_i f_i\|_{p_i}^{1-\frac{p_i}{M}}, \tag{7}$$

where $\frac{1}{P} = \sum_{i=1}^{n-1} \frac{1}{p_i}$ and

$$\widetilde{F}_n(\mathbf{x}) = \frac{K^{\frac{1}{p_n}}(\mathbf{x})}{(\phi_{nn}\omega_n)^P(x_n)} \left(\int_{\hat{\mathbf{\Omega}}^n} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d\hat{\mu}^n(\mathbf{x})\right)^{P-1} \prod_{j=1}^n \phi_{nj}(x_j).$$

Our main aim in this paper is to obtain a refinement of Hilbert-type inequality in discrete case.

Techniques that will be used in the proofs are mainly based on the claaical real analysis. Further, throughout this paper all the functions are assumed to be non-negative and measurable. Also, all series and integrals are assumed to be convergent.

2. Main results

Rewrite the inequalities (5) and (7) for the counting measure on $\mathbb{N}$, with $\mathbf{m} := (m_1, \dots, m_n) \in \mathbb{N}^n$, and the sequences $(a_{m_i}^{(i)})$, $a_{m_i}^{(i)} \geq 0$, $m_i \in \mathbb{N}$, $i = 1, \dots, n$. Thus, in the above setting we have

$$\omega_i^{p_i}(m_i) = \sum_{m_n=1}^{\infty} \cdots \sum_{m_{i+1}=1}^{\infty} \sum_{m_{i-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} K(\mathbf{m}) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{p_i}(m_j), \tag{8}$$

$$F_i(\mathbf{m}) = K^{1/p_i}(\mathbf{m}) a_{m_i}^{(i)} \prod_{j=1}^{n} \phi_{ij}(x_j), \quad i = 1, 2, \dots, n, \tag{9}$$

and

$$G(F_1, \dots, F_n) = \sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left(\sum_{i=1}^{n} \frac{F_i^{p_i}(\mathbf{m})}{p_i \|F_i\|_{p_i}^{p_i}} \right) \left[\frac{\prod_{i=1}^{n} F_i^{\frac{p_i}{n}}(\mathbf{m})}{\sum_{i=1}^{n} \frac{F_i^{p_i}(\mathbf{m})}{\|F_i\|_{p_i}^{p_i}}} \right]^{\frac{n}{M}}, \tag{10}$$

where $M = \max_{1 \leq i \leq n} p_i$.

Now, regarding the above notations and definitions we have the following theorems.

Theorem 1. *Let p_i, P, ϕ_{ij}, $i, j = 1, \dots, n$, be defined as in Theorem A. Let $K : \mathbb{R}_+^n \to \mathbb{R}$ be non-negative function strictly decreasing in each variable and $a^{(i)} = (a_{m_i}^{(i)})_{m_i \in \mathbb{N}}$, $(a_{m_i}^{(i)} \geq 0)$, $i = 1, \dots, n$. Then the following inequalitiy holds*

$$\sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} K(\mathbf{m}) \prod_{i=1}^{n} a_{m_i}^{(i)} \leq n^{\frac{n}{M}} G(F_1, \dots, F_n) \prod_{i=1}^{n} \|\phi_{ii} \omega_i a^{(i)}\|_{p_i}^{1 - \frac{p_i}{M}}, \tag{11}$$

where $\omega_i(m_i)$, F_i, $i = 1, 2, \dots, n$, and $G(F_1, \dots, F_n)$ are defined by (8), (9) *and* (10) *respectively.*

By using Theorem B we obtain the refinement of Hardy-Hilbert's type inequality.

Theorem 2. *Suppose that the assumptions as in Theorem 1 are fulfilled. Then,*

$$\left[\sum_{m_n=1}^{\infty} \frac{1}{(\phi_{nn}\omega_n)(m_n)} \left(\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} K(\mathbf{m}) \prod_{i=1}^{n-1} a_{m_i}^{(i)} \right)^{P} \right]^{\frac{1}{P} + \frac{1}{M}}$$
$$\leq n^{\frac{n}{M}} G(F_1, F_2 \dots F_{n-1}, \widetilde{F}_n) \prod_{i=1}^{n-1} \|\phi_{ii} \omega_i a^{(i)}\|_{p_i}^{1 - \frac{p_i}{M}}, \tag{12}$$

where G is defined by (10) *and*

$$\widetilde{F}_n(\mathbf{m}) = \frac{K^{\frac{1}{p_n}}(\mathbf{m})}{(\phi_{nn}\omega_n)(m_n)} \left(\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} K(\mathbf{m}) \prod_{i=1}^{n-1} a_{m_i}^{(i)} \right)^{P-1} \prod_{j=1}^{n} \phi_{nj}(m_j). \tag{13}$$

If we put the parameters

$$A_{ii} = \frac{p_i - 1}{p_i^2}, \quad A_{ij} = -\frac{1}{p_i p_j}, i \neq j, \quad i, j = 1, \dots, n, \tag{14}$$

and the kernel $K(x_1,\dots,x_n)=(x_1+\beta_1^2x_2)^{-1}(x_1+\beta_2^2x_3)^{-1}\dots(x_1+\beta_{n-1}^2x_n)^{-1}$, $\beta_i>0$, $i=1,\dots,n-1$, in inequalities (5) and (7), then we obtain the following result.

Corollary 3. *Let P, M, p_i and $a^{(i)}=(a^{(i)}_{m_i})_{m_i\in\mathbb{N}}$, $i=1,\dots,n$, be defined as in Theorem 1. Suppose that $\beta_i>0$, $i=1,\dots,n-1$. Then the following inequalities hold*

$$\sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^n a^{(i)}_{m_i}}{(m_1+\beta_1^2m_2)(m_1+\beta_2^2m_3)\dots(m_1+\beta_{n-1}^2m_n)} \le L\cdot G(F_1,F_2,\dots,F_n)\prod_{i=1}^n\|a^{(i)}\|_{p_i}^{1-\frac{p_i}{M}}, \tag{15}$$

and

$$\left[\sum_{m_n=1}^{\infty}\left(\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n-1} a^{(i)}_{m_i}}{(m_1+\beta_1^2m_2)(m_1+\beta_2^2m_3)\dots(m_1+\beta_{n-1}^2m_n)}\right)^P\right]^{\frac{1}{P}+\frac{1}{M}} \le L\cdot G(F_1,F_2,\dots,F_{n-1},\widetilde{F}_n)\prod_{i=1}^{n-1}\|a^{(i)}\|_{p_i}^{1-\frac{p_i}{M}}, \tag{16}$$

where

$$L=\pi^{n-1}\prod_{i=2}^n\frac{\beta_i^{\frac{2(1-p_i)}{p_i}}}{\sin\frac{\pi}{p_i}}, \tag{17}$$

and F_i, $i=1,2,\dots,n$, G, and $\widetilde{F}_n$ are defined by (9), (10) *and* (13) *respectively.*

Proof. Set $\phi_{ij}(x_j)=x_j^{A_{ij}}$ in Theorem 1, where the parameters A_{ij}, $i,j=1,\dots,n$, are defined by (14). Since these parameters A_{ij} are symmetric one obtains $\alpha_i=\sum_{j=1}^n A_{ij}=0$, for $i=1,\dots,n$. It is enough to calcute the functions $\omega_i(m_i)$, $i=1,\dots,n$. Since the kernel $K(x_1,\dots,x_n)=\prod_{i=2}^n(x_1+\beta_{i-1}^2x_i)^{-1}$ is strictly decreasing in each variable, we conclude that the functions $\omega_i(m_i)$, $i=1,\dots,n$, are strictly decreasing. Hence, we have

$$\omega_1^{p_1}(m_1)=\sum_{m_n=1}^{\infty}\cdots\sum_{m_2=1}^{\infty}\frac{\prod_{j=2}^n m_j^{-\frac{1}{p_j}}}{(m_1+\beta_1^2m_2)(m_1+\beta_2^2m_3)\dots(m_1+\beta_{n-1}^2m_n)} \le\int_{\mathbb{R}_+^{n-1}}\frac{\prod_{j=2}^n x_j^{-\frac{1}{p_j}}}{(m_1+\beta_1^2x_2)(m_1+\beta_2^2x_3)\dots(m_1+\beta_{n-1}^2x_n)}dx_2\dots dx_n, \tag{18}$$

since the left-hand side of this inequality is obviously the lowe Darboux sum for the integral on the right-hand side of inequality. Further, by using the substitution

$t_i = x_i/m_1$, $i = 2, \ldots, n$, from (18) we get

$$\omega_1^{p_1}(m_1) \le m_1^{\frac{1}{p_1}-1} \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{j=2}^n t_j^{-\frac{1}{p_j}}}{(1+\beta_1^2 t_2)(1+\beta_2^2 t_3)\ldots(1+\beta_{n-1}^2 t_n)} dt_2 \ldots dt_n$$
$$= m_1^{\frac{1}{p_1}-1} \prod_{i=2}^n \left(\int_{\mathbb{R}_+} \frac{t_i^{-\frac{1}{p_i}}}{1+\beta_{i-1}^2 t_i} dt_i \right) = L m_1^{\frac{1}{p_1}-1}, \tag{19}$$

where the constant L is defined by (17). By using the same arguments as for the function $\omega_1(m_1)$, we also get

$$\omega_2(m_2) \le \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{j=2}^n x_j^{-\frac{1}{p_j}}}{(x_1+\beta_1^2 m_2)(x_1+\beta_2^2 x_3)\ldots(x_1+\beta_{n-1}^2 x_n)} dx_1 dx_3 \ldots dx_n. \tag{20}$$

Now, let J denotes the right-hand side of the inequality (20). It is easy to see that the transformation of variables

$$x_1 = m_2 \frac{1}{t_2}, \quad x_i = m_2 \frac{t_i}{t_2}, \quad i = 3, \ldots, n,$$

yields

$$\frac{\partial(x_1, x_3, \ldots, x_n)}{\partial(t_2, t_3, \ldots, t_n)} = m_2^{n-1} t_2^{-n},$$

where $\frac{\partial(x_1,x_3,\ldots,x_n)}{\partial(t_2,t_3,\ldots,t_n)}$ denotes the Jacobian of the transformation. Now, by using the above change of variables, we have

$$J = \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{j=2}^n x_j^{-\frac{1}{p_j}}}{x_1^{n-1}\left(1+\beta_1^2 \frac{m_2}{x_1}\right)\left(1+\beta_2^2 \frac{x_3}{x_1}\right)\cdots\left(1+\beta_{n-1}^2 \frac{x_n}{x_1}\right)} dx_1 dx_3 \ldots dx_n$$
$$= m_2^{\frac{1}{p_2}-1} \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{j=2}^n t_j^{-\frac{1}{p_j}}}{(1+\beta_1^2 t_2)(1+\beta_2^2 t_3)\ldots(1+\beta_{n-1}^2 t_n)} dt_2 dt_3 \ldots dt_n$$
$$= L m_2^{\frac{1}{p_2}-1},$$

where the constant L is defined by (17). In a similar manner we obtain

$$\omega_i^{p_i}(m_i) \le L m_i^{\frac{1}{p_i}-1}, \quad i = 3, \ldots, n.$$

The estimates for the functions $\omega_i(m_i)$, $i = 1, \ldots, n$, yields

$$\|\phi_{ii}\omega_i a^{(i)}\|_{p_i} \le L^{\frac{1}{p_i}} \|a^{(i)}\|_{p_i}, \quad i = 1, \ldots, n.$$

Now, from Theorems 1 and 2 we get the inequalities (15) and (16). □

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