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Mathematical model of mosquito populations dynamics with logistic growth in a periodic environment

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Abstract. In this paper, we analyze the effect of climate change on the dynamics of mosquito population. The model is formulated as a nonautonomous system of ordinary differential equations with Verhulst-Pearl logistic growth. We show that the global dynamics of the model is determined by the vectorial reproduction ratio, $\mathcal{R}_v$ which is defined through the spectral radius of a linear integral. Indeed, we show that if the threshold $\mathcal{R}_v$ is greater than 1, then the mosquito-free equilibrium is globally asymptotically stable; but if it is smaller than 1, then the mosquitoes persist and the system admits at least one positive periodic solution. Finally, we perform some numerical simulations in order to illustrate our mathematical results.

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1. Introduction

Mosquito-borne diseases are a major public health concern because they cause substantial mortality and morbidity worldwide. Indeed, mosquito abundance is a key determining factor that affects the persistence or resurgence of mosquito-borne diseases in population. Hence, it is crucial to study the dynamics of mosquitoes, and devise effective and realistic methods for controlling mosquito population in communities. Since mosquito biology and disease ecology are strongly linked to environmental conditions, then the impacts of mosquito-borne diseases increase with warm temperatures and extreme precipitations. Several mathematical models have been established in order to describe the dynamic behavior of the transmission of mosquito-borne diseases [6, 9, 4, 19]; but many of these models, do not take into account the life cycle of mosquitoes [1, 5, 8].

In [14], the authors have formulated a mathematical model of the dynamics of vector population with constant coefficients in the study of the transmission of chikungunya disease. To describe the dynamics of mosquito population, they use the stage structure model. Their model has three compartments that correspond to the different stages of mosquito evolution. The compartment E for eggs, the compartment L which combine larvae and pupae and the compartment A for adult mosquitoes. Through the mathematical analysis, they have found the threshold between the extinction and the persistence of the mosquito population.

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However, it must be noticed that climate change affects the distribution and seasonal dynamics of mosquito populations [7, 12, 2, 17]. Thus, to model disease seasonality and persistence, it is essential that epidemiological models be coupled with accurate seasonal predictions of vector density. So, it would be more realistic to consider the climatic factors in the modeling of the mosquito population dynamics. Moreover, even if the larval stage and the pupal stage are two neighboring states, they have several different biological structures. So, it would be more realistic to distinguish them [10, 13].

Our model is an extension of the model studied in [14] in the sense that on the one hand, it takes into account the effect of temperature on the mosquitoes development cycle and the other hand, the main evolution stages of mosquitoes are distinguished. Particularly, the larval and pupal stages are clearly distinguished. Thus, the model is formulated by using ordinary differential equations [3] with periodic coefficients.

The paper is structured as follows. In section 2, we present the model of mosquito populations. In section 3, we use the theory of uniform persistence, the Floquet theory and the theorem of comparison to analyze the global behavior of the model. Numerical simulations are provided in section 4 in order to illustrate our theoretical results. In the final section, we conclude and give some future works.

2. Model formulation

2.1. Model description.

In the evolutionary cycle of mosquitoes, there are mainly two major stages: the mature stage and the immature stage. The immature stage is largely aquatic and the mature stage is aerial. The aquatic stage is composed of three main stages: egg, larval and pupal. Each of these stages is morphologically different from the other, with even the habitat of each stage differing. Thus, our model is composed of four compartments: the compartment of eggs, E, the compartment of larvae, L, the compartment of pupae, P and the compartment of adult mosquitoes, N_v.

The eggs are laid by female mosquitoes on the surface of water. Some days after, depending on the temperature, they hatch to larva. Mosquito larvae live in water from seven to fourteen days depending on the water temperature. Soon after they hatch out eggs, the larva begins to feed on bacteria and decaying organic matter on the water surface. Moreover, it has been observed that the larvae of some species such that *Aedes* are cannibal because they are able to eat earlier-stage larva under certain conditions, [15, 16]. After the larvae have completed moulting, they become pupae. The pupal stage is a resting non-feeding stage. After a few days, depending on the temperature and other circumstances, the adult mosquito emerges.

In each compartment, mosquitoes leave the population through natural death rates which depend on the climatic profile. Hence, from the above assumptions, the transfer diagram of our model is given by Figure 1.

The biological descriptions of the model parameters are given in the Table 1.

2.2. Mathematical formulation.

To describe the mosquito populations dynamics, we use the stage structured model which consists of four stages: eggs (E), larvae (L), pupae (P) and adults females (N_v); these stages represent the state variables of the model. Hence, assuming that the number of laid eggs is proportional to the number

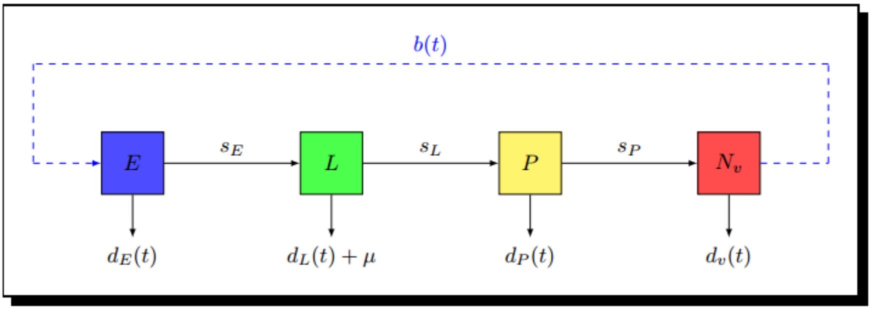


FIGURE 1. Transfer diagram: the solid arrows represent the transition from one class to another and the dashed arrow represents the eggs laying of female adult mosquitoes.

Parameters	**Biological descriptions**
K_E	available breeder sites occupied by eggs
K_L	available breeder sites occupied by larvae
K_P	available breeder sites occupied by pupae
s_E	transfer rate from eggs to larvae
s_L	transfer rate from larvae to pupae
s_P	transfer rate from pupae to adult mosquitoes
μ	death rate of larva due to cannibalism
$b(t)$	number of eggs laid per female at unit time t
$d_E(t)$	density-independent, temperature driven, mortality rate of eggs
$d_L(t)$	density-independent, temperature driven, mortality rate of larvae
$d_P(t)$	density-independent, temperature driven, mortality rate of pupae
$d_v(t)$	density-independent, temperature driven, mortality rate of adult mosquitoes

TABLE 1. Parameters of the model.

of female mosquitoes, we obtain the following system:

$$\begin{cases} \dot{E}(t) = b(t)N_v(t) - (s_E + d_E(t))E(t), \\ \dot{L}(t) = s_E E(t) - (s_L + \mu + d_L(t))L(t), \\ \dot{P}(t) = s_L L(t) - (s_P + d_P(t))P(t), \\ \dot{N}_v(t) = s_P P(t) - d_v(t)N_v(t). \end{cases}$$

Moreover, as we said above, mosquitoes lay their eggs on water or any moist surface, but they can also breed in natural habitats like vegetation or near rivers. However, if there are too much eggs in the oviposition habitat or too few nutrients and water resources, then females laid less eggs or choose another site. Therefore, larvae and pupae need water or nutrients to complete their development.

Mathematically, to express this biological phenomenon in the model, we introduce the availability of nutrients and the occupation by eggs, larvae and pupae of the available breeder sites. Hence, at any time,

- the per capita oviposition rate is given by,
$$b(t)\left(1-\frac{E(t)}{K_E}\right)N_v(t),$$
- the number of eggs that hatch and survive is given by,
$$s_E\left(1-\frac{L(t)}{K_L}\right)E(t),$$
- the number of larvae that survive to become pupae is given by,
$$s_L\left(1-\frac{P(t)}{K_P}\right)L(t).$$

Assuming that the temperature varies as a function of time, the proportion of individuals which survive from recruitment into one class, to maturation to the next, is defined by the following sequence of ODEs which leads to the final mathematical model:

$$\begin{cases} \dot{E}(t) = b(t)\left(1-\frac{E(t)}{K_E}\right)N_v(t) - (s_E + d_E(t))E(t), \\ \dot{L}(t) = s_E\left(1-\frac{L(t)}{K_L}\right)E(t) - (s_L + \mu + d_L(t))L(t), \\ \dot{P}(t) = s_L\left(1-\frac{P(t)}{K_P}\right)L(t) - (s_P + d_P(t))P(t), \\ \dot{N}_v(t) = s_P P(t) - d_v(t)N_v(t). \end{cases} \tag{1}$$

Let

$$Z(t) = (E(t), L(t), P(t), N_v(t))^T$$

and

$$f(t, Z(t)) = \begin{pmatrix} b(t)\left(1-\frac{E(t)}{K_E}\right)N_v(t) - (s_E + d_E(t))E(t) \\ s_E\left(1-\frac{L(t)}{K_L}\right)E(t) - (s_L + \mu + d_L(t))L(t) \\ s_L\left(1-\frac{P(t)}{K_P}\right)L(t) - (s_P + d_P(t))P(t) \\ s_P P(t) - d_v(t)N_v(t) \end{pmatrix}.$$

Then, the system (1) can be written as follows:

$$\dot{Z}(t) = f(t, Z(t)), \tag{2}$$

where the function $f : \mathbb{R}_+ \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is classe $\mathcal{C}^1$.

3. Mathematical analysis

We assume that:

(H1): the parameters of the model are positive except the cannibalism death rate, μ which is assumed to be nonnegative,

(H2): $d_E(t), d_L(t), d_P(t), b(t)$ and $d_v(t)$ are positive, continuous and periodic functions with the same period ω,

(H3): all adult mosquitoes measure refer to female mosquitoes.

For a continuous positive ω-periodic function $Q(t)$, we define

$$\hat{Q} = \sup_{t\in[0,\omega]} Q(t) \quad \text{and} \quad \bar{Q} = \inf_{t\in[0,\omega]} Q(t).$$

Remark 3.1. Mathematically, the system (1) is well defined over the whole $\mathbb{R}^4$. Nevertheless, the region of biological interest is given by

$$\Gamma = \left\{(E, L, P, N_v) \in \mathbb{R}^4_+ \ : \ E \leq K_E, \ L \leq K_L, \ P \leq K_P, \ N_v \leq \frac{s_P}{\bar{d}_v} K_P \right\}.$$

3.1. Positivity and boundedness of solutions.

Lemma 3.1. *The compact* Γ *is a positively invariant set, which attracts all positive orbits in* $\mathbb{R}^4_+$*. Moreover, all the solutions are bounded.*

Proof. From the system (1), we have

$$\begin{array}{rcl}
\dot{E}(t) & = & b(t)\left(1 - \dfrac{E(t)}{K_E}\right)N_v(t) - (s_E + d_E(t))E(t), \\
& \leq & \dfrac{\hat{b}s_P K_P}{\bar{d}_v} - \dfrac{\hat{b}s_P K_P}{\bar{d}_v K_E} E(t). \\
\dot{L}(t) & = & s_E\left(1 - \dfrac{L(t)}{K_L}\right)E(t) - (s_L + \mu + d_L(t))L(t), \\
& \leq & s_E K_E - \dfrac{s_E K_E}{K_L} L(t). \\
\dot{P}(t) & = & s_L\left(1 - \dfrac{P(t)}{K_P}\right)L(t) - (s_P + d_P(t))P(t), \\
& \leq & s_L K_L - \dfrac{s_L K_L}{K_P} P(t). \\
\dot{N}_v(t) & = & s_P P(t) - d_v(t)N_v(t) \\
& \leq & s_P K_P - \bar{d}_v N_v(t).
\end{array}$$

Thus, if

$$E(t) > K_E, \ L(t) > K_L, \ P(t) > K_P, \ N_v(t) > \frac{s_P K_P}{\bar{d}_v}$$

then,

$$\dot{N}_v(t) < 0, \ \dot{E}(t) < 0, \dot{L}(t) < 0, \dot{P}(t) < 0.$$

Let us consider the following auxiliary ordinary differential equations:

$$\begin{array}{rcl}
\dot{E}(t) & = & \dfrac{\hat{b}s_P K_P}{\bar{d}_v} - \dfrac{\hat{b}s_P K_P}{\bar{d}_v K_E} E(t), \\
\dot{L}(t) & = & s_E K_E - \dfrac{s_E K_E}{K_L} L(t), \\
\dot{P}(t) & = & s_L K_L - \dfrac{s_L K_L}{K_P} P(t) \\
\dot{N}_v(t) & = & s_P K_P - \bar{d}_v N_v(t).
\end{array}$$

Using the constant variation formula, we obtain the following solutions:

$$\begin{aligned} E(t) &= K_E + \Big(E(0) - K_E\Big) \exp\Big(- \frac{\hat{b} s_P K_P}{\bar{d}_v K_E} t\Big), \\ L(t) &= K_L + \Big(L(0) - K_L\Big) \exp\Big(- \frac{s_E K_E}{K_L} t\Big), \\ P(t) &= K_P + \Big(P(0) - K_P\Big) \exp\Big(- \frac{s_L K_L}{K_P} t\Big), \\ N_v(t) &= \frac{s_P K_P}{\bar{d}_v} + \Big(N_v(0) - \frac{s_P K_P}{\bar{d}_v}\Big) \exp(-\bar{d}_v t). \end{aligned}$$

By applying the standard comparison theorem, it then follows that :

$$\begin{aligned} E(t) &\leq K_E, \text{ if } E(0) \leq K_E, \\ L(t) &\leq K_L, \text{ if } L(0) \leq K_L, \\ P(t) &\leq K_P, \text{ if } P(0) \leq K_P, \\ N_v(t) &\leq \frac{s_P K_P}{\bar{d}_v}, \text{ if } N_v(0) \leq \frac{s_L K_L}{\bar{d}_v}. \end{aligned}$$

Thus, the compact set Γ is positively invariant and all the solutions are non-negative and bounded. □

Remark 3.2. The system (1) has always the mosquito-free equilibrium $\mathcal{E} = (0,0,0,0)$.

3.2. Threshold dynamics. The *vectorial reproduction ratio* of the nonautonomous model (1) can be computed using the technique in [20]. Linearizing the system (1) at the mosquito-free equilibrium, $\mathcal{E} = (0,0,0,0)$, we obtain the following system:

$$\begin{cases} \dot{E}(t) = b(t)N_v(t) - (s_E + d_E(t))E(t), \\ \dot{L}(t) = s_E E(t) - (s_L + \mu + d_L(t))L(t), \\ \dot{P}(t) = s_L L(t) - (s_P + d_P(t))P(t), \\ \dot{N}_v(t) = s_P P(t) - d_v(t)N_v(t). \end{cases} \tag{3}$$

The system (3) can be written as follows:

$$\dot{U}(t) = \Big(F(t) - V(t)\Big)U(t), \tag{4}$$

with

$$U(t) = \big(E(t), L(t), P(t), N_v(t)\big)^T,$$

$$V(t) = \begin{pmatrix} (s_E + d_E(t)) & 0 & 0 & 0 \\ -s_E & (s_L + \mu + d_L(t)) & 0 & 0 \\ 0 & -s_L & (s_P + d_P(t)) & 0 \\ 0 & 0 & -s_P & d_v(t) \end{pmatrix},$$

and

$$F(t) = \begin{pmatrix} 0 & 0 & 0 & b(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For all $t \geq s$, let $Y(t,s)$ be the evolution operator of the linear periodic system

$$\dot{y} = -V(t)y.$$

For each $s \in \mathbb{R}$, the 4×4 matrix $Y(t,s)$ satisfies the equation

$$\dot{Y}(t,s) = -V(t)Y(t,s), \forall t \geq s, \quad Y(s,s) = I, \tag{5}$$

where I is the 4×4 identity matrix.

Let C_ω be the ordered Banach space of all ω-periodic functions from $\mathbb{R}$ to $\mathbb{R}^4$ which is equipped with the maximum norm $\|.\|$ and the positive cone

$$\mathrm{C}_\omega^+ := \big\{\phi \in \mathrm{C}_\omega : \phi(t) \geq 0, \forall t \in \mathbb{R}\big\}.$$

Suppose $F(s)\phi(s) \in \mathrm{C}_\omega$ is the rate of generation (hatching) of new eggs in the breeding habits at time s in the individuals compartment in the periodic environment at time s.
Thus,

$$\psi(t) = \int_{-\infty}^{t} Y(t,s)F(s)\phi(s)ds = \int_0^\infty Y(t,t-a)F(t-a)\phi(t-a)da,$$

is the distribution of new eggs at time t, hatched by all female adult mosquitoes $\phi(s)$ introduced at the previous time.

Let $\mathcal{L} : C_\omega \longrightarrow C_\omega$ be the linear operator defined by

$$(\mathcal{L}\phi)(t) = \int_0^\infty Y(t,t-a)F(t-a)\phi(t-a)da, \quad \forall t \in \mathbb{R}, \ \phi \in C_\omega. \tag{6}$$

Then, *the vectorial reproduction ratio* is $\mathcal{R}_v := \rho(\mathcal{L})$, the spectral radius of $\mathcal{L}$.

In order to calculate $\mathcal{R}_v$, we consider the following linear ω-periodic system:

$$\dot{w}(t) = \left[\frac{1}{z}F(t) - V(t)\right]w(t), \quad \forall t \in \mathbb{R}_+, z \in (0,\infty). \tag{7}$$

Let $W(t,s,z),\ t \geq s,\ s \in \mathbb{R}$ be the evolution operator of the system (7) on $\mathbb{R}^4$. Clearly, we have $W(t,0,1) = \Phi_{F-V}(t),\ \forall\ t \geq 0$. Thus, the following result will be used in our numerical calculation of the *vectorial reproduction ratio.*

Lemma 3.2. [20]

(i) *If* $\rho(W(\omega,0,z)) = 1$ *has a positive solution* z_0*, then* z_0 *is an eigenvalue of* $\mathcal{L}$*, and hence* $\mathcal{R}_v > 0$.

(ii) *If* $\mathcal{R}_v > 0$*, then* $z = \mathcal{R}_v$ *is the unique solution of* $\rho(W(\omega,0,z)) = 1$.

(iii) $\mathcal{R}_v = 0$ *if and only if* $\rho(W(\omega,0,z)) < 1$*, for all* $z > 0$.

Thanks to this Lemma, we have the following main result, which shows that $\mathcal{R}_v$ is in scale with eggs laying, $b(t)$.

Theorem 3.3. *Let* $\sigma > 0$*. If* $\tilde{\mathcal{R}}_v$ *is the vectorial reproduction ratio corresponding to eggs laying rate* $\tilde{b}(t) = \sigma b(t)$*, then* $\tilde{\mathcal{R}}_v = \sigma\mathcal{R}_v$.

Proof. If $\tilde{b}(t) = \sigma b(t)$ then, the linear system (7) becomes:

$$\dot{w}(t) = \left[\frac{\sigma}{z}F(t) - V(t)\right]w(t), \quad \forall t \in \mathbb{R}_+, z \in (0,\infty).$$

Let $\tilde{F}(t) = \sigma F(t)$ and $\tilde{V}(t) = V(t)$. Let $\tilde{W}(\omega, 0, z)$ be the monodromy matrix of the following system:

$$\dot{\tilde{w}}(t) = \left[\frac{1}{z}\tilde{F}(t) - \tilde{V}(t)\right]\tilde{w}(t), \quad \forall t \in \mathbb{R}_+, z \in (0, \infty).$$

It is easy to remark that $\tilde{W}(\omega, 0, z) = W(\omega, 0, \frac{z}{\sigma})$. Thus, it then follows that $\rho(\tilde{W}(\omega, 0, \tilde{\mathcal{R}}_v)) = 1 \Longleftrightarrow \rho(W(\omega, 0, \frac{\tilde{\mathcal{R}}_v}{\sigma})) = 1$. Hence, $\tilde{\mathcal{R}}_v = \sigma \mathcal{R}_v$. □

3.3. Stability of mosquito-free equilibrium. To study the stability of the equilibrium $\mathcal{E}$, we use Floquet theory, [18] with comparison principle.

Lemma 3.4. [2]

i) $\mathcal{R}_v = 1$ *if and only if* $\rho(\Phi_{F-V}(\omega)) = 1$.
ii) $\mathcal{R}_v < 1$ *if and only if* $\rho(\Phi_{F-V}(\omega) < 1$.
iii) $\mathcal{R}_v > 1$ *if and only if* $\rho(\Phi_{F-V}(\omega)) > 1$.

Then, the mosquito-free equilibrium $\mathcal{E} = (0, 0, 0, 0)$ *is locally asymptotically stable if* $\mathcal{R}_v < 1$ *and unstable if* $\mathcal{R}_v > 1$.

Lemma 3.5. [21] *Let* $r = \dfrac{1}{\omega} \ln \rho(\Phi_{\mathcal{B}(.)}(\omega))$, *then there exists a positive* ω*-periodic function* $v(t)$ *such that* $e^{rt}v(t)$ *is a solution of* $\dot{x}(t) = \mathcal{B}(t)x(t)$.

Theorem 3.6. *The mosquito-free equilibrium* $\mathcal{E} = (0, 0, 0, 0)$ *is globally asymptotically stable if* $\mathcal{R}_v < 1$.

Proof. For all $t \geq 0$, we have:

$$1 - \frac{E(t)}{K_E} \leq 1, \quad 1 - \frac{L(t)}{K_L} \leq 1 \ \text{ and } \ 1 - \frac{P(t)}{K_P} \leq 1. \tag{8}$$

Thus, the system (1) can be rewritten as follows:

$$\begin{cases} \dot{E}(t) \leq b(t)N_v(t) - (s_E + d_E(t))E(t), \\ \dot{L}(t) \leq s_E E(t) - (s_L + \mu + d_L(t))L(t), \\ \dot{P}(t) \leq s_L L(t) - (s_P + d_P(t))P(t), \\ \dot{A}(t) = s_P P(t) - d_v(t)N_v(t). \end{cases}$$

Let us consider the following auxiliary system:

$$\dot{\tilde{Z}}_1(t) = \mathcal{A}(t)\tilde{Z}_1(t), \tag{9}$$

with

$$\tilde{Z}_1(t) = (\tilde{E}(t), \tilde{L}(t), \tilde{P}(t), \tilde{N}_v(t))^T$$

and

$$\mathcal{A}(t) = \begin{pmatrix} -(s_E + d_E(t)) & 0 & 0 & b(t) \\ s_E & -(s_L + \mu + d_L(t)) & 0 & 0 \\ 0 & s_L & -(s_P + d_P(t)) & 0 \\ 0 & 0 & s_P & -d_v(t) \end{pmatrix}.$$

From Lemma 3.4, if $\mathcal{R}_v < 1$, then $\rho(\Phi_{F-V}(\omega)) < 1$. Moreover, from the Lemma 3.5, there exists a positive ω-periodic function $v(t)$ such that $\tilde{Z}_1(t) = v(t)e^{rt}$ with $r = \dfrac{1}{\omega} \ln \rho(\Phi_{\mathcal{A}}(\omega))$. Since the function $v(t)$ is bounded and $\rho(\Phi_{\mathcal{A}}(\omega)) = \rho(\Phi_{F-V}(\omega)) < 1$

then, $r < 0$ and $\tilde{Z}_1(t) \to 0$ as $t \to \infty$. By applying the comparison theorem [11] on system (9), we get

$$\lim_{t\to+\infty}(E(t), L(t), P(t), N_v(t)) = (0, 0, 0, 0).$$

Hence, the mosquito-free equilibrium, $\mathcal{E}$ is globally attractive. □

3.4. Existence of positive periodic solutions. Let us consider the following sets:

$$\begin{array}{rcl} X & := & \mathbb{R}^4_+, \\ X_0 & := & \operatorname{int}(\mathbb{R}^4_+), \\ \partial X_0 & := & X \setminus X_0. \end{array}$$

Let $u(t, \varphi)$ be the unique solution of (1) with initial condition φ, $\Phi(t)$ the periodic semiflow generated by periodic system (1) and $S : X \longrightarrow X$ the Poincaré map associated with system (1), namely:

$$\begin{array}{rcl} S(\varphi) & = & \Phi(\omega)\varphi = u(\omega, \varphi), \ \forall \varphi \in X. \\ S^n(\varphi) & = & \Phi(n\omega)\varphi = u(n\omega, \varphi), \forall n \geq 0. \end{array}$$

We notice that $\forall \varphi \in X_0, \Phi(t)(\varphi) = u(t, \varphi) \in X_0$. Thus, $\Phi(t)(X_0) \subset X_0, \forall t \geq 0$. So, X_0 and ∂X_0 are positively invariant. Therefore, Lemma 3.1 implies that discrete-time system $S : X \longrightarrow X$ is point dissipative.

Lemma 3.7. *If* $\mathcal{R}_v > 1$, *there exists* $\delta > 0$ *such that when* $\|\varphi - \mathcal{E}\| \leq \delta$, *for any* $\varphi \in X_0$, *one has* $\limsup_{k\to\infty} d\big(S^k(\varphi), \mathcal{E}\big) \geq \delta$, *where* $\varphi = (E(0), L(0), P(0), N_v(0))$.

Proof. Suppose by contradiction that $\limsup_{k\to\infty} d\big(S^k(\varphi), \mathcal{E}\big) < \delta$ for some $\varphi \in X_0$. Then, there exists an integer k_2 such that for all $k \geq k_2, d\big(S^k(\varphi), \mathcal{E}\big) < \delta$. By the continuity of the solution $u(t, \varphi)$, we have

$$\|u(t, S^k(\varphi)) - u(t, \mathcal{E})\| \leq \alpha, \forall t \geq 0 \text{ and } \alpha > 0.$$

Let $t = k\omega + t_1$, where $t_1 \in [0, \omega]$ and $k = [\frac{t}{\omega}]$. $[\frac{t}{\omega}]$ is the greatest integer less than or equal to $\frac{t}{\omega}$. If $\|\varphi - \mathcal{E}\| \leq \delta$, then we get

$$\begin{array}{rcl} \|u(t, \varphi) - u(t, \mathcal{E})\| & = & \|u(t_1 + k\omega, \varphi) - u(t_1 + k\omega, \mathcal{E})\| \\ & = & \big\|u(t_1, S^k(\varphi)) - u(t_1, \mathcal{E})\big\| \leq \alpha, \text{ for any } t \geq 0. \end{array}$$

It then follows that $0 \leq E(t) \leq \alpha, 0 \leq L(t) \leq \alpha$ and $0 \leq P(t) \leq \alpha$, for all $t \geq 0$. Thus, there exists $\alpha^* = \max\left\{\frac{\alpha}{K_E}, \frac{\alpha}{K_L}, \frac{\alpha}{K_P}\right\}$ such that

$$1 - \frac{E(t)}{K_E} \geq 1 - \alpha^*, \ \ 1 - \frac{L(t)}{K_L} \geq 1 - \alpha^* \text{ and } 1 - \frac{P(t)}{K_P} \geq 1 - \alpha^*, \text{ for all } t \geq 0.$$

Furthermore, the system (1) can be rewritten as follows:

$$\left\{\begin{array}{l} \dot{E}(t) \geq b(t)(1 - \alpha^*)N_v(t) - (s_E + d_E(t))E(t), \\ \dot{L}(t) \geq s_E(1 - \alpha^*)E(t) - (s_L + \mu + d_L(t))L(t), \\ \dot{P}(t) \geq s_L(1 - \alpha^*)L(t) - (s_P + d_P(t))L(t), \\ \dot{N}_v(t) = s_P P(t) - d_v(t)N_v(t). \end{array}\right. \tag{10}$$

Let us consider the following auxiliary linear system

$$\dot{\tilde{Z}}_2(t) = M_{\alpha^*}(t)\tilde{Z}_2(t), \tag{11}$$

where

$$\tilde{Z}_2 = (\tilde{E}(t), \tilde{L}(t), \tilde{P}(t), \tilde{N}_v(t))^T$$

and

$$M_{\alpha^*}(t) = \begin{pmatrix} -(s_E + d_E(t)) & 0 & 0 & b(t)(1-\alpha^*) \\ s_E(1-\alpha^*) & -(s_L + \mu + d_L(t)) & 0 & 0 \\ 0 & s_L(1-\alpha^*) & -(s_P + d_P(t)) & 0 \\ 0 & 0 & s_P & -d_v(t) \end{pmatrix}.$$

We have $\lim\limits_{\alpha^*\to 0^+} \Phi_{M_{\alpha^*}}(\omega) = \Phi_{F-V}(\omega)$ and by the continuity of the spectral radius, we have $\lim\limits_{\alpha^*\to 0^+} \rho\big(\Phi_{M_{\alpha^*}}(\omega)\big) = \rho(\Phi_{F-V}(\omega))$. From Lemma 3.4, if $\mathcal{R}_v > 1$ then $\rho(\Phi_{F-V}(\omega)) > 1$ and then $\lim\limits_{\alpha^*\to 0^+} \rho\big(\Phi_{M_{\alpha^*}}(\omega)\big) > 1$. It follows that there exists $\alpha_1 > 0$ such that $\rho\big(\Phi_{M_{\alpha^*}}(\omega)\big) > 1$, for all $\alpha \in [0, \alpha_1[$. From Lemma 3.5, there exists a positive ω-periodic function $v(t)$ such that $\tilde{Z}_2 = e^{rt}v(t)$ with $r = \dfrac{1}{\omega}\ln\rho\big(\Phi_{M_{\alpha^*}}(\omega)\big)$. Since $\rho\big(\Phi_{M_{\alpha^*}}(\omega)\big) > 1$ and the function $v(t)$ is bounded, then $r > 0$ and then $\tilde{Z}_2 \to \infty$ as $t \to \infty$. By applying the comparison theorem on system (11), we get $\lim\limits_{t\to\infty} |(E(t), L(t), P(t), N_v(t))| = \infty$, which contradicts the fact that the solutions are bounded. □

Theorem 3.8. *If $\mathcal{R}_v > 1$, there exists $\eta > 0$ such that any solution $(E(t), L(t), P(t), N_v(t))$ with initial condition $\varphi \in X_0$ satisfies*

$$\liminf_{t\to\infty} E(t) \geq \eta, \quad \liminf_{t\to\infty} L(t) \geq \eta, \quad \liminf_{t\to\infty} P(t) \geq \eta, \quad \liminf_{t\to\infty} N_v(t) \geq \eta$$

and the system (1) has at least one positive periodic solution.

Proof. Denote

$$M_\partial = \{\phi \in \partial X_0 : S^k(\phi) \in \partial X_0, k \geq 0\}.$$

At first, we prove that $M_\partial = \{(0,0,0,0)\}$. Indeed, it is obvious that $\{(0,0,0,0)\} \subset M_\partial$. So, we only need to show that $M_\partial \subset \{(0,0,0,0)\}$, that means that for any initial condition

$$\varphi \in \partial X_0, E(k\omega)L(k\omega)P(k\omega)N_v(k\omega) = 0, \forall k \geq 0.$$

Let $\varphi \in \partial X_0$. Suppose by contradiction that there exists an integer $k_1 \geq 0$ such that $(E(k_1\omega), L(k_1\omega), P(k_1\omega), N_v(k_1\omega))^T > 0$. Using the constant variation, we derive that

$$E(t) = \exp\Big(-\int_0^t H_1(s)ds\Big)\Big[E(0) + \int_0^t b(s)N_v(s)\exp\Big(\int_0^s H_1(\tau)d\tau\Big)ds\Big] \tag{12}$$

$$L(t) = \exp\Big(-\int_0^t H_2(s)ds\Big)\Big[L(0) + \int_0^t s_E E(s)\exp\Big(\int_0^s H_2(\tau)d\tau\Big)ds\Big] \tag{13}$$

$$P(t) = \exp\Big(-\int_0^t H_3(s)ds\Big)\Big[P(0) + \int_0^s s_L L(s)\exp\Big(\int_0^s H_3(\tau)d\tau\Big)ds\Big] \tag{14}$$

$$N_v(t) = \exp\Big(-\int_0^t d_v(s)ds\Big)\Big[N_v(0) + \int_0^t s_P P(s)\exp\Big(\int_0^s d_v(\tau)d\tau\Big)ds\Big] \tag{15}$$

where

$$\begin{aligned} H_1(t) &= \frac{b(t)}{K_E} N_v(t) + d_E(t) + s_E, \quad H_2(t) = \frac{s_E}{K_L} E(t) + d_L(t) + s_L, \\ H_3(t) &= \frac{s_L}{K_P} L(t) + d_P(t) + \mu + s_P. \end{aligned}$$

Thus, taking $k_1\omega$ as the initial time in (12)−(15), we obtain that $E(t) > 0, L(t) > 0, P(t) > 0$ and $N_v(t) > 0$, for all $t > k_1\omega$, which contradicts the fact that ∂X_0 is positively invariant so $M_\partial = \{(0,0,0,0)\}$.

The equality $M_\partial = \{(0,0,0,0)\}$ implies that $\mathcal{E}$ is a fixed point of S and acyclic in M_∂, every solution in M_∂ approaches to $\mathcal{E}$. Moreover, Lemma 3.7 implies that $\mathcal{E}$ is an isolate invariant set in X and $W^s(\mathcal{E}) \cap X_0 = \emptyset$, where $W^s(\mathcal{E}) = \big\{x \in X : \limsup_{k\to\infty} d\big(S^k(x), \mathcal{E}\big) = 0\big\}$ is the stable set of $\mathcal{E}$. By the acyclicity theorem on uniform persistence for maps [[22], Theorem 3.1.1], it follows that S is uniformly persistent with respect to $(X_0, \partial X_0)$. So, the periodic semiflow $\Phi(t)$ is also uniformly persistent. It follows that there exists $\eta > 0$ such that any solution $(E(t), L(t), P(t), N_v(t))$ with initial condition $(E(0), L(0), P(0), N_v(0)) \in X_0$ satisfies

$$\liminf_{t\to\infty} E(t) \geq \eta, \quad \liminf_{t\to\infty} L(t) \geq \eta, \quad \liminf_{t\to\infty} P(t) \geq \eta, \quad \liminf_{t\to\infty} N_v(t) \geq \eta.$$

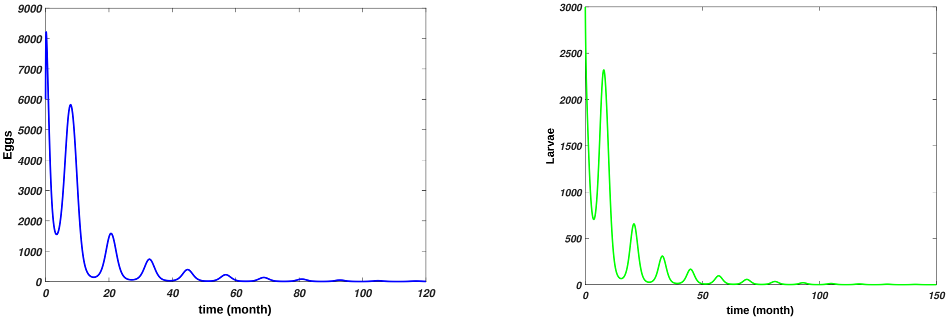
Furthermore, thanks to Theorem 1.3.6 in [22], the system (1) has at least one periodic solution $u_p(t, \varphi^*)$ with $\phi^* = (E^*(0), L^*(0), P^*(0), N_v^*(0)) \in X_0$. Since $\varphi^* \in X_0$, then from equations (12) - (15) we obtain that $E^*(t) > 0$, $L^*(t) > 0$, $P^*(t) > 0$ and $N_v^*(t) > 0$ for all $t \geq 0$. Thus, the ω-periodic solution $u_p(t, \varphi^*)$ is positive. □

4. Numerical simulations

In this section, we make some numerical simulations to support our mathematical results. Moreover, we perform some sensitivity analysis to determine the influence of the parameter, $b(t)$ on the threshold dynamics, $\mathcal{R}_v$. Our numerical simulation will be performed using the MATLAB technical computing software with the fourth-order Runge Kutta method.

Parameters	**Values**	**References**	**Dimensions**
ω	12	estimated	/month
K_E	10000	estimated	-
K_L	8000	estimated	-
K_P	6000	estimated	-
s_E	-	[14]	/month
s_L	-	[14]	/month
s_p	-	[14]	/month
μ	-	[12]	/month

Table 2. Values for constant parameters of the model.



(a) Density of eggs.

(b) Density of larvae.

FIGURE 2. Extinction of eggs and larvae with $s_E = 0.6,\ s_L = 0.4,\ s_P = 0.5,\ \mu = 0.1,\ a_1 = 25,\ a_2 = 1.5,\ a_3 = 0.7,\ a_4 = 0.9,\ a_5 = 0.8,\ b_1 = 2,\ b_2 = 1.3,\ b_3 = 0.5,\ b_4 = 0.6,\ b_5 = 0.7$ and initial conditions $E(0) = 6000,\ L(0) = 3000,\ P(0) = 2000,\ N_v(0) = 5000$. We obtain $\mathcal{R}_v = 0.8864 < 1$.

By using the periodicity of the functions $d_E(t),\ d_L(t),\ d_P(t),\ d_v(t)$ and $b(t)$, we can write them in the following forms:

$$\begin{array}{rcl} b(t) & = & a_1 + b_1 \cos\left(\dfrac{\pi t}{6}\right), \quad d_E(t) = a_2 + b_2 \cos\left(\dfrac{\pi t}{6}\right), \\ d_L(t) & = & a_3 + b_3 \cos\left(\dfrac{\pi t}{6}\right), \quad d_v(t) = a_4 + b_4 \cos\left(\dfrac{\pi t}{6}\right), \\ d_P(t) & = & a_5 + b_5 \cos\left(\dfrac{\pi t}{6}\right), \end{array}$$

where the constant numbers a_i and b_i are positive for all $i \in \{1, 2, 3, 4, 5\}$.

4.1. Illustration of mathematical results. We give some numerical results in order to illustrate our theoretical results.

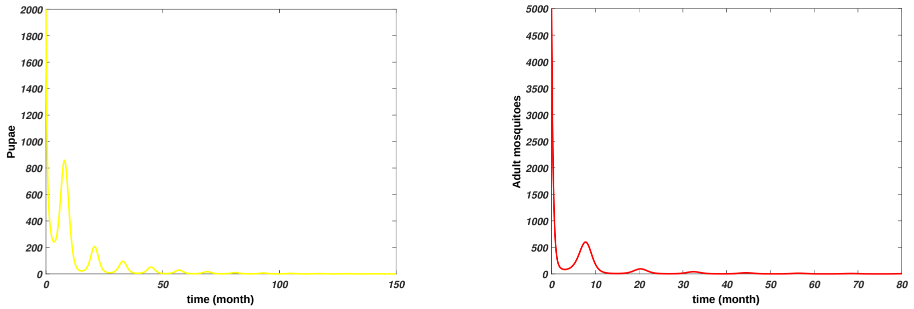
Figures 2 and 3 show the extinction of eggs, larvae, pupae and adult mosquitoes. Moreover, we observe that the mosquito-free equilibrium $\mathcal{E} = (0, 0, 0, 0)$ is globally stable, which illustrate the result of our Theorem 3.6.

Now, we examine the uniform persistence of mosquito populations and the existence of periodic solutions.

Figures 4 and 5 show the persistence of eggs density 4(a), larvae density 4(b), pupae density 5(a) and adult mosquitoes density 5(b). Furthermore, we remark that the model (1) admits a periodic solution which illustrate our mathematical result of subsection 3.3.

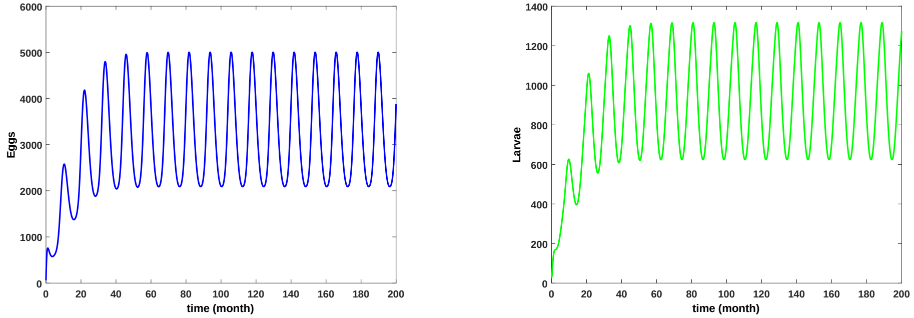
Next, we analyze numerically the global dynamics of the system for $\mathcal{R}_v = 1$.

Remark 4.1. In our mathematical analysis, we have not established the stability of the mosquito-free equilibrium for $\mathcal{R}_v = 1$. However, through numerical simulations, we observe that the mosquito-free equilibrium, $\mathcal{E}$ is globally asymptotically stable if $\mathcal{R}_v = 1$ (see Figures 6 and 7).



(a) Density of pupae. (b) Density of adult mosquitoes.

FIGURE 3. Extinction of pupae and adult mosquitoes with $s_E = 0.6,\ s_L = 0.4,\ s_P = 0.5,\ \mu = 0.1,\ a_1 = 25,\ a_2 = 1.5,\ a_3 = 0.7,\ a_4 = 0.9,\ a_5 = 0.8,\ b_1 = 2,\ b_2 = 1.3,\ b_3 = 0.5,\ b_4 = 0.6,\ b_5 = 0.7$ and initial conditions $E(0) = 6000,\ L(0) = 3000,\ P(0) = 2000,\ N_v(0) = 5000$. We obtain $\mathcal{R}_v = 0.8864 < 1$.



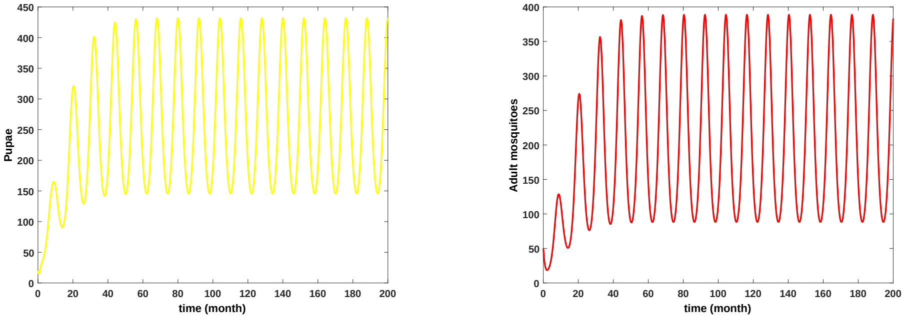
(a) Density of eggs. (b) Density of larvae.

FIGURE 4. Persistence of eggs and larvae with $s_E = 0.5,\ s_L = 0.3,\ s_P = 0.4,\ \mu = 0.4,\ a_1 = 25,\ a_2 = 0.55,\ a_3 = 0.4,\ a_4 = 0.85,\ a_5 = 0.65,\ b_1 = 15,\ b_2 = 0.35,\ b_3 = 0.25,\ b_4 = 0.7,\ b_5 = 0.45$ and initial conditions $E(0) = 60,\ L(0) = 30,\ P(0) = 20,\ N_v(0) = 50$. We obtain $\mathcal{R}_v = 1.7952 > 1$.

4.2. Sensitivity analysis. Now, we perform some sensitivity analysis to determine the influence of eggs laying rate, $b(t)$ on the dynamics of mosquito population.

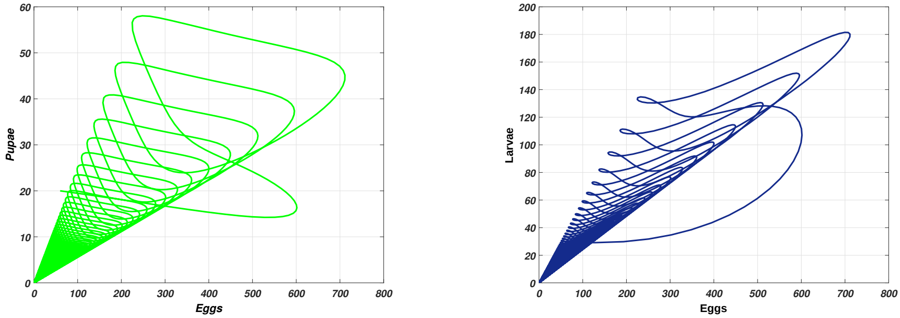
If we replace the eggs laying rate, $b(t)$ by $\tilde{b}(t) = \sigma b(t)$ in the model (1), we obtain the following numerical results.

- if $\sigma = 0.9$, then $\tilde{b}(t) = 0.9b(t)$ and $\tilde{\mathcal{R}}_v = 0.9 \times 1.7952 = 1.6157$.
- if $\sigma = 0.8$, then $\tilde{b}(t) = 0.8b(t)$ and $\tilde{\mathcal{R}}_v = 0.8 \times 1.7952 = 1.4361$.
- if $\sigma = 0.7$, then $\tilde{b}(t) = 0.7b(t)$ and $\tilde{\mathcal{R}}_v = 0.7 \times 1.7952 = 1.2566$.
- if $\sigma = 0.6$, then $\tilde{b}(t) = 0.6b(t)$ and $\tilde{\mathcal{R}}_v = 0.6 \times 1.7952 = 1.0771$.
- if $\sigma = 0.5$, then $\tilde{b}(t) = 0.5b(t)$ and $\tilde{\mathcal{R}}_v = 0.5 \times 1.7952 = 0.8976$.



(a) Density of pupae. (b) Density of adult mosquitoes.

FIGURE 5. Persistence of pupae and adult mosquitoes with $s_E = 0.5,\ s_L = 0.3,\ s_P = 0.4,\ \mu = 0.4,\ a_1 = 25,\ a_2 = 0.55,\ a_3 = 0.4,\ a_4 = 0.85,\ a_5 = 0.65,\ b_1 = 15,\ b_2 = 0.35,\ b_3 = 0.25,\ b_4 = 0.7,\ b_5 = 0.45$ and initial conditions $E(0) = 60,\ L(0) = 30,\ P(0) = 20,\ N_v(0) = 50$. We obtain $\mathcal{R}_v = 1.7952 > 1$.

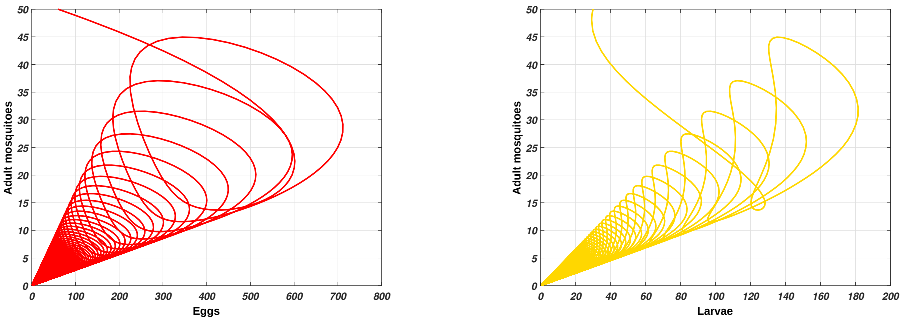


(a) Density of pupae/eggs. (b) Density of larvae/eggs.

FIGURE 6. Extinction of eggs, pupae and larvae with $s_E = 0.5,\ s_L = 0.3,\ s_P = 0.4,\ \mu = 0.4,\ a_1 = 18.8219216,\ a_2 = 0.7,\ a_3 = 0.4,\ a_4 = 0.85,\ a_5 = 0.65,\ b_1 = 15,\ b_2 = 0.35,\ b_3 = 0.25,\ b_4 = 0.7,\ b_5 = 0.45$ and initial conditions $E(0) = 60,\ L(0) = 30,\ P(0) = 20,\ N_v(0) = 50$. We obtain $\mathcal{R}_v = 1$.

- if $\sigma = 0.4$, then $\tilde{b}(t) = 0.4b(t)$ and $\tilde{\mathcal{R}}_v = 0.4 \times 1.7952 = 0.7180$.
- if $\sigma = 0.3$, then $\tilde{b}(t) = 0.3b(t)$ and $\tilde{\mathcal{R}}_v = 0.3 \times 1.7952 = 0.5385$.
- if $\sigma = 0.2$, then $\tilde{b}(t) = 0.2b(t)$ and $\tilde{\mathcal{R}}_v = 0.2 \times 1.7952 = 0.3590$.
- if $\sigma = 0.1$, then $\tilde{b}(t) = 0.1b(t)$ and $\tilde{\mathcal{R}}_v = 0.1 \times 1.7952 = 0.1795$.

Our numerical results show that the dynamics of mosquito population depends considerably on the eggs laying rate of female mosquitoes, $b(t)$. The larger $b(t)$ is, the larger the vectorial reproduction ratio becomes (see Figure 8), that illustrates our mathematical result of Theorem 3.3. Therefore, this parameter may be an excellent means in fighting against the proliferation of mosquitoes.



(a) Density of adult mosquitoes/eggs. (b) Density of adult mosquitoes/larvae.

FIGURE 7. Extinction of eggs, larvae and adult mosquitoes with $s_E = 0.5,\ s_L = 0.3,\ s_P = 0.4,\ \mu = 0.4,\ a_1 = 18.8219216,\ a_2 = 0.7,\ a_3 = 0.4,\ a_4 = 0.85,\ a_5 = 0.65,\ b_1 = 15,\ b_2 = 0.35,\ b_3 = 0.25,\ b_4 = 0.7,\ b_5 = 0.45$ and initial conditions $E(0) = 60,\ L(0) = 30,\ P(0) = 20,\ N(0) = 50$. We obtain $\mathcal{R}_v = 1$.

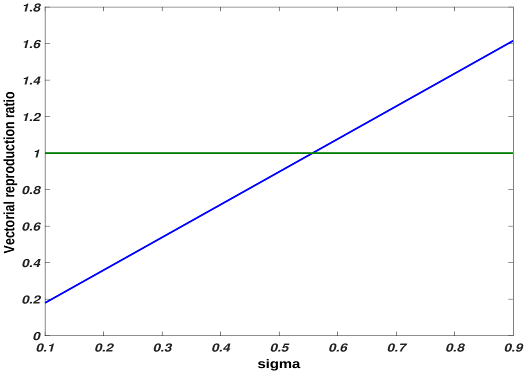


FIGURE 8. Relationship between $\mathcal{R}_v$ and σ

.

5. Conclusion

In this paper, we have formulated a mathematical model based on ordinary differential equations to describe the dynamics of mosquito population in a periodic environment. It emerges from our mathematical analysis that the dynamics of mosquito population is determined by the threshold dynamics, $\mathcal{R}_v$. Indeed, we have shown that if $\mathcal{R}_v < 1$, the mosquito-free equilibrium $(0, 0, 0, 0)$ is globally stable and if $\mathcal{R}_v > 1$, the system admits at least a positive periodic solution. Furthermore, numerical simulations have indicated that on the one hand the equilibrium $(0, 0, 0, 0)$ is globally stable if $\mathcal{R}_v = 1$ and on the other hand the periodic solution is globally stable if $\mathcal{R}_v > 1$. We have also shown that the dynamics of mosquito population depends on the eggs laying rate. This parameter may be used to reduce the number of mosquitoes in endemic regions.

In future works, one can construct a more realistic model of mosquito-borne diseases by coupling the model (1) to the transmission model. In addition, the numerical result about the stability of the periodic solution will be another further work.

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