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# Applications of Riesz mean and lacunary sequences to generate Banach spaces and AK-BK spaces

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Abstract. In this paper we establish some wide-ranging spaces of sequences generated by Riesz mean associated with lacunary sequences and multiplier sequences of Orlicz function. We have encompassed some topological and algebraic properties of these sequence spaces. We also make an effort to prove that these spaces are Banach and AK-BK spaces. Finally, we prove that these sequence spaces are topologically isomorphic.

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## 1. Introduction

Let $l_\infty, c$ and $c_0$, respectively, denotes the spaces of all bounded, convergent and null sequences. Also by $l_1, l(p), cs$ and $bs$ we denote the space of all absolutely, p-absolutely convergent, convergent and bounded series, respectively. Let $w$ be the space of all real or complex sequences. Any linear subspace of $w$ is called a sequence space. A sequence space $X$ with linear topology is called a $K$-space provided each of map $p_k : X \to \mathbb{R}$ defined by $p_k(x) = x_k$ is continuous for all $k \in \mathbb{N}$. A $K$-space $X$ is called an $FK$-space provided $X$ is a complete linear metric space. In other words, we say that $X$ is $FK$-space, if $X$ is Fréchet space with continuous coordinate projection, we mean if $x^n \to x$ $(n \to \infty)$ in the metric of $X$, then $x_k^{(n)} \to x_k$ $(n \to \infty)$ for each $k \in \mathbb{N}$. That is, for each $k \in \mathbb{N}$, the linear functional $p_k(x) = x_k$ is such that $p_k$ is continuous on $X$. An $FK$-space whose topology is normable is called a $BK$-space [3]. The space $l_p (1 \le p < \infty)$ is a $BK$-space with $\|x\|_p = \Big(\sum_{k=0}^{\infty} |x_k|^p\Big)^{\frac{1}{p}}$ and $c_0$, $c$ and $l_\infty$ are $BK$-spaces with $\|x\|_\infty = \sup_k |x_k|$.

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for any sequence space $E$, the *multiplier sequence space* $E(\Lambda)$ of $E$, associated with the multiplier sequence $\Lambda$ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

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The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. G. Goes and S. Goes defined the differentiated sequence spaces $dE$ and integrated sequence space $\int E$ for a given sequence space $E$, using the multiplier sequence $(\frac{1}{k})$ and $(k)$ in [8], respectively. A multiplier sequence can be used to accelerate the convergence of sequences in some spaces. In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction. Sometimes the associated multiplier sequence delays the rate of convergence of a sequence. Thus, it also covers a larger class of sequences for study.

An *Orlicz function* being continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$ was used by Lindenstrauss and Tzafriri [14] to define the following sequence space. Let $w$ be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. The space $\ell_M$ is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1\Big\}.$$

[14] reflects that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p (p \ge 1)$. For given set of values of $x \ge 0$ and $L > 1$, the $\Delta_2-$condition is equivalent to $M(Lx) \le kLM(x)$. An Orlicz function $M$ can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where $\eta$ is known as the kernel of $M$, is a right differentiable for $t \ge 0$, $\eta(0) = 0$, $\eta(t) > 0$, $\eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$ whenever $\frac{M(x)}{x} \uparrow \infty$ as $x \uparrow \infty$.
Consider the kernel $\eta$ associated with the Orlicz function $M$ and let

$$\zeta(s) = \sup\{t : \eta(t) \le s\}$$

Then, $\zeta$ possesses the same properties as the function $\eta$. Now suppose

$$N(x) = \int_0^x \zeta(s)ds.$$

Then, $N$ is an Orlicz function. The functions $M$ and $N$ are called *mutually complementary Orlicz functions.*
Let $M$ and $N$ be mutually complementary Orlicz functions. Then, we have
(i) For all $x, y \ge 0$,

$$xy \le M(x) + N(y). \tag{1}$$

(ii) For all $x \ge 0$,

$$x\eta(x) = M(x) + N(\eta(x)), \quad \text{(Young's inequality)}. \tag{2}$$

(iii) For all $x \ge 0$ and $0 < \lambda < 1$,

$$M(\lambda u) < \lambda M(u). \tag{3}$$

Two Orlicz functions $M_1$ and $M_2$ are said to be *equivalent* if there are positive constants $\alpha, \beta$ and $b$ such that

$$vM_1(\alpha x) \leq M_2(x) \leq M_1(\beta x). \tag{4}$$

For relevant terminology and additional knowledge on the Orlicz sequence spaces and related topics, the reader may refer to ([1], [4], [6], [9], [10], [15], [18], [23]).

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function* (see [16], [17]). Complementary function where $\mathcal{N} = (N_k)$, defined as

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is derived from the Musielak-Orlicz function $\mathcal{M}$.
Initially introduced by Kızmaz [11], the notion of *difference sequence spaces* was conceptualized as $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$. Further, the notion was generalized by Et and Çolak [5] as they introduced the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$.
Let $m$, $n$ be non-negative integers, then for $Z$ a given sequence space, we have

$$Z(\Delta^n_m) = \{x = (x_k) \in w : (\Delta^n_m x_k) \in Z\}$$

for $Z = c, c_0$ and $l_\infty$ where $\Delta^n_m x = (\Delta^n_m x_k) = (\Delta^{n-1}_m x_k - \Delta^{n-1}_m x_{k+m})$ and $\Delta^0_m x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n_m x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+mv}.$$

If $m = 1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$ as studied by Et and Çolak [5]. If $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$ as introduced and studied by Kızmaz [11]. For more details about sequence spaces (see [12], [13], [19], [20], [21], [22]) and references therein.
An increasing sequence of non-negative integers $h_r = (i_r - i_{r-1}) \to \infty$ as $r \to \infty$ can be made through *lacunary sequence* $\theta = (i_r)$, $r = 0, 1, 2, \cdots$, where $i_0 = 0$. The intervals determined by $\theta$ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio $i_r/i_{r-1}$ will be denoted by $q_r$. Freedman [7] defined the space of lacunary strongly convergent sequences $N_\theta$ as:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0 \ \text{ for some } \ L\Big\}.$$

Let $t = (t_k)$ be a sequence of non-negative real numbers with $t_0 > 0$ and let us write, $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then the matrix $R^t = (r^t_{nk})$ of the Riesz mean $(R, t_n)$ is given by

$$r^t_{nk} = \begin{cases} \frac{t_k}{T_n}, & \text{if } \ 0 \leq k \leq n, \\ \\ 0 & \text{if } \ k > n \end{cases}$$

The Riesz mean $(R, t_n)$ is regular if and only if $T_n \to \infty$ as $n \to \infty$ (see [2]).

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $\delta(\mathcal{M}, x) = \sum_k M_k(|x_k|)$ where $x = (x_k) \in w$. Let $t = (t_k)$ be a sequence of positive real numbers, $p = (p_k)$ be a bounded sequence of positive real numbers. Let $\theta$ be a lacunary sequence. Then, we

define the following new sequence spaces of lacunary convergent sequences combining with Riesz mean:
$\tilde{l}_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s) =$

$$\Big\{x = (x_k) \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{T_k}\Big)^{p_k} < \infty\Big\}$$

and

$$\tilde{l}_{\mathcal{M}} = \{x = (x_k) \in w : \delta(\mathcal{M}, x) < \infty.$$

Let $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ be mutually complementary functions. Then we define the sequence space as:
$l_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s) =$

$$\Big\{x = (x_k) \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big(\frac{\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j}{T_k}\Big)^{p_k} y_k \quad \text{converges for all } y = (y_k) \in \tilde{l}_{\mathcal{N}}\Big\}.$$

The paper lays an emphasis upon the introduction of new sequence spaces $\tilde{l}_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$, $l_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$, $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ and $h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. An attempt has been made to study some algebraic and topological properties between these sequence spaces. We also study some inclusion relations between these sequence spaces and prove that these spaces are normed linear space, Banach space and AK-BK space.

## 2. Main Results

**Theorem 2.1.** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $t = (t_k)$ be a sequence of positive real numbers, $p = (p_k)$ be a bounded sequence of positive real numbers and $\theta$ be a lacunary sequence. Then the inclusion $\tilde{l}_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s) \subset l_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ holds.*

*Proof.* Let $x = (x_k) \in \tilde{l}_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. Then we have,

$$\lim_{r} \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{T_k}\Big)^{p_k} < \infty.$$

By Young's inequality, we have

$$\begin{aligned} \frac{1}{h_r}\Big|\sum_{k\in I_r} \Big(\frac{\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j}{T_k}\Big)^{p_k} y_k\Big| &\le \frac{1}{h_r} \sum_{k\in I_r} \Big|\Big(\frac{\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j}{T_k}\Big)^{p_k} y_k\Big| \\ &\le \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(\Big|\frac{\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j}{T_k}\Big|\Big)^{p_k} + \frac{1}{h_r} \sum_{k\in I_r} N_k(|y_k|) \\ &< \infty \end{aligned}$$

for every $y = (y_k) \in \tilde{l}_{\mathcal{N}}$. Thus, $x = (x_k) \in l_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. □

**Theorem 2.2.** *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $t=(t_k)$ be a sequence of positive real numbers, $p=(p_k)$ be a bounded sequence of positive real numbers and $\theta$ be a lacunary sequence. For each $x=(x_k)\in l_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$,*

$$\sup_r\Big\{\frac{1}{h_r}\Big|\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^k\lambda_j t_j\Delta^s x_j}{T_k}\Big)^{p_k}y_k\Big|:\delta(\mathcal{N},y)\le 1\Big\}<\infty. \tag{5}$$

*Proof.* Suppose equation (5) does not hold. Then for each $n\in\mathcal{N}$, there exists $y^n$ with $\delta(\mathcal{N},y^n)\le 1$ such that

$$\frac{1}{h_r}\Big|\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^k\lambda_j t_j\Delta^s x_j}{T_k}\Big)^{p_k}y_k^n\Big|>2^{n+1}.$$

With loss of generality, we can assume that $\frac{\sum_{j=0}^k\lambda_j t_j\Delta^s x_j}{T_k}, y^n\ge 0$. Now, we can define a sequence $z=(z_k)$ by

$$z_k=\sum_n\frac{1}{2^{n+1}}y_k^n \text{ for all } k\in\mathbb{N}.$$

By the convexity of $\mathcal{N}=(N_k)$, we have

$$\begin{aligned}N_k\Big(\sum_{n=0}^l\frac{1}{2^{n+1}}y_k^n\Big) &\le \frac{1}{2}\Big[N_k(y_k^0)+N_k\Big(y_k^1+\frac{y_k^2}{2}+...+\frac{y_k^l}{2^{l-1}}\Big)\Big]\\ &\le \sum_{n=0}^l\frac{1}{2^{n+1}}N_k(y_k^n)\end{aligned}$$

for any positive integer $l$. Hence, using the continuity of $\mathcal{N}=(N_k)$, we have

$$\delta(\mathcal{N},z)=\sum_k N_k z_k\le\sum_k\sum_n\frac{1}{2^{n+1}}N_k(y_k^n)\le\sum_n\frac{1}{2^{n+1}}=1.$$

But for every $l\in\mathbb{N}$, it holds

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^k\lambda_j t_j\Delta^s x_j}{T_k}\Big)^{p_k}z_k &\ge \frac{1}{h_r}\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^k\lambda_j t_j\Delta^s x_j}{T_k}\Big)^{p_k}\sum_{n=0}^l\frac{1}{2^{n+1}}y_k^n\\ &= \frac{1}{h_r}\sum_{n=0}^l\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^k\lambda_j t_j\Delta^s x_j}{T_k}\Big)^{p_k}\sum_{n=0}^l\frac{1}{2^{n+1}}y_k^n\\ &\ge l\end{aligned}$$

Hence, $\frac{1}{h_r}\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^k\lambda_j t_j\Delta^s x_j}{T_k}\Big)^{p_k}z_k$ diverges and this implies that $x\notin l_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$, a contradiction. This leads us to the required result. □

**Theorem 2.3.** *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $t=(t_k)$ be a sequence of positive real numbers, $p=(p_k)$ be a bounded sequence of positive real numbers and $\theta$ be a lacunary sequence. Then the following statements hold:*

*(i) $l_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$ is a normed linear spaces under the norm $\|.\|_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$ defined by*

$$\|.\|_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)=\sup_r\Big\{\frac{1}{h_r}\Big|\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^{k}\lambda_j t_j\Delta^s x_j}{T_k}\Big)^{p_k}y_k\Big|:\delta(\mathcal{N},y)\le 1\Big\}. \quad (6)$$

*(ii) $l_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$ is a Banach space under the norm defined by (6).*
*(iii) $l_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$ is a BK-space under the norm defined by (6).*

*Proof.* It is easy to verify that $l_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$ is a linear space with respect to the coordinatewise addition and scalar multiplication of sequences. Now we show that $\|.\|_{\mathcal{M}}(R^t,\Lambda,$
$\theta,p,\Delta^s)$ is a norm on the space $l_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$. If $x=0$, then obviously $\|.\|_{\mathcal{M}}(R^t,\Lambda,$
$\theta,p,\Delta^s)=0$.
Conversely, assume $\|.\|_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)=0$. Then using the definition of the norm given by (6), we have

$$\sup_r\Big\{\frac{1}{h_r}\Big|\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^{k}\lambda_j t_j\Delta^s x_j}{T_k}\Big)^{p_k}y_k\Big|:\delta(\mathcal{N},y)\le 1\Big\}=0.$$

This implies that

$$\frac{1}{h_r}\Big|\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^{k}\lambda_j t_j\Delta^s x_j}{T_k}\Big)^{p_k}y_k\Big|=0.$$

for all $y$ such that $\delta(\mathcal{N},y)\le 1$. Now considering $y=e^k$ if $\mathcal{N}(1)\le 1$ otherwise considering $y=e^k/\mathcal{N}(1)$ and $s=0$ so that $\lambda_k t_k\Delta^s x_k=0$ for all $k\in\mathbb{N}$, where $e^k$ is a sequence whose only non-zero terms is 1 in $k^{th}$ place for each $k\in\mathbb{N}$. Hence, we have $x_k=0$ for all $k\in\mathbb{N}$. Since $(\lambda_k)$ is a sequence of non-zero scalars and $t=(t_k)$ is a sequence of non-negative real numbers with $t_0>0$. Thus, $x=0$. It is easy to show that $\|\alpha x\|_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)=|\alpha|\|x\|_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$ and $\|x+y\|_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)\le\|x\|_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)+\|y\|_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$ for $\alpha\in\mathbb{C}$ and $x,y\in l_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$.

(ii) Let $(x^m)$ be any Cauchy sequence in the space $l_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)$. Then for any $\epsilon>0$, there exists a positive integer $n_0$ such that $\|x^m-x^n\|_{\mathcal{M}}(R^t,\Lambda,\theta,p,\Delta^s)<\epsilon$ for all $m,n\ge n_0$. Using the definition of norm given by (6), we get

$$\sup_r\Big\{\frac{1}{h_r}\Big|\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^{k}\lambda_j t_j\Delta^s(x_j^m-x_j^n)}{T_k}\Big)^{p_k}y_k\Big|:\delta(\mathcal{N},y)\le 1\Big\}<\epsilon$$

for all $m,n\ge n_0$. This implies that

$$\frac{1}{h_r}\Big|\sum_{k\in I_r}\Big(\frac{\sum_{j=0}^{k}\lambda_j t_j\Delta^s(x_j^m-x_j^n)}{T_k}\Big)^{p_k}y_k\Big|<\epsilon$$

for all $y$ such that $\delta(\mathcal{N},y)\le 1$ and for all $m,n\ge n_0$. Now suppose $y=e^k$ if $\mathcal{N}(1)\le 1$, otherwise considering $y=e^k/\mathcal{N}(1)$ we have $\{\lambda_k t_k\Delta^s x_k^m\}_k$ is a Cauchy sequence in $\mathbb{C}$ for all $k\in\mathbb{N}$. Hence, it is a convergent sequence in $\mathbb{C}$ for all $k\in\mathbb{N}$. Let $\lim_{m\to\infty}\lambda_k t_k\Delta^s x_k^m=x_k$ for each $k\in\mathbb{N}$. Using the continuity of the modulas, we can derive for all $m \geq n_0$ as $n \to \infty$, that

$$\sup_r \Big\{ \frac{1}{h_r} \Big| \sum_{k \in I_r} \Big( \frac{\sum_{j=0}^{k} \lambda_j t_j \Delta^s (x_j^m - x_j)}{T_k} \Big)^{p_k} y_k \Big| : \delta(\mathcal{N}, y) \leq 1 \Big\} \leq \epsilon.$$

It follows that $(x^m - x) \in l_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. Since $(x^m)$ is in the space $l_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ and $l_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is a linear space, we have $x = (x_k) \in l_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$.

(iii) From the above proof, one can easily conclude that $\|x^m\|_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s) \to 0$ implies that $x_k^m \to 0$ for each $m \in \mathbb{N}$ which leads us to the desired result.
Therefore, the proof of the theorem is completed. □

**Theorem 2.4.** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $t = (t_k)$ be a sequence of positive real numbers, $p = (p_k)$ be a bounded sequence of positive real numbers and $\theta$ be a lacunary sequence. Then $l_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is a normed linear space under the norm $\|.\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s)$ defined by*

$$\|x\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s) = \inf \Big\{ \rho > 0 : \frac{1}{h_r} \sum_{k \in I_r} M_k \Big( \frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho T_k} \Big)^{p_k} \leq 1 \Big\}. \quad (7)$$

*Proof.* Clearly, $\|x\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s) = 0$ if $x = 0$. Now suppose that $\|x\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s) = 0$. Then, we have

$$\|.\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s) = \inf \Big\{ \rho > 0 : \frac{1}{h_r} \sum_{k \in I_r} M_k \Big( \frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho T_k} \Big)^{p_k} \leq 1 \Big\} = 0.$$

This yields the fact for a given $\epsilon > 0$ that there exists some $\rho_\epsilon \in (0, \epsilon)$ such that

$$\sup_{k \in \mathbb{N}} M_k \Big( \frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho_\epsilon T_k} \Big)^{p_k} \leq 1$$

which implies that

$$M_k \Big( \frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho_\epsilon T_k} \Big)^{p_k} \leq 1$$

for all $k \in \mathbb{N}$. Thus,

$$M_k \Big( \frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\epsilon T_k} \Big)^{p_k} \leq M_k \Big( \frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho_\epsilon T_k} \Big)^{p_k} \leq 1$$

for all $k \in \mathbb{N}$. Suppose $\dfrac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\epsilon T_k} \neq 0$ for some $k \in \mathbb{N}$. Then, $\dfrac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\epsilon T_k} \to \infty$ as $\epsilon \to 0$.
It follows that $M_k \Big( \frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\epsilon T_k} \Big)^{p_k} \to \infty$ as $\epsilon \to 0$ for some $k \in \mathbb{N}$, which is a contradiction. Therefore, $\dfrac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\epsilon T_k} = 0$ for all $k \in \mathbb{N}$. It follows that $\lambda_k t_k \Delta^s x_k = 0$ for all $k \in \mathbb{N}$. Hence $x = 0$. Since $(\lambda_k)$ is a sequence of non-zero scalars and $t = (t_k)$ be a sequence of non-negative real numbers with $t_0 > 0$.

Let $x = (x_k)$ and $y = (y_k)$ be two elements of $l_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. Then there exists $\rho_1, \rho_2 > 0$ such that

$$\frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho_1 T_k}\Big)^{p_k} \leq 1$$

and

$$\frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s y_j|}{\rho_2 T_k}\Big)^{p_k} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then, by the convexity of $\mathcal{M} = (M_k)$, we have
$\frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s (x_j + y_j)|}{\rho T_k}\Big)^{p_k}$

$$\begin{aligned} &\leq \Big(\frac{\rho_1}{\rho_1 + \rho_2}\Big) \frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho_1 T_k}\Big)^{p_k} \\ &+ \Big(\frac{\rho_2}{\rho_1 + \rho_2}\Big) \frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s y_j|}{\rho_2 T_k}\Big)^{p_k}. \end{aligned}$$

Hence, we have

$$\|x+y\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s) = \inf\Big\{\rho > 0 : \frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s (x_j + y_j)|}{\rho T_k}\Big)^{p_k} \leq 1\Big\}$$

$$\leq \inf\Big\{\rho_1 > 0 : \frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho_1 T_k}\Big)^{p_k} \leq 1\Big\}$$

$$+ \inf\Big\{\rho_2 > 0 : \frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s y_j|}{\rho_2 T_k}\Big)^{p_k} \leq 1\Big\}.$$

Therefore,

$$\|x + y\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s) \leq \|x\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s) + \|y\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s).$$

Finally, let $\alpha$ be any scalar and define $r$ by $r = \frac{\rho}{|\alpha|}$. Then,

$$\begin{aligned} \|\alpha x\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s) &= \inf\Big\{\rho > 0 : \frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s \alpha x_j|}{\rho T_k}\Big)^{p_k} \leq 1 \Big\} \\ &= \inf\Big\{r|\alpha| > 0 : \frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{r T_k}\Big)^{p_k} \leq 1\Big\} \\ &= |\alpha| \|x\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s). \end{aligned}$$

This completes the proof. □

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $\delta(\mathcal{M}, x) = \sum_k M_k(|x_k|)$ where $x = (x_k) \in w$. Let $t = (t_k)$ be a sequence of positive real numbers, $p = (p_k)$ be a bounded sequence of positive real numbers. Let $\theta$ be a lacunary sequence. Then, we

define the following sequence space:
$l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s) =$

$$\Big\{x = (x_k) \in w : \lim_r \frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} < \infty \text{ for some } \rho > 0\Big\}.$$

**Theorem 2.5.** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $t = (t_k)$ be a sequence of positive real numbers, $p = (p_k)$ be a bounded sequence of positive real numbers and $\theta$ be a lacunary sequence. Then the following statements hold:*
*(i) $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is a normed linear spaces under the norm $\|x\|_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ defined by* (7).
*(ii) $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is a Banach space under the norm defined by* (7).
*(iii) $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is a BK- space under the norm defined by* (7).

*Proof.* (i) Since the proof is similar to the proof of theorem (2.4), so we omit the detail.

(ii) Let $(x^m)$ be any Cauchy sequence in the space $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. Let $\delta > 0$ be fixed and $r > 0$ be given such that $0 < \epsilon < 1$ and $r\delta \geq 1$. Then, there exists a positive integer $n_0$ such that $\|x^m - x^n\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s) < \frac{\epsilon}{r\delta}$ for all $m, n \geq n_0$. Using the definition of the norm given by (7), we get

$$\inf\Big\{\rho > 0 : \frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s (x_j^m - x_j^n)|}{\rho T_k}\Big)^{p_k} \leq 1\Big\} < \frac{\epsilon}{r\delta}$$

for all $m, n \geq n_0$. This implies that

$$\frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s (x_j^m - x_j^n)|}{\|x^m - x^n\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s)}\Big)^{p_k} \leq 1$$

for all $m, n \geq n_0$. It follows that

$$M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s (x_j^m - x_j^n)|}{\|x^m - x^n\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s)}\Big)^{p_k} \leq 1$$

for all $m, n \geq n_0$ and for all $k \in \mathbb{N}$. For $r > 0$ with $M_k(\frac{r\delta}{2}) \geq 1$, we have

$$M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s (x_j^m - x_j^n)|}{\|x^m - x^n\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s)}\Big)^{p_k} \leq M_k(\frac{r\delta}{2})$$

for all $m, n \geq n_0$ and for all $k \in \mathbb{N}$. Since $\mathcal{M} = (M_k)$ is non-decreasing, we have

$$\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s (x_j^m - x_j^n)|}{T_k}\Big)^{p_k} \leq \frac{r\delta}{2}.\frac{\epsilon}{r\delta} = \frac{\epsilon}{2}$$

for all $m, n \geq n_0$ and for all $k \in \mathbb{N}$. Hence, $\{\lambda_j t_j \Delta^s x_k^m\}_k$ is a Cauchy sequence in $\mathbb{C}$ for all $k \in \mathbb{N}$ which implies that it is a convergent sequence in $\mathbb{C}$ for all $k \in \mathbb{N}$. Let $\lim_{m \to \infty} \lambda_j t_j \Delta^s x_k^m = x_k$ for each $k \in \mathbb{N}$. Using the continuity of an Musielak-Orlicz function and modulus, we have

$$\inf\Big\{\rho > 0 : \frac{1}{h_r} \sum_{k \in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s (x_j^m - x_j)|}{\rho T_k}\Big)^{p_k} \leq 1\Big\} < \epsilon$$

for all $m \geq n_0$ as $n \to \infty$. It follows that $(x^m - x) \in l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. Since $x^m$ is in the space $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ and this space is a linear space, we have $x = (x_k) \in l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$.
(iii) From the above proof, we can easily conclude that $\|x^m\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s) \to 0$ as $m \to \infty$ which implies that $x_k^m \to 0$ as $k \to \infty$ for each $m \in \mathbb{N}$. This completes the proof. □

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, let $t = (t_k)$ be a sequence of positive real numbers, $p = (p_k)$ be a bounded sequence of positive real numbers. Let $\theta$ be a lacunary sequence. Then, we define the following sequence space:
$h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s) =$

$$\Big\{x = (x_k) \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} < \infty \text{ for each } \rho > 0\Big\}.$$

Clearly, $h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is a subspace of $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. We write $\|.\|$ instead of $\|.\|_{(\mathcal{M})}(R^t, \Lambda, \theta, p, \Delta^s)$, here the topology $h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is induced by $\|.\|$.

**Theorem 2.6.** *The inequality* $\frac{1}{h_r} \sum_{k\in I_r} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\|x\|_{(\mathcal{M})}(R^t,\Lambda,\theta,p,\Delta^s) T_k}\Big)^{p_k} \leq 1$ *holds for all* $x = (x_k) \in l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$.

**Theorem 2.7.** *Let* $\mathcal{M} = (M_k)$ *be a Musielak-Orlicz function,* $t = (t_k)$ *be a sequence of positive real numbers,* $p = (p_k)$ *be a bounded sequence of positive real numbers. Let* $\theta$ *be a lacunary sequence. Then,* $h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ *is an AK-BK space.*

*Proof.* We first show that $h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is an AK-space. Let $x = (x_k) \in h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. Then, for each $\epsilon \in (0, 1)$, we can find $n_0$ such that

$$\frac{1}{h_r} \sum_{\substack{k\in I_r \\ k\geq n_0}} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\epsilon T_k}\Big)^{p_k} \leq 1.$$

Define the $n^{th}$ section $x^{[n]}$ of a sequence $x = (x_k)$ by $x^{[n]} = \sum_{k=0}^{n} x_k e_k$. Hence for $n \geq n_0$, it holds

$$\begin{aligned} \|x - x^{[n]}\| &= \inf\Big\{\rho > 0 : \frac{1}{h_r} \sum_{\substack{k\in I_r \\ k\geq n_0}} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} \leq 1\Big\} \\ &= \inf\Big\{\rho > 0 : \frac{1}{h_r} \sum_{\substack{k\in I_r \\ k\geq n}} M_k\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} \leq 1\Big\} < \epsilon. \end{aligned}$$

Thus, we can conclude that $h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is an AK-space.
Next, to show that $h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is a BK-space, it is enough to show that $h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is a closed subspace of $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. For this, let $(x^n)$ be a sequence in $h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ such that $\|x^n - x\| \to 0$ as $n \to \infty$ where $x = (x_k) \in l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. To complete the proof we need to show that $x =$

$(x_k) \in h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ i.e,

$$\frac{1}{h_r}\sum_{k\in I_r} M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} < \infty \text{ for all } \rho > 0.$$

There is $l$ corresponding to $\rho > 0$ such that $\|x^l - x\| \leq \frac{\rho}{2}$. Then by using the convexity of $\mathcal{M}$, we have by Theorem 2.6., that

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r} M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} \\
&= \frac{1}{h_r}\sum_{k\in I_r} M_k\Big(\frac{2|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j^l| - 2\big(|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j^l| - |\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|\big)}{2\rho T_k}\Big)^{p_k} \\
&\leq \frac{1}{2h_r}\sum_{k\in I_r} M_k\Big(\frac{2|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} + \frac{1}{2h_r}\sum_{k\in I_r} M_k\Big(\frac{2|\sum_{j=0}^{k}\lambda_j t_j \Delta^s (x_j^l - x_j)|}{\rho T_k}\Big)^{p_k} \\
&\leq \frac{1}{2h_r}\sum_{k\in I_r} M_k\Big(\frac{2|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} + \frac{1}{2h_r}\sum_{k\in I_r} M_k\Big(\frac{2|\sum_{j=0}^{k}\lambda_j t_j \Delta^s (x_j^l - x_j)|}{\|x^l - x\| T_k}\Big)^{p_k} \\
&< \infty.
\end{aligned}$$

Hence, $x = (x_k) \in h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ and consequently $h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is a BK-space. □

**Theorem 2.8.** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. If $\mathcal{M}$ satisfies the $\Delta_2$-condition at 0, then $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ is an AK-space.*

*Proof.* We shall show that $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s) = h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$ if $(M_k)$ stisfies the $\Delta_2-$condition at 0. For this it is enough to show $l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s) \subset h_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. Let $x = (x_k) \in l'_{\mathcal{M}}(R^t, \Lambda, \theta, p, \Delta^s)$. Then foe some $\rho > 0$,

$$\frac{1}{h_r}\sum_{k\in I_r} M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} < \infty.$$

This implies that

$$\frac{1}{h_r}\lim_{k\to\infty} M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} = 0. \tag{8}$$

Take an arbitrary $l > 0$. If $\rho \leq l$, then

$$\frac{1}{h_r}\sum_{k\in I_r} M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{l T_k}\Big)^{p_k} < \infty.$$

Now, let $l < \rho$ and put $k = \frac{\rho}{l}$. Since $\mathcal{M} = (M_k)$ satisfies the $\Delta_2$-condition at 0, there exists $R \equiv R_k > 0$ and $r \equiv r_k > 0$ with $M_k(kx) \leq RM_k(x)$ for all $x \in (0, r]$. By (8), there exists a positive integer $n_1$ such that

$$M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} < q\Big(\frac{r}{2}\Big)\frac{r}{2} \text{ for all } k \geq n_1.$$

We claim that $\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k} \le r$ for all $k \ge n_1$. Otherwise, we can find $d > n_1$ with

$$\frac{|\sum_{j=0}^{d}\lambda_j t_j \Delta^s x_j|}{\rho T_d} > r$$

and thus

$$\begin{aligned} M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_d}\Big)^{p_k} &\ge \int_{\frac{r}{2}}^{\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_d}} q(t)dt \\ &> q\Big(\frac{r}{2}\Big)\frac{r}{2}, \end{aligned}$$

a contradiction. Hence, our claim is true. Then, we can find that

$$\frac{1}{h_r}\sum_{\substack{k\in I_r\\ k\ge n_1}} M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{lT_k}\Big)^{p_k} \le \frac{R}{h_r}\sum_{\substack{k\in I_r\\ k\ge n_1}} M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k}.$$

Hence, $\frac{1}{h_r}\sum_{k\in I_r} M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{lT_k}\Big)^{p_k} < \infty$ for all $l > 0$. □

**Theorem 2.9.** *Let $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ be two Musielak-Orlicz function. If $\mathcal{M}'$ and $\mathcal{M}''$ are equivalent, then $l'_{\mathcal{M}'}(R^t,\Lambda,\theta,p,\Delta^s) = l'_{\mathcal{M}''}(R^t,\Lambda,\theta,p,\Delta^s)$ and the identity map $I : \big(l'_{\mathcal{M}'}(R^t,\Lambda,\theta,p,\Delta^s), \|.\|_{\mathcal{M}'}(R^t,\Lambda,\theta,p,\Delta^s)\big) \to \big(l'_{\mathcal{M}''}(R^t,\Lambda,\theta,p,\Delta^s),$ $\|.\|_{\mathcal{M}''}(R^t,\Lambda,\theta,p,\Delta^s)\big)$ is a topological isomorphism.*

*Proof.* Let $\alpha, \beta$ and $b$ be constants from (4). Since $(M_k')$ and $(M_k'')$ are equivalent, it is obvious that (4) holds. Let $x = (x_k) \in l'_{\mathcal{M}''}(R^t,\Lambda,\theta,p,\Delta^s)$. Then

$$\frac{1}{h_r}\sum_{k\in I_r} M_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k} < \infty \text{ for some } \rho > 0.$$

Hence, for some $l \ge 1$, $\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{l_\rho T_k}\Big)^{p_k} \le b$ for all $k \in \mathbb{N}$.
Therefore,

$$\frac{1}{h_r}\sum_{k\in I_r} M_k'\Big(\frac{\alpha|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{l\rho T_k}\Big)^{p_k} \le \frac{1}{h_r}\sum_{k\in I_r} M_k''\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\rho T_k}\Big)^{p_k}$$

which shows that the inclusion

$$l'_{\mathcal{M}''}(R^t,\Lambda,\theta,p,\Delta^s) \subset l'_{\mathcal{M}'}(R^t,\Lambda,\theta,p,\Delta^s) \tag{9}$$

holds. We can easily see in the same way that the inclusion

$$l'_{\mathcal{M}'}(R^t,\Lambda,\theta,p,\Delta^s) \subset (l'_{\mathcal{M}''}(R^t,\Lambda,\theta,p,\Delta^s) \tag{10}$$

also holds. By combining the inclusions (9) and (10), we conclude that $l'_{\mathcal{M}'}(R^t,\Lambda,\theta,p,$ $\Delta^s) = l'_{\mathcal{M}''}(R^t,\Lambda,\theta,p,\Delta^s)$.
For simplicity in notation, let us write $\|.\|_1$ and $\|.\|_2$ instead of $\|.\|_{\mathcal{M}'}(R^t,\Lambda,\theta,p,\Delta^s)$ and $\|.\|_{\mathcal{M}''}(R^t,\Lambda,\theta,p,\Delta^s)$ respectively. For $x = (x_k) \in l'_{\mathcal{M}''}(R^t,\Lambda,\theta,p,\Delta^s)$, we get

$$\frac{1}{h_r}\sum_{k\in I_r} M_k''\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\|x\|_2 T_k}\Big) \le 1.$$

One can fixed $\mu > 1$ with

$$\frac{b}{2}\mu p_2\big(\frac{b}{2}\big) \geq 1$$

where $p_2$ is the kernel associated with $\mathcal{M}''$. Hence,

$$M''_k\Big(\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\|x\|_2 T_k}\Big) \leq \frac{b}{2}\mu p_2\big(\frac{b}{2}\big)$$

for all $k \in \mathbb{N}$. This implies that

$$\frac{|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\mu\|x\|_2 T_k} \leq b$$

for all $k \in \mathbb{N}$. Therefore,

$$\frac{1}{h_r}\sum_{k\in I_r} M'_k\Big(\frac{\alpha|\sum_{j=0}^{k}\lambda_j t_j \Delta^s x_j|}{\mu\|x\|_2 T_k}\Big) < 1.$$

Hence, $\|x\|_1 \leq (\frac{\mu}{\alpha})\|x\|_2$. Similarly, we can show that $\|x\|_2 \leq \beta\gamma\|x\|_1$ by choosing $\gamma$ with $\gamma\beta > 1$ such that $\gamma\beta(\frac{b}{2})p_1(\frac{b}{2}) \geq 1$. Thus,

$$\frac{\alpha}{\mu}\|x\|_1 \leq \|x\|_2 \leq \beta\gamma\|x\|_1$$

which establish that $I$ is a topological isomorphism. □

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