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Notes on regular filters in residuated lattices

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Abstract. In the mathematical literature, many types of filters in residuated lattices have been studied. In [2] we proposed a new approach for the study of these filters. In this paper, in the spirit of [2], we establish some connections between regular filters and other types of filters in residuated lattices. Also, we prove that a residuated lattice is a Stonean residuated lattice if and only if every filter is a Stonean filter and we show that a residuated lattice has the Double Negation Condition if and only if every filter is a regular filter.

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1. Introduction

The filter theory of the logical algebras plays an important role in studying these algebras. From a logical point of view, various filters correspond to various sets of provable formulas. Sometimes, a filter is also called deductive system ([10]). At present, the filter theory of residuated lattices has been studied and some important results have been obtained ([2], [3], [6], [12], [13]).

In [2] we proposed a new approach for the study of filters in residuated lattices.

In this paper we work in the general case of residuated lattices and we establish some relationships between regular filters and other filters: Boolean filters, MV filters, Stonean filters, divisible filters and by some examples we show that these filters are different.

Also, we prove that a residuated lattice L is a Stonean residuated lattice if and only if every filter of L is a Stonean filter and we show that a residuated lattice L has the Double Negation Condition if and only if every filter is a regular filter.

2. Preliminaries of residuated lattices

In this section we recall some definitions, properties and results relative to residuated lattices.

Definition 2.1. ([1], [7], [10], [11]) A *residuated lattice* is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:

(LR_1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice relative to the order $\leq$;

(LR_2) $(L, \odot, 1)$ is a commutative monoid;

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(LR_3) The operations $\odot$ and $\to$ form an *adjoint pair*, i.e. $a \odot x \le y$ if and only if $a \le x \to y$ for every $a, x, y \in L$.

In what follows, we denote by L a residuated lattice.

For $x \in L$ we define $x^* = x \to 0, x^{**} = (x^*)^*$.

The following rules of calculus in a residuated lattice L can be found, for example, in [5], [7], [9], [11]:

(c_1) $1 \to x = x, x \to x = 1, x \to 1 = 1$;
(c_2) $x \le y$ if and only if $x \to y = 1$;
(c_3) If $x \le y$, then $x \odot z \le y \odot z, z \to x \le z \to y$ and $y \to z \le x \to z$;
(c_4) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;
(c_5) $(x \vee y)^* = x^* \wedge y^*$;
(c_6) $(x \odot y)^* = x \to y^* = x^{**} \to y^*$;
(c_7) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$, $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, for every $x, y, z \in L$.

A *Boolean algebra* is a residuated lattice L in which $x \vee x^* = 1$ for every $x \in L$ (see [6]).

A residuated lattice L is called *divisible* if $x \odot (x \to y) = x \wedge y$, for every $x, y \in L$ ([7], [10]).

A divisible residuated lattice in which $(x \to y) \vee (y \to x) = 1$, for every $x, y \in L$ is called a *BL algebra* ([7], [10]).

A residuated lattice L is called an *MV algebra* if $(x \to y) \to y = (y \to x) \to x$, for every $x, y \in L$ ([7], [10]).

A residuated lattice L is called *Stonean* if $x^* \vee x^{**} = 1$, for every $x \in L$ ([4]).

A residuated lattice L has the *Double Negation condition* (DN) ([8]) if $x^{**} = x$, for every $x \in L$. A residuated lattice verifying (DN) condition is also called a *regular residuated lattice* or a *Girard monoid* (see [8]).

Definition 2.2. ([10]) A nonempty subset F of a residuated lattice L is called *deductive system* if :

(F_1) $1 \in F$;
(F_2) If $x, x \to y \in F$, then $y \in F$.

An equivalent definition for a deductive system is ([10]):

(F_1') If $x \le y$ and $x \in F$, then $y \in F$;
(F_2') If x, $y \in F$, then $x \odot y \in F$.

Following this equivalence, a deductive system of L is also called an *implicative filter* (or *filter*, for short).

If F is a filter of a residuated lattice L, then the relation $\sim_F$ defined on L by $(x, y) \in\sim_F$ if and only if $x \to y, y \to x \in F$ if and only if $(x \to y) \odot (y \to x) \in F$ is a congruence relation on L (see [10]).

The quotient algebra $L/\sim_F$ denoted by L/F becomes a residuated lattice in a natural way, with the operations induced from those of L.

For $x \in L$ we denote by x/F the congruence class of x modulo $\sim_F$. So, the order relation on L/F is given by $x/F \le y/F$ if and only if $x \to y \in F$. Clearly, $x/F = 1/F$ if and only if $x \in F$.

3. Classes of filters in residuated lattices

Definition 3.1. A filter F of a residuated lattice L is called *Boolean filter* if L/F is a Boolean algebra.

We denote by $\mathbf{BF(L)}$ the set of all Boolean filters of L.

Theorem 3.1. ([13]) *For a filter F of a residuated lattice L the following conditions are equivalent:*
(i) $F \in \mathbf{BF(L)}$;
(ii) $x \vee x^* \in F$, *for every* $x \in L$.

Definition 3.2. A filter F of a residuated lattice L is called *Stonean filter* if L/F is a Stonean residuated lattice.

We denote by $\mathbf{StF(L)}$ the set of all Stonean filters of L.

Theorem 3.2. ([3]) *For a filter F of a residuated lattice L the following conditions are equivalent:*
(i) $F \in \mathbf{StF(L)}$;
(ii) $x^* \vee x^{**} \in F$, *for every* $x \in L$.

Theorem 3.3. *Let L be a residuated lattice. L is a Stonean residuated lattice if and only if any filter of L is a Stonean filter.*

Proof. If we suppose that L is a Stonean residuated lattice then $x^* \vee x^{**} = 1$, for every $x \in L$. So, $x^* \vee x^{**} \in F$, for any filter F of L. Thus, $(x/F)^* \vee (x/F)^{**} = 1/F$, So, L/F is a Stonean residuated lattice and $F \in \mathbf{StF(L)}$.

Conversely, if any filter of L is a Stonean filter, then $F = \{1\}$ is a Stonean filter, so $L/\{1\} = L$ is a Stonean residuated lattice. $\square$

Corollary 3.4. *A residuated lattice L is Stonean if and only if $\{1\}$ is a Stonean filter of L.*

In [3] it is proved that $\mathbf{BF(L)} \subseteq \mathbf{StF(L)}$.

Definition 3.3. A filter F of a residuated lattice L is called *divisible filter* if L/F is a divisible residuated lattice.

We denote by $\mathbf{DivF(L)}$ the set of all divisible filters of L.

Theorem 3.5. ([3]) *For a filter F of a residuated lattice L the following conditions are equivalent:*
(i) $F \in \mathbf{DivF(L)}$*;*
(ii) $(x \wedge y) \to [x \odot (x \to y)] \in F$, *for every* $x, y \in L$.

Theorem 3.6. *For a filter F of a residuated lattice L the following are equivalent:*
(i) $F \in \mathbf{DivF(L)}$*;*
(ii) $[y \odot (y \to x)] \to [x \odot (x \to y)] \in F$, *for every* $x, y \in L$.

Proof. $(i) \Rightarrow (ii)$. Let $F \in \mathbf{DivF(L)}$. From Theorem 3.5, $(x \wedge y) \to [x \odot (x \to y)] \in F$, for every $x, y \in L$. Since $y \odot (y \to x) \leq x \wedge y$, from (c_3), we deduce that $(x \wedge y) \to [x \odot (x \to y)] \leq [y \odot (y \to x)] \to [x \odot (x \to y)]$. But F is a filter, so $[y \odot (y \to x)] \to [x \odot (x \to y)] \in F$, for every $x, y \in L$.

$(ii) \Rightarrow (i)$. Conversely, we suppose that $[y \odot (y \to x)] \to [x \odot (x \to y)] \in F$, for every $x, y \in L$. Changing x with y we deduce that $[y \odot (y \to x)] \sim_F [x \odot (x \to y)]$, i.e., $y/F \odot (y/F \to x/F) = x/F \odot (x/F \to y/F) \overset{not}{=} a/F$.

Since $[y \odot (y \to x)] \to y = [x \odot (x \to y)] \to x = 1 \in F$, we obtain that $a/F \leq x/F$ and $a/F \leq y/F$.

To prove that $a/F = x/F \wedge y/F$, let $z/F \in L/F$ such that $z/F \leq x/F$ and $z/F \leq y/F$.

We have $z/F = z/F \odot 1/F = z/F \odot (z/F \to y/F) = y/F \odot (y/F \to z/F)$. Since $z/F \leq x/F$ we deduce that $y/F \to z/F \leq y/F \to x/F$, so $z/F \leq y/F \odot (y/F \to x/F) = a/F$. Then $a/F = x/F \wedge y/F$, that is, $(x \wedge y) \to [x \odot (x \to y)] \in F$, for every $x, y \in L$. $\square$

Definition 3.4. A filter F of a residuated lattice L is called *BL filter* if L/F is a BL algebra.

We denote by $\mathbf{BLF(L)}$ the set of all BL filters of L.
In [3] it is proved that $\mathbf{BLF(L)} \subseteq \mathbf{DivF(L)}$.

Definition 3.5. A filter F of a residuated lattice L is called *MV filter* if L/F is an MV algebra.

We denote by $\mathbf{MVF(L)}$ the set of all MV filters of L.

Theorem 3.7. ([13]) *For a filter F of a residuated lattice L the following conditions are equivalent:*
(i) $F \in \mathbf{MVF(L)}$;
(ii) $((x \to y) \to y) \to ((y \to x) \to x) \in F$, *for every* $x, y \in L$.

Definition 3.6. A filter F of a residuated lattice L is called *regular filter* if L/F is a regular residuated lattice.

We denote by $\mathbf{RF(L)}$ the set of all regular filters of L.

Remark 3.1. ([3]) $\mathbf{MVF(L)} \subseteq \mathbf{RF(L)}$ and $\mathbf{BF(L)} \subseteq \mathbf{RF(L)}$.

Theorem 3.8. ([3]) *For a filter F of a residuated lattice L the following conditions are equivalent:*
(i) $F \in \mathbf{RF(L)}$*;*
(ii) $x^{**} \to x \in F$, *for every* $x \in L$.

Proposition 3.9. ([3]) *Let* $F \in \mathbf{RF(L)}$. *Then* $(x \wedge y)^* \to (x^* \vee y^*) \in F$, *for every* $x, y \in L$.

Remark 3.2. Since $x^* \vee y^* \leq (x \wedge y)^*$, for every $x, y \in L$ we deduce that $(x \wedge y)^* \sim_F (x^* \vee y^*)$, for every $F \in \mathbf{RF(L)}$.

Proposition 3.10. *Let* $F \in \mathbf{RF(L)}$. *Then for all* $x, y \in L$,

$$x \to y \in F \text{ if and only if } y^* \to x^* \in F.$$

Proof. Since $F \in \mathbf{RF(L)}$, L/F is a regular residuated lattice, so $(x/F)^{**} = x/F$ and $(y/F)^{**} = y/F$. Then we have $(x/F) \to (y/F) = (x/F)^{**} \to (y/F)^{**} \overset{(c_6)}{=} [(x/F) \odot (y/F)^*]^* = [(y/F)^* \odot (x/F)]^* = (y/F)^* \to (x/F)^*$. $\square$

Proposition 3.11. *Let $F \in \mathbf{RF}(\mathbf{L})$. Then for all $x, y \in L$,*

$$(x^* \vee y^*)^* \to (x \wedge y) \in F.$$

Proof. From (c_5), $(x^* \vee y^*)^* = x^{**} \wedge y^{**}$. Since $F \in \mathbf{RF}(\mathbf{L})$, using Theorem 3.8, $x^{**} \to x \in F, y^{**} \to y \in F$.

So we have $(x^* \vee y^*)^* \to (x \wedge y) = (x^{**} \wedge y^{**}) \to (x \wedge y) \overset{(c_7)}{=} [(x^{**} \wedge y^{**}) \to x] \wedge [(x^{**} \wedge y^{**}) \to y]$. But, $x^{**} \to x \in F, y^{**} \to y \in F$ and $x^{**} \wedge y^{**} \leq x^{**}, y^{**}$ so from (c_3), $x^{**} \to x \leq (x^{**} \wedge y^{**}) \to x$ and $y^{**} \to y \leq (x^{**} \wedge y^{**}) \to y$. Since F is a filter we deduce that $(x^* \vee y^*)^* \to (x \wedge y) \in F$. $\square$

Remark 3.3. Since $x \wedge y \leq (x^* \vee y^*)^*$ for every $x, y \in L$ we deduce that $x \wedge y \sim_F (x^* \vee y^*)^*$, for every $F \in \mathbf{RF}(\mathbf{L})$.

Proposition 3.12. *Let L be a residuated lattice and $F \in \mathbf{RF}(\mathbf{L})$. Then for every $x, y, z \in L$,*

$$[(x \wedge y) \to z] \sim_F [(x \to z) \vee (y \to z)] \textit{ iff } [z \to (x \vee y)] \sim_F [(z \to x) \vee (z \to y)].$$

Proof. $'' \Rightarrow''$. First, we suppose that $[(x \wedge y) \to z] \sim_F [(x \to z) \vee (y \to z)]$, i.e., $[(x/F \wedge y/F) \to z/F] = [(x/F \to z/F) \vee (y/F \to z/F)]$, for every $x, y, z \in L$. From the proof of Proposition 3.10 we deduce that $z/F \to x/F = (x/F)^* \to (z/F)^*$ and $z/F \to y/F = (y/F)^* \to (z/F)^*$, so $(z/F \to x/F) \vee (z/F \to y/F) = [(x/F)^* \to (z/F)^*] \vee [(y/F)^* \to (z/F)^*] = [(x/F)^* \wedge (y/F)^*] \to (z/F)^* \overset{(c_5)}{=} [(x/F) \vee (y/F)]^* \to (z/F)^* = z/F \to (x/F \vee y/F)$, so $[z \to (x \vee y)] \sim_F [(z \to x) \vee (z \to y)]$, for every $x, y, z \in L$.

$'' \Leftarrow''$. Conversely, if we suppose that $[z \to (x \vee y)] \sim_F [(z \to x) \vee (z \to y)]$, for every $x, y, z \in L$, then $(z/F \to x/F) \vee (z/F \to y/F) = z/F \to (x/F \vee y/F)$. From Remark 3.2 we have $(x/F \to z/F) \vee (y/F \to z/F) = [(z/F)^* \to (x/F)^*] \vee [(z/F)^* \to (y/F)^*] = (z/F)^* \to [(x/F)^* \vee (y/F)^*] = (z/F)^* \to [(x/F) \wedge (y/F)]^* = (x/F \wedge y/F) \to z/F$, so $[(x \wedge y) \to z] \sim_F [(x \to z) \vee (y \to z)]$, for every $x, y, z \in L$. $\square$

Remark 3.4. If $F \in \mathbf{RF}(\mathbf{L})$, then for every $x, y, z \in L$, $[(x \wedge y) \to z] \to [(x \to z) \vee (y \to z)] \in F$ if and only if $[z \to (x \vee y)] \to [(z \to x) \vee (z \to y)] \in F$.

Proposition 3.13. *Let L be a residuated lattice, $x, y, z \in L$ and $F \in \mathbf{RF}(\mathbf{L})$. If*

$$[(x \wedge y) \to z] \sim_F [(x \to z) \vee (y \to z)] \textit{ then } [x \vee (y \wedge z)] \sim_F [(x \vee y) \wedge (x \vee z)].$$

Proof. Obviously, $x/F, y/F \wedge z/F \leq (x/F \vee y/F) \wedge (x/F \vee z/F)$. Let $t/F \in L/F$ such that $x/F, y/F \wedge z/F \leq t/F$. From (c_7), $(x/F \vee y/F) \to t/F = (x/F \to t/F) \wedge (y/F \to t/F) = 1/F \wedge (y/F \to t/F) = y/F \to t/F$ and $(x/F \vee z/F) \to t/F = (x/F \to t/F) \wedge (z/F \to t/F) = 1/F \wedge (z/F \to t/F) = z/F \to t/F$.

Then $[(x/F \vee y/F) \wedge (x/F \vee z/F)] \to t/F = [(x/F \vee y/F) \to t/F] \vee [(x/F \vee z/F) \to t/F] = (y/F \to t/F) \vee (z/F \to t/F) = (y/F \wedge z/F) \to t/F = 1/F$, that is, $(x/F \vee y/F) \wedge (x/F \vee z/F) \leq t/F$. We deduce that $(x/F \vee y/F) \wedge (x/F \vee z/F)$ is the least upper bound of x/F and $y/F \wedge z/F$.

We conclude that $x/F \vee (y/F \wedge z/F) = (x/F \vee y/F) \wedge (x/F \vee z/F)$, for every $x, y, z \in L$, i.e., $[x \vee (y \wedge z)] \sim_F [(x \vee y) \wedge (x \vee z)]$. $\square$

For $F = \{1\}$ we deduce that:

Theorem 3.14. *If L is a regular residuated lattice and $[(x \wedge y) \to z] = [(x \to z) \vee (y \to z)]$, for every $x, y, z \in L$, then L is a distributive residuated lattice.*

Theorem 3.15. *Let L be a residuated lattice. L is a regular residuated lattice if and only if any filter of L is a regular filter.*

Proof. We suppose that L is regular residuated lattice and let F be a filter of L. Then $x^{**} = x$, for every $x \in L$, so $x^{**} \to x = 1 \in F$, for every $x \in L$, thus from Theorem 3.8, $F \in \mathbf{RF(L)}$.

Conversely, if any filter of L is a regular filter, then $F = \{1\}$ is a regular filter, so $x^{**} \to x \in F = \{1\}$, for every $x \in L$ so $x^{**} = x$, for every $x \in L$. We conclude that L has (DN) condition. □

Corollary 3.16. *A residuated lattice L is regular if and only if $\{1\}$ is a regular filter of L.*

Proposition 3.17. *Let L be a residuated lattice and $F, G \in \mathbf{RF(L)}$. Then $F \cap G \in \mathbf{RF(L)}$.*

Proof. Let $x \in L$. Since $F, G \in \mathbf{RF(L)}$, then by Theorem 3.8, $x^{**} \to x$ is in F and also in G, so $x^{**} \to x \in F \cap G$, that is, $F \cap G \in \mathbf{RF(L)}$. □

Proposition 3.18. *Suppose that F and G are two filters of a residuated lattice L and $F \subseteq G$. If $F \in \mathbf{RF(L)}$, then $G \in \mathbf{RF(L)}$.*

Proof. Let $x \in L$. Since $F \in \mathbf{RF(L)}$, from Theorem 3.8, we deduce that $x^{**} \to x \in F$. By hypothesis, $F \subseteq G$, so $x^{**} \to x \in G$, for every $x \in L$, thus $G \in \mathbf{RF(L)}$. □

4. New connections between regular filters and other filters

In [3] it is proved that $\mathbf{BLF(L)} \cap \mathbf{RF(L)} = \mathbf{MVF(L)}$.

Theorem 4.1. *In any residuated lattice L,*

$$\mathbf{DivF(L)} \cap \mathbf{RF(L)} = \mathbf{MVF(L)}.$$

Proof. If we consider $F \in \mathbf{DivF(L)} \cap \mathbf{RF(L)}$, then L/F is a divisible and regular residuated lattice and from Theorem 3.15, $x/F = (x/F)^{**}$ for every $x \in L$. From Propositions 3.9 and 3.10, we have $x/F \vee y/F = (x/F)^{**} \vee (y/F)^{**} = [(x/F)^* \wedge (y/F)^*]^* = [(x/F)^* \odot ((x/F)^* \to (y/F)^*)]^* = [(x/F)^* \to (y/F)^*] \to (x/F)^{**} = [(x/F)^* \to (y/F)^*] \to (x/F) = [(y/F) \to (x/F)] \to (x/F)$.

Obviously, $x/F \vee y/F = [(y/F) \to (x/F)] \to (x/F) = [(x/F) \to (y/F)] \to (y/F)$, for every $x, y \in L$ so L/F is an MV algebra and F is an MV filter.

Conversely, since $\mathbf{BLF(L)} \subseteq \mathbf{DivF(L)}$ and $\mathbf{BLF(L)} \cap \mathbf{RF(L)} = \mathbf{MVF(L)}$ we deduce that $\mathbf{MVF(L)} \subseteq \mathbf{DivF(L)} \cap \mathbf{RF(L)}$. □

Remark 4.1. In any residuated lattice L, $\mathbf{MVF(L)} \subsetneq \mathbf{DivF(L)}$. To show that $\mathbf{MVF(L)} \neq \mathbf{DivF(L)}$ we consider the following example ([8]) of a finite divisible residuated lattice which is not an MV-algebra. Let $L = \{0, a, b, c, 1\}$, with $0 < c < a, b < 1$, but a, b are incomparable.

Define on L the following operations:

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

,

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

.

It is easy to see that $F = \{1, a\}$ is a BL filter, so is a divisible filter of L. Since a BL filter F is an MV filter if and only if $x^{**} \to x \in F$, for every $x \in L$ (see [3]) and in this case, $c^{**} \to c = 1 \to c = c \notin F$, we deduce that F is not an MV filter.

Remark 4.2. In any residuated lattice L, $\mathbf{MVF(L)} \subsetneq \mathbf{RF(L)}$. To show that $\mathbf{MVF(L)} \neq \mathbf{RF(L)}$ we consider (see ([8])) the finite residuated lattice $L = \{0, a, b, c, d, e, f, 1\}$. Lattice ordering is such that $0 < d < c < b < a < 1$, $0 < d < e < f < a < 1$ and elements $\{b, f\}$ and $\{c, e\}$ are pairwise incomparable. The operations of implication and multiplication are given by the tables below :

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	1	a	b	c	d	e	f	1

,

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

Clearly, L is a regular residuated lattice, so $F = \{1\}$ is a regular filter of L. Since $[(b \to f) \to f] \to [(f \to b) \to b] = (f \to f) \to (a \to b) = 1 \to a = a \notin F$, from Theorem 3.7 we deduce that $\{1\}$ is not an MV filter of L.

Theorem 4.2. *In any residuated lattice L,*

$$\mathbf{StF(L)} \cap \mathbf{RF(L)} = \mathbf{BF(L)}.$$

Proof. Since $\mathbf{BF(L)} \subseteq \mathbf{StF(L)}$ and $\mathbf{BF(L)} \subseteq \mathbf{RF(L)}$, see [3], we deduce that

$$\mathbf{BF(L)} \subseteq \mathbf{StF(L)} \cap \mathbf{RF(L)}.$$

Conversely, let $F \in \mathbf{StF(L)} \cap \mathbf{RF(L)}$. Then L/F is a regular and a Stonean residuated lattice. So, we have $(x/F)^* \vee (x/F)^{**} = 1/F$ for every $x \in L$. But $(x/F)^{**} = x/F$, so we deduce that $(x/F)^* \vee x/F = 1/F$, thus $x \vee x^* \in F$ for every $x \in L$, and by Theorem 3.1, $F \in \mathbf{BF(L)}$. □

Using Remark 3.1 and Theorem 4.2, we deduce:

Corollary 4.3. *In any residuated lattice L,*

$$\mathbf{StF(L)} \cap \mathbf{MVF(L)} = \mathbf{BF(L)}.$$

Remark 4.3. In any residuated lattice L, $\mathbf{BF(L)} \subsetneq \mathbf{RF(L)}$. To show that $\mathbf{RF(L)} \neq \mathbf{BF(L)}$, we consider an example of a finite residuated lattice which is an MV algebra ([8]). Let $L = \{0, a, b, c, d, 1\}$ with $0 < a, b < c < 1, 0 < b < d < 1$, but a, b and, respective c, d are incomparable. We define on L the following operations:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

.

It is easy to see that $F = \{1\}$ is a regular filter of L. Since $b \vee b^* = b \vee c = c \notin F$, from Theorem 3.1 we deduce that F is not a Boolean filter of L.

5. Conclusion

In the last years there was a big boom of papers about special types of filters on different subvarieties of residuated lattices. In [2] we proposed a new approach for the study of filters in residuated lattices.

In this paper we establish some relationships between regular filters in residuated lattices and Boolean filters, MV filters, Stonean filters and divisible filters. Also, by some examples we show that these filters are different.

Finally, the connections studied in this paper, between regular filters and other filters are resumed in the following Figures:

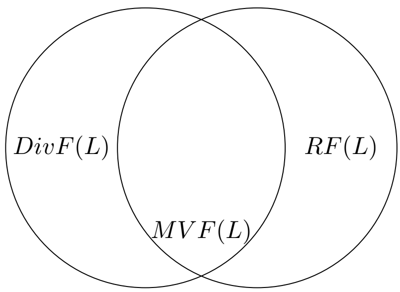

Figure 1. Theorem 19.

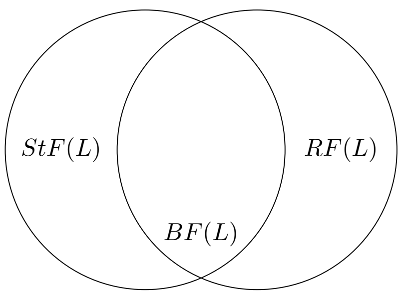

Figure 2. Theorem 20.

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