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A finite difference scheme in Hilbert spaces

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Abstract. We study the existence for a class of difference inclusions associated with maximal monotone operators. They are the discrete versions of some second order evolution equations in Hilbert spaces on a finite interval.

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1. Introduction

Let H be a real Hilbert space with the scalar product $(.,.)$ and the corresponding norm $||.||$. We study the existence and uniqueness of the solution for the finite difference scheme

$$\begin{cases} u_{i+1} - (1+\theta_i) u_i + \theta_i u_{i-1} \in c_i A u_i + f_i, \ i = \overline{1,N} \\ u_1 - u_0 \in \alpha (u_0 - a), \ u_{N+1} - u_N \in -\beta (u_{N+1} - b), \end{cases} \tag{1}$$

where $\alpha,\ \beta$ and A are maximal monotone operators in $H,\ A$ is also strongly monotone, $a, b \in H$ and $(f_i)_{i=\overline{1,N}} \in H^N,\ \theta_i \in (0,1),\ 0 < c_i,\ i = \overline{1,N}$ are finite sequences.

Denote by $H^N_{a_i}$ the space H^N with the weight sequence $(a_i)_{i=\overline{0,N}}$, where $a_0 = 1,\ a_i = 1/\theta_1\theta_2...\theta_i$, for $i = \overline{1,N}$. This sequence is nondecreasing and $a_{i-1} = \theta_i a_i,\ i = \overline{1,N}$. Therefore, the scalar product in $H^N_{a_i}$ is

$$< (u_i)_{i=\overline{1,N}}, (v_i)_{i=\overline{1,N}} >= \sum_{i=1}^{N} a_i (u_i, v_i), \tag{2}$$

for all $(u_i)_{i=\overline{1,N}}$, $(v_i)_{i=\overline{1,N}} \in H^N$ and the norm is

$$\left| (u_i)_{i=\overline{1,N}} \right| = \left(\sum_{i=1}^{N} a_i ||u_i||^2 \right)^{1/2}. \tag{3}$$

Since $1 = a_0 \leq a_1 \leq ... \leq a_N$, the spaces H^N and $H^N_{a_i}$ contains the same sequences and have equivalent norms. The reason we have introduced the space $H^N_{a_i}$ is that the operator B given by

$$B\left((u_i)_{i=\overline{1,N}}\right) = (-u_{i+1} + (1+\theta_i) u_i - \theta_i u_{i-1})_{i=\overline{1,N}}, \tag{4}$$

$$D(B) = \{(u_i)_{i=\overline{1,N}} \in H^N,\ u_1 - u_0 \in \alpha (u_0 - a), \\ u_{N+1} - u_N \in -\beta (u_{N+1} - b)\} \tag{5}$$

is maximal monotone in $H^N_{a_i}$ (see Proposition 2.1). This is the main tool in the proof of our existence result.

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Problem (1) is the discrete variant of the problem

$$\begin{cases} pu'' + ru' \in Au + f, \text{ a.e. on } [0,T] \\ u'(0) \in \alpha(u(0) - a), \ u'(T) \in -\beta(u(T) - b), \end{cases} \tag{6}$$

which was studied by A. Aftabizadeh & N. Pavel [1]. Different particular cases of (6) were analyzed before by V. Barbu [4], [5], H. Brézis [6], N. Pavel [9], [10], L. Véron [13], N. Apreutesei [2]. Taking

$$j(x) = \begin{cases} 0, & x = 0 \\ +\infty, & \text{otherwise} \end{cases}$$

and $\alpha(x) = \beta(x) = \partial j(x)$, where ∂j is the subdifferential mapping of the convex function j, one obtains the bilocal problem

$$\begin{cases} u_{i+1} - (1+\theta_i) u_i + \theta_i u_{i-1} \in c_i A u_i + f_i, \ i = \overline{1,N} \\ u_0 = a, \quad u_{N+1} = b. \end{cases} \tag{7}$$

This equation together with the problem

$$\begin{cases} u_{i+1} - (1+\theta_i) u_i + \theta_i u_{i-1} \in c_i A u_i + f_i, \ i \geq 1 \\ u_0 = a, \quad \sup_{i \geq 1} ||u_i|| < \infty, \end{cases} \tag{8}$$

was the subject of many papers. G. Morosanu [8] and E. Mitidieri & G. Morosanu [7] proved the existence and the asymptotic behavior of the solution to (7) and (8) for $\theta_i \equiv 1, \ f_i \equiv 0$ in Hilbert spaces, while E. Poffald & S. Reich established similar results in Banach spaces [11], [12]. For arbitrary $\theta_i \geq 1$, equations (7) and (8) were studied by N. Apreutesei [3].

In this paper we suppose $\theta_i \in (0,1)$. This corresponds to the case $r > 0$ on $[0,T]$ in equation (6). The boundary conditions in (1) are new for difference equations.

In section 2 we give an auxiliary result, namely we show the maximal monotonicity of the operator B defined by $(4)-(5)$. We use the Yosida approximation of A to prove the existence and uniqueness of the solution of problem (1). This is the subject of section 3.

2. The maximal monotonicity of B in $H_{a_i}^N$

The aim of this section is to prove that the operator B defined by $(4)-(5)$ is maximal monotone in $H_{a_i}^N$. We use an idea from A. Aftabizadeh & N. Pavel [1]. Denoting

$$a_0 = 1, \ a_i = \frac{1}{\theta_1 \theta_2 ... \theta_i}, \ i = \overline{1,N} \tag{9}$$

and

$$\varphi_i = a_{i-1}(u_i - u_{i-1}), \ i = \overline{1,N}, \tag{10}$$

we can write B under the form

$$\begin{aligned} B\left((u_i)_{i=\overline{1,N}}\right) &= \left(-u_{i+1} + (1+\theta_i) u_i - \theta_i u_{i-1}\right)_{i=\overline{1,N}} = \\ &= \left(-\frac{1}{a_i}(\varphi_{i+1} - \varphi_i)\right)_{i=\overline{1,N}}, \end{aligned} \tag{11}$$

$$\begin{aligned} D(B) = \{&(u_i)_{i=\overline{1,N}} \in H^N, \ u_1 - u_0 \in \alpha(u_0 - a), \\ &u_{N+1} - u_N \in -\beta(u_{N+1} - b)\}. \end{aligned} \tag{12}$$

We begin with

Lemma 2.1. *Let* $(\theta_i)_{i=\overline{1,N}}$ *be a given sequence in* $(0,1)$ *and* $c>0$ *a constant. Then, the problem*

$$\begin{cases} \xi_{i+1}-(2+\theta_i)\,\xi_i+\theta_i\xi_{i-1}=0,\ i=\overline{1,N} \\ \xi_0=0, \quad \xi_1=c \end{cases} \tag{13}$$

has a strictly increasing solution $\xi_i>0$, *for all* $i=\overline{1,N+1}$ *and the problem*

$$\begin{cases} \eta_{i+1}-(2+\theta_i)\,\eta_i+\theta_i\eta_{i-1}=0,\ i=\overline{1,N} \\ \eta_{N+1}=0, \quad \eta_N=-c \end{cases} \tag{14}$$

has a strictly increasing solution $\eta_i<0,\ i=\overline{0,N}$.

The proof is obvious.

Now we are able to state the main result of this section.

Proposition 2.1. *If* $(\theta_i)_{i=\overline{1,N}}$ *is a finite sequence of real numbers,* $\theta_i\in(0,1)$ *for all* $i=\overline{1,N},\ a,b\in H$ *and* $\alpha,\ \beta$ *are maximal monotone operators in* H, *then the operator* B *given by* $(11)-(12)$ *is maximal monotone in* $H_{a_i}^N$.

Proof. Let $(u_i)_{i=\overline{1,N}}$, $(v_i)_{i=\overline{1,N}}$ be two given sequences in $D(B)$ and $\varphi_i = a_{i-1}(u_i-u_{i-1})\,,\ \psi_i=a_{i-1}(v_i-v_{i-1})\,,\ i=\overline{1,N}$. If $<.,.>$ is the scalar product in $H_{a_i}^N$ defined by (2), we have

$$< B\left((u_i)_{i=\overline{1,N}}\right)-B\left((v_i)_{i=\overline{1,N}}\right),(u_i-v_i)_{i=\overline{1,N}} >=$$

$$=-\sum_{i=1}^N(\varphi_{i+1}-\varphi_i-\psi_{i+1}+\psi_i,u_i-v_i)=\sum_{i=1}^N a_i||u_{i+1}-u_i-v_{i+1}+v_i||^2+$$

$$+\sum_{i=1}^N\left[(\varphi_i-\psi_i,u_i-v_i)-(\varphi_{i+1}-\psi_{i+1},u_{i+1}-v_{i+1})\right],$$

so

$$< B\left((u_i)_{i=\overline{1,N}}\right)-B\left((v_i)_{i=\overline{1,N}}\right),(u_i-v_i)_{i=\overline{1,N}} >=$$

$$=\sum_{i=1}^N a_i||u_{i+1}-u_i-v_{i+1}+v_i||^2-$$

$$-a_N(u_{N+1}-u_N-v_{N+1}+v_N,u_{N+1}-v_{N+1})+$$

$$+(u_1-u_0-v_1+v_0,u_1-v_1). \tag{15}$$

Since $(u_i)_{i=\overline{1,N}}$, $(v_i)_{i=\overline{1,N}}\in D(B)$, by the monotonicity of α and β, it follows that

$$(u_1-u_0-v_1+v_0,u_1-v_1)=||u_1-u_0-v_1+v_0||^2+$$

$$+(u_1-u_0-v_1+v_0,u_0-v_0)\geq 0 \tag{16}$$

and

$$(u_{N+1}-u_N-v_{N+1}+v_N,u_{N+1}-v_{N+1})\leq 0. \tag{17}$$

Using (16) and (17) in (15), one obtains that B is monotone in $H_{a_i}^N$.

Now we show that B is maximal monotone, that is $R(I+B)=H_{a_i}^N$ or, equivalently, $(\forall)\ (g_i)_{i=\overline{1,N}}\in H^N$, there is a sequence $(u_i)_{i=\overline{1,N}}\in H^N$ such that

$$\begin{cases} u_{i+1}-(2+\theta_i)\,u_i+\theta_iu_{i-1}=g_i,\ i=\overline{1,N} \\ u_1-u_0\in\alpha\,(u_0-a)\,,\ u_{N+1}-u_N\in-\beta\,(u_{N+1}-b)\,. \end{cases} \tag{18}$$

We are looking for the solution of (18) of the form

$$u_i=w_i+x\xi_i+y\eta_i,\ i=\overline{1,N}, \tag{19}$$

where $w_i,\ \xi_i,\ \eta_i$ are the solutions of the problems

$$\begin{cases} w_{i+1} - (2+\theta_i)\, w_i + \theta_i w_{i-1} = g_i,\ i = \overline{1,N} \\ w_0 = 0, \quad w_1 = 0 \end{cases} \tag{20}$$

and (13), (14) respectively. This u_i verifies the equation from (18) for all $x, y \in H$. We find $x, y \in H$ such that u_i satisfies also the boundary condition in (18). These conditions become

$$\xi_1 x + (\eta_1 - \eta_0)\, y \in \alpha\,(\eta_0 y - a)\,, \tag{21}$$

$$(\xi_{N+1} - \xi_N)\, x + cy \in -\beta\,(w_{N+1} + \xi_{N+1} x - b) - w_{N+1} + w_N, \tag{22}$$

or equivalently

$$((\xi_{N+1} - \xi_N)\, x + cy + z_1,\ \xi_1 x + (\eta_1 - \eta_0)\, y + z_2) \ni$$

$$\ni (-w_{N+1} + w_N, 0)\,, \tag{23}$$

where

$$z_1 \in \beta\,(w_{N+1} + \xi_{N+1} x - b)\,, \tag{24}$$

$$z_2 \in -\alpha\,(\eta_0 y - a)\,. \tag{25}$$

This can be written as

$$F\,(x,y) + G\,(x,y) \ni (-w_{N+1} + w_N, 0)\,, \tag{26}$$

where

$$F\,(x,y) = ((\xi_{N+1} - \xi_N)\, x + cy,\ \xi_1 x + (\eta_1 - \eta_0)\, y)\,, \tag{27}$$

$$G\,(x,y) = (z_1, z_2)\,. \tag{28}$$

It is easy to check that F is everywhere defined, linear, continuous and strongly monotone. We show that G is maximal monotone in $H \times H$. Denote by $((.,.))$ its scalar product. Let G$(x,y) = (z_1, z_2)$, $G\,(u,v) = (z_3, z_4)$, where

$$z_1 \in \beta\,(w_{N+1} + \xi_{N+1} x - b)\,,\ z_2 \in -\alpha\,(\eta_0 y - a)\,, \tag{29}$$

$$z_3 \in \beta\,(w_{N+1} + \xi_{N+1} u - b)\,,\ z_4 \in -\alpha\,(\eta_0 v - a)\,. \tag{30}$$

Then,

$$((G\,(x,y) - G\,(u,v)\,, (x,y) - (u,v))) =$$

$$(z_1 - z_3, x - u) + (z_2 - z_4, y - v) =$$

$$= \frac{1}{\xi_{N+1}}\,(z_1 - z_3, (w_{N+1} + \xi_{N+1} x - b) - (w_{N+1} + \xi_{N+1} u - b)) +$$

$$+ \frac{1}{\eta_0}\,(z_2 - z_4, (\eta_0 y - a) - (\eta_0 v - a)) \geq 0,$$

so G is monotone in $H \times H$.

To prove that G is maximal monotone in $H \times H$, consider $(\lambda, \mu) \in H \times H$ and show that there is $(x,y) \in D\,(\beta) \times D\,(\alpha)$ such that

$$x + z_1 = \lambda, \quad y + z_2 = \mu, \tag{31}$$

where $z_1,\ z_2$ satisfy (29). Denoting by $l = w_{N+1} + \xi_{N+1} x - b$ and $m = \eta_0 y - a$, relations (31) can be written as

$$\lambda + \frac{w_{N+1} - b}{\xi_{N+1}} \in \beta\,(l) + \frac{1}{\xi_{N+1}} l, \tag{32}$$

$$-\mu + \frac{a}{\eta_0} \in \alpha\,(m) - \frac{1}{\eta_0} m. \tag{33}$$

But α is maximal monotone and $-1/\eta_0 > 0$, so $R\left(\alpha - \frac{1}{\eta_0} I\right) = H$. This implies that (33) has a solution $m \in D\,(\alpha)$. Analogously, (32) has a solution $l \in D\,(\beta)$. Therefore, there are $x = (l - w_{N+1} + b) / \xi_{N+1}$ and $y = (m + a) / \eta_0$, such that (31). This means that G is maximal monotone in $H \times H$.

Consequently, $F + G$ is maximal monotone and coercive and thus (26) has a solution, hence B is maximal monotone, as claimed.

3. The main result

In this section we establish the existence and uniqueness of the solution to the finite difference inclusion (1), under the hypothesis that A is strongly monotone in H. Denote by J_λ and A_λ the resolvent and the Yosida approximation of A, respectively: $J_\lambda = (I + \lambda A)^{-1}$ and $A_\lambda = (I - J_\lambda) / \lambda$. Now we state the main result.

Theorem 3.1. *Let $A : D(A) \subseteq H \to H$ be a maximal monotone and strongly monotone operator in the real Hilbert space H, with $0 \in D(A)$ and $0 \in A0$. Suppose that α, β are maximal monotone in H, $0 \in D(\alpha)$, $0 \in D(\beta)$, $0 \in \alpha(0) \cap \beta(0)$. Consider the sequences $c_i > 0$, $\theta_i \in (0,1)$ and $f_i \in H$, for all $i = \overline{1,N}$. Then, for all $a, b \in H$, problem (1) admits a unique solution $(u_i)_{i=\overline{1,N}} \in D(A)^N$.*

Proof. By hypothesis, there exists $\omega > 0$ such that for all $x, y \in D(A)$ and $x' \in Ax$, $y' \in Ay$, we have

$$(x' - y', x - y) \geq \omega ||x - y||^2. \tag{34}$$

Then A_λ satisfies the inequality

$$(A_\lambda x - A_\lambda y, x - y) \geq \frac{\omega}{1 + \lambda\omega} ||x - y||^2 \geq \frac{\omega}{2} ||x - y||^2, \tag{35}$$

for $0 < \lambda < 1/\omega$. Denoting by $\mathcal{A}$ the operator

$$\mathcal{A}\left((u_i)_{i=\overline{1,N}}\right) = (c_1 A u_1, ..., c_N A u_N) \tag{36}$$

and by B the operator defined by $(4) - (5)$, problem (1) can be written as

$$0 \in B\left((u_i)_{i=\overline{1,N}}\right) + \mathcal{A}\left((u_i)_{i=\overline{1,N}}\right) + (f_i)_{i=\overline{1,N}}. \tag{37}$$

Since B is maximal monotone in $H_{a_i}^N$, $B + \mathcal{A}_\lambda$ is also maximal monotone in $H_{a_i}^N$. By (35), it follows that $B + \mathcal{A}_\lambda$ is coercive, so it is surjective from $D(B)$ to $H_{a_i}^N$. Thus, for $(f_i)_{i=\overline{1,N}} \in H^N$, there is $\left(u_i^\lambda\right)_{i=\overline{1,N}} \in D(B)$ such that $B\left((u_i^\lambda)_{i=\overline{1,N}}\right) + \mathcal{A}_\lambda\left((u_i^\lambda)_{i=\overline{1,N}}\right) = -(f_i)_{i=\overline{1,N}}$, that is

$$\begin{cases} u_{i+1}^\lambda - (1 + \theta_i) u_i^\lambda + \theta_i u_{i-1}^\lambda = c_i A_\lambda u_i^\lambda + f_i, \ i = \overline{1,N} \\ u_1^\lambda - u_0^\lambda \in \alpha\left(u_0^\lambda - a\right), \ u_{N+1}^\lambda - u_N^\lambda \in -\beta\left(u_{N+1}^\lambda - b\right). \end{cases} \tag{38}$$

We prove the boundedness in $H_{a_i}^N$ of $\left(u_i^\lambda\right)_{i=\overline{1,N}}$ with respect to λ. To do this, one multiplies (38) by $a_i u_i^\lambda$ and obtains

$$\sum_{i=1}^N a_i \left(u_{i+1}^\lambda - u_i^\lambda, u_i^\lambda\right) - \sum_{i=1}^N a_i \theta_i \left(u_i^\lambda - u_{i-1}^\lambda, u_i^\lambda\right) =$$

$$= \sum_{i=1}^N c_i a_i \left(A_\lambda u_i^\lambda, u_i^\lambda\right) + \sum_{i=1}^N a_i \left(f_i, u_i^\lambda\right).$$

Since $a_i\theta_i = a_{i-1}$ and $0 \in A0$, using (35) we deduce

$$\frac{\omega}{2}\sum_{i=1}^{N} c_i a_i ||u_i^\lambda||^2 \leq \sum_{i=1}^{N}[a_i\left(u_{i+1}^\lambda - u_i^\lambda, u_i^\lambda\right) - a_{i-1}\left(u_i^\lambda - u_{i-1}^\lambda, u_{i-1}^\lambda\right)] -$$

$$-\sum_{i=1}^{N} a_{i-1}||u_i^\lambda - u_{i-1}^\lambda||^2 - \sum_{i=1}^{N} a_i\left(f_i, u_i^\lambda\right), \quad (39)$$

so

$$\frac{\omega}{2}\sum_{i=1}^{N} c_i a_i ||u_i^\lambda||^2 \leq a_N\left(u_{N+1}^\lambda - u_N^\lambda, u_N^\lambda\right) - \left(u_1^\lambda - u_0^\lambda, u_0^\lambda\right) -$$

$$-\sum_{i=1}^{N} a_{i-1}||u_i^\lambda - u_{i-1}^\lambda||^2 - \sum_{i=1}^{N} a_i\left(f_i, u_i^\lambda\right). \quad (40)$$

Since α and β are monotone and $0 \in \alpha 0,\ 0 \in \beta 0$, from (38) we have

$$-\left(u_1^\lambda - u_0^\lambda, u_0^\lambda\right) \leq ||u_1^\lambda - u_0^\lambda||.||a||, \quad (41)$$

$$\left(u_{N+1}^\lambda - u_N^\lambda, u_N^\lambda\right) \leq ||u_{N+1}^\lambda - u_N^\lambda||.||b||. \quad (42)$$

Hence (40) implies

$$\frac{\omega}{2}\sum_{i=1}^{N} c_i a_i ||u_i^\lambda||^2 + \sum_{i=1}^{N} a_{i-1}||u_i^\lambda - u_{i-1}^\lambda||^2 \leq a_N||u_{N+1}^\lambda - u_N^\lambda||.||b|| +$$

$$+||u_1^\lambda - u_0^\lambda||.||a|| + \left(\sum_{i=1}^{N} a_i||f_i||^2\right)^{1/2}\left(\sum_{i=1}^{N} a_i||u_i^\lambda||^2\right)^{1/2} \quad (43)$$

and thus

$$\sum_{i=1}^{N} a_i||u_i^\lambda||^2 \leq \frac{8}{\omega^2 c^2}\sum_{i=1}^{N} a_i||f_i||^2 +$$

$$+\frac{4a_N}{\omega c}||u_{N+1}^\lambda - u_N^\lambda||.||b|| + \frac{4}{\omega c}||u_1^\lambda - u_0^\lambda||.||a||. \quad (44)$$

By $u_{N+1}^\lambda - u_N^\lambda \in -\beta\left(u_{N+1} - b\right)$ and $0 \in \beta 0$, we find

$$||u_{N+1}^\lambda||^2 \leq \left(||u_N^\lambda|| + ||b||\right)||u_{N+1}^\lambda|| + ||u_N^\lambda||.||b||,$$

which leads to

$$||u_{N+1}^\lambda|| \leq \frac{1}{\sqrt{a_N}}\left(\sum_{i=1}^{N} a_i||u_i^\lambda||^2\right)^{1/2} + ||b|| + \frac{||b||}{\sqrt[4]{a_N}}\left(\sum_{i=1}^{N} a_i||u_i^\lambda||^2\right)^{1/4}. \quad (45)$$

Similarly, we have

$$||u_0^\lambda|| \leq \frac{1}{\sqrt{a_1}}\left(\sum_{i=1}^{N} a_i||u_i^\lambda||^2\right)^{1/2} + ||a|| + \frac{||a||}{\sqrt[4]{a_1}}\left(\sum_{i=1}^{N} a_i||u_i^\lambda||^2\right)^{1/4}. \quad (46)$$

Next,

$$||u_N^\lambda|| \leq \frac{1}{\sqrt{a_N}}\left(\sum_{i=1}^{N} a_i||u_i^\lambda||^2\right)^{1/2} \quad (47)$$

and

$$||u_1^\lambda|| \leq \frac{1}{\sqrt{a_1}} \left(\sum_{i=1}^{N} a_i ||u_i^\lambda||^2 \right)^{1/2}. \tag{48}$$

Using (45), (46), (47) and (48) in (44), we find that $\sum_{i=1}^{N} a_i||u_i^\lambda||^2$ is bounded with respect to λ. Inequalities $(45) - (48)$, lead us to the boundedness of $||u_{N+1}^\lambda||$, $||u_0^\lambda||$, $||u_N^\lambda||$ and $||u_1^\lambda||$ and by (43), $\sum_{i=1}^{N} a_{i-1}||u_i^\lambda - u_{i-1}^\lambda||^2$ is also bounded. Now we use (38) to get the boundedness with respect to λ of $A_\lambda u_i^\lambda$.

We are going to pass to the limit in (38). To this end, let $\lambda,\ \mu > 0$ be fixed. One subtracts the equation (38) for for λ and for μ, one multiplies the difference by $a_i \left(u_i^\lambda - u_i^\mu\right)$ and one sums from $i = 1$ to $i = N$:

$$\begin{aligned}\sum_{i=1}^{N} a_i(u_{i+1}^\lambda - u_{i+1}^\mu - u_i^\lambda + u_i^\mu, u_i^\lambda - u_i^\mu)-\\ -\sum_{i=1}^{N} a_{i-1}(u_i^\lambda - u_i^\mu - u_{i-1}^\lambda + u_{i-1}^\mu, u_i^\lambda - u_i^\mu) =\\ = \sum_{i=1}^{N} a_i c_i (A_\lambda u_i^\lambda - A_\mu u_i^\mu, u_i^\lambda - u_i^\mu).\end{aligned} \tag{49}$$

Denote by M_1 and M_2 the left hand side and the right hand side in (49). Since $a_i\theta_i = a_{i-1}$, we have

$$\begin{aligned}M_1 = a_N(u_{N+1}^\lambda - u_{N+1}^\mu - u_N^\lambda + u_N^\mu, u_{N+1}^\lambda - u_{N+1}^\mu)-\\ a_N||u_{N+1}^\lambda - u_{N+1}^\mu - u_N^\lambda + u_N^\mu||^2-\\ -(u_1^\lambda - u_1^\mu - u_0^\lambda + u_0^\mu, u_0^\lambda - u_0^\mu) - \sum_{i=1}^{N} a_{i-1}||u_i^\lambda - u_i^\mu - u_{i-1}^\lambda + u_{i-1}^\mu||^2,\end{aligned} \tag{50}$$

so

$$M_1 \leq -\sum_{i=1}^{N} a_{i-1}||u_i^\lambda - u_i^\mu - u_{i-1}^\lambda + u_{i-1}^\mu||^2. \tag{51}$$

On the other hand, since

$$J_\lambda u_i^\lambda + \lambda A_\lambda u_\lambda^i = u_i^\lambda, \tag{52}$$

we get

$$\begin{aligned}M_2 = \sum_{i=1}^{N} a_i c_i (A_\lambda u_i^\lambda - A_\mu u_i^\mu, J_\lambda u_i^\lambda - J_\mu u_i^\mu)+\\ +\sum_{i=1}^{N} a_i c_i (A_\lambda u_i^\lambda - A_\mu u_i^\mu, \lambda A_\lambda u_i^\lambda - \mu A_\mu u_i^\mu) \geq \frac{\omega c}{2} \sum_{i=1}^{N} a_i ||J_\lambda u_i^\lambda - J_\mu u_i^\mu||^2+\\ +\sum_{i=1}^{N} a_i c_i (\lambda ||A_\lambda u_i^\lambda||^2 + \mu ||A_\mu u_i^\mu||^2) - (\lambda + \mu) \sum_{i=1}^{N} a_i c_i (A_\lambda u_i^\lambda, A_\mu u_i^\mu).\end{aligned} \tag{53}$$

Using (51), (53) and the boundedness of $\sum_{i=1}^{N} a_i||A_\lambda u_i^\lambda||^2$ in (49), one obtains

$$\frac{\omega c}{2} \sum_{i=1}^{N} a_i ||J_\lambda u_i^\lambda - J_\mu u_i^\mu||^2 \leq k_1 \left(\lambda + \mu\right), \tag{54}$$

where k_1 is a positive constant. So $J_\lambda u_\lambda^i$ is strongly convergent in H as $\lambda \searrow 0$, say $J_\lambda u_\lambda^i \to u_i$. This, together with (52), gives us that $u_i^\lambda \to u_i$ as $\lambda \searrow 0$ in H. Let $A_\lambda u_i^\lambda \rightharpoonup w_i$ as $\lambda \searrow 0$ (weakly) in H. Since A is maximal monotone, we may pass to the limit in the inclusion $A_\lambda u_i^\lambda \in A\left(J_\lambda u_i^\lambda\right)$ and find $u_i \in D(A)$ and $w_i \in Au_i$, $i = \overline{1,N}$.

Passing to the limit in (38), it follows that u_i verifies the problem (1) and thus the existence is proved.

If $(u_i)_{i=\overline{1,N}}$, $(v_i)_{i=\overline{1,N}}$ are two solutions of (1) and $x_i = u_i - v_i$, then, subtracting the equations for u_i and for v_i, multiplying by $a_i x_i$ and summing from $i = 1$ to $i = N$, by (34) we get

$$\sum_{i=1}^{N}[a_i(x_{i+1} - x_i, x_i) - a_{i-1}(x_i - x_{i-1}, x_{i-1})]-$$

$$-\sum_{i=1}^{N} a_{i-1}||x_i - x_{i-1}||^2 \geq c\omega \sum_{i=1}^{N} a_i||x_i||^2. \tag{55}$$

This implies

$$c\omega \sum_{i=1}^{N} a_i||x_i||^2 \leq -a_N||x_{N+1} - x_N||^2 +$$

$$+a_N\left(x_{N+1} - x_N, x_{N+1}\right) - (x_1 - x_0, x_0) \leq 0,$$

because u_i and v_i verify the boundary conditions of problem (1). Therefore $x_i = 0$, i.e. we proved the uniqueness.

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