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A mathematical model with a trophic chain predation based on the ODEs to describe fish and plankton dynamics

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Abstract. The aim of this paper is the formulation and the study of a prey-predator model to describe fish and plankton population dynamics, with three developmental stages of the fish population (larva, juvenile and adult). First, we develop a mathematical model based on the ODEs, describing the dynamics of the various classes for the fish population depending on the plankton in a general framework. Then, we are interested in the model in the case of a trophic chain predation for the fish population. Finally, we continue our study through numerical simulations of the model in different fishing areas. The obtained numerical results confirm the mathematical analysis and allow us to have an idea on the development of the fish population in a fishing area.

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1. Introduction

A situation of overexploitation of fishery resources has taken place because of the high demand for fish products and the lack of adequate and coherent policies for a sustainable management of exploitable resources, [2, 13, 7]. At the present stage of knowledge, research on fish raises a set of problems in terms of biology and populations dynamics, [6, 11, 13, 8, 10, 15]. The biology and ecology related issues of the fish population require a comprehensive study and reliable use of existing knowledge at the applied level (management and planning).

It is in this line of thoughts that we are interested here in the study of fish population dynamics with a single food resource, i.e plankton. Our goal is to understand how the dynamics behaves in a fishing area with a trophic chain predation on the population of fish, in order to predict the state of fish stock, [6, 11, 15].

The paper is organized as follows. In section 2, we present the general mathematical model of our problem. Section 3 provides the mathematical analysis of the model at the general form. Also, we look at a particular case of our model in the section 4. Computational simulations are performed in section 5 and finally, in the last section, the paper concludes with a discussion of the work and a number of suggestions as to how it may be extended.

2. General mathematical model of fish-plankton system

In this section, we proceed to the construction of a fish-plankton model by taking into account three weight classes in the fish population: larva, juvenile and adult.

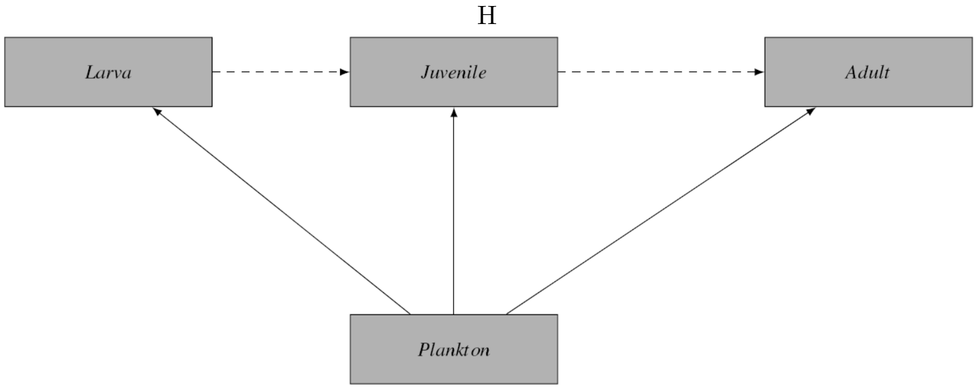


FIGURE 1. Interaction model of the larva-juvenile-fish-plankton system

Each of these weight classes can be perceived as a developmental stage of the fish population, [5, 11, 14, 15]. The dynamics of the larva-juvenile-adult-plankton system is represented by Figure 1.

The state variables are: the plankton density P, the adult density A, the juvenile density J and the larva density L. Let m_L, m_J and m_A ($m_L < m_J < m_A$) respectively larva, juvenile and fish adult weight. Let $H = (H_L, H_J, H_A) = (L, J, A)$ the vector of the fish class. Similarly, we define the energy r_a used to grow up and to reproduce. We also define ε, the fraction of hatched eggs. Thus, according to Figure 1, we obtain the following system of four differential equations

$$\begin{cases} \dfrac{dP}{dt} = \varphi(P) - f_{L,P}(P,H)L - f_{J,P}(P,H)J - f_{A,P}(P,H)A, \quad P(0) = P_0 > 0 \\ \dfrac{dL}{dt} = bf_A(P,H)A - \tau_1 f_L(P,H)L - \mu_L L - \mu_L(P,H), \quad L(0) = L_0 > 0 \\ \dfrac{dJ}{dt} = \tau_1 f_L(P,H)L - \tau_2 f_J(P,H)J - \mu_J J - \beta_J J - \mu_J(P,H), \quad J(0) = J_0 > 0 \\ \dfrac{dA}{dt} = \tau_2 f_J(P,H)J - \mu_A A - \beta_A A - \mu_A(P,H), \quad A(0) = A_0 > 0 \end{cases} \tag{1}$$

where

- f_i is the given function in biomass per time unit for $i \in \{L, J, A\}$,
- $f_{L,P}$ is the energy acquired per time unit for the larva on the plankton population,
- $f_{J,P}$ is the energy acquired per time unit for the juvenile on the plankton population,
- $f_{A,P}$ is the energy acquired per time unit for the fish adult on the plankton population,
- $\mu_L(P,H)$ is the larva mortality rate due to cannibalism,
- $\mu_J(P,H)$ is the juvenile mortality rate due to cannibalism,
- $\mu_A(P,H)$ is the adult mortality rate due to cannibalism,
- $\tau_1 = \dfrac{r_a}{m_J - m_L}$ is the transfer-rate of the larva in juvenile population,
- $\tau_2 = \dfrac{r_a}{m_A - m_J}$ is the transfer-rate of the juvenile in adult population,
- $b = \dfrac{\varepsilon r_a}{m_L}$ is the fertility-rate of the adult population,

- $\varphi(P)$ is the natural growth-rate of the plankton quantity,
- q_i is the coefficient of catchability for the class i, $i \in \{J, A\}$,
- E_i is the fishing effort for the class i, $i \in \{J, A\}$,
- $\beta_J = q_J E_J$ is the mortality rate due to fishing of the juvenile population,
- $\beta_A = q_A E_A$ is the mortality rate due to fishing of the adult population,
- μ_L is the natural mortality rate of the larva population,
- μ_J is the natural mortality rate of the juvenile population,
- μ_A is the natural mortality rate of the adult population.

The model presented here is general and it is necessary to make choices, particularly for the functions: $f_A, f_J, f_L, f_{L,P}, f_{A,P}$, and $f_{J,P}$.

In the remainder of the paper, we will study a particular case where we will assign a precise expression to these functions. We will, for example, deal with the situation where each class has a unique source of food and therefore one can easily assume that these functions become functions of one variable, which will bring us to the classic choice of linear functions or Holling type II, [13, 16, 17, 15]. In the situation where there are several sources of food, one possibility is to assume that the functions $f_{L,P}$, $f_{J,P}$ and $f_{A,P}$ depend only on P and H, [2, 4, 13, 18].

3. Mathematical results of the general model

Here, we will give some general results on the model defined in (1). The goal of this section is to answer the questions on the existence of local and global solutions [1, 2, 3, 15, 18].

The functions f_i in biomass per unit of time and cannibalism mortalities μ_i in number of individuals per unit of time are in the following form for all $i \in \{L, J, A\}$:

$$f_i(P, H) = f_{i,P}(P, H) + \sum_{j \in \{L,J,A\}} f_{i,j}(P, H) \tag{2}$$

$$\mu_i(P, H) = \frac{1}{m_i} \sum_{j \in \{L,J,A\}} f_{j,i}(P, H) H_j \tag{3}$$

The functions $f_{i,j}$, $j \in \{L, J, A\}$ correspond to the acquired biomass per unit of time by individual of the size class i by predation on the size class j. Additionally, we rewrite the system (1) in the following form:

$$\begin{cases} \dfrac{dP}{dt} = g_1(P, L, J, A) \\ \dfrac{dL}{dt} = g_2(P, L, J, A), \\ \dfrac{dJ}{dt} = g_3(P, L, J, A) \\ \dfrac{dA}{dt} = g_4(P, L, J, A), \end{cases} \tag{4}$$

$N(0) = N_0 > 0$ for $N = P, L, J, A$.

3.1. Hypothesis. We will formulate the following assumptions, for the remainder of the work.

(H_1) : $m_L < m_J < m_A$,

(H_2) : $\mu_i > 0$ for any $i \in \{L, J, A\}$ and $\beta_j \geq 0$ for any $j \in \{J, A\}$, where μ_i is the natural mortality,

(H_3) : $r_a, \varepsilon \in]0, 1]$,

(H_4) : φ is continuously differentiable on $[0, K_p]$ and verify:

- (a) $\varphi(0) = \varphi(K_p) = 0$, where K_p is the maximal carrying capacity of the plankton.
- (b) $\forall P \in]0, K_p[$, $\varphi(P) > 0$.
- (c) $\varphi'(0) > 0$ and $\varphi'(K_p) < 0$.

(H_5) : The functions $f_{i,j}$ and $f_{i,P}$ for $i \in \{L, J, A\}$, are positive and continuously differentiable. In addition:

- (a) $H_j = 0 \Rightarrow f_{i,j}(P, H) = 0$ and $P = 0 \Rightarrow f_{i,P}(P, H) = 0$ for any $i, j \in \{L, J, A\}$.
- (b) At least one of the following functions (by fixing the other variables) is strictly positive for $i \in \{L, J, A\}$: $H_j \longmapsto \dfrac{\partial f_{i,j}}{\partial H_j}(P, H)$ for $j \in \{L, J, A\}$ and $P \longmapsto \dfrac{\partial f_{i,P}}{\partial P}(P, H)$ for $i \in \{L, J, A\}$.

Remark 3.1. From the biological point of view, the assumption (H_5)-(a) means that if there is no fish in a size class (or if there is no plankton), the energy obtained by predation on this class (resp. the plankton) will be zero. The hypothesis (H_5)-(b) requires that each fish feeds by predation on at least one size class or on the plankton. The more there are fish in the predated size class (resp. the plankton is more abundant), the more the energy obtained is important in a known external environment, i.e for known amounts of fish in the other size classes.

3.2. Some preliminary results.

The initial condition is taken in the domain $\Omega := \left]0, K_p\right[\times \mathbb{R}_+^{*3}$. Let $\widehat{\mu}_J = \mu_J + \beta_J$ and $\widehat{\mu}_A = \mu_A + \beta_A$.

Proposition 3.1. *The domain Ω is positively invariant by the positive semi-wave generated by* (1).

Proof. By seeing that the system (4) can be rewritten as follows:

$$\frac{d}{dt}\begin{pmatrix} P \\ L \\ J \\ A \end{pmatrix} = \begin{pmatrix} g_1(P, L, J, A) \\ g_2(P, L, J, A) \\ g_3(P, L, J, A) \\ g_4(P, L, J, A) \end{pmatrix} = G(P, L, J, A)$$

and using the previous hypotheses, we have:

$$\begin{aligned}
&g_1(P = 0, L, J, A) = 0 \text{ for } L, J, A \geq 0,\\
&g_1(P = K_p, L, J, A) = -f_{L,P}(K_p, H)L + f_{J,P}(K_p, H)J - f_{A,P}(K_p, H)A \leq 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } L, J, A \geq 0,\\
&g_2(P, L = 0, J, A) = b f_A(P, H)A \geq 0 \text{ for } P, J, A \geq 0,\\
&g_3(P, L, J = 0, A) = \tau_1 f_L(P, H)L \geq 0 \text{ for } P, L, A \geq 0,\\
&g_4(P, L, J, A = 0) = \tau_2 f_J(P, H)A \geq 0 \text{ for } P, L, J, \geq 0,
\end{aligned}$$

So, the field is not outgoing on $\mathrm{Fr}(\Omega)$. □

Proposition 3.2. *For all initial condition in Ω, the trajectory resulting from this initial condition remains in a bounded domain included in Ω.*

Proof. We consider the total biomass of the system at a time t i.e the following function:

$$Q(t) = P(t) + m_L L(t) + m_J J(t) + m_A A(t)$$

The derivative of Q with respect to time is:

$$\dot{Q} = \varphi(P) - (1 - r_a)(f_L(P, H)L + f_J(P, H)J) - (1 - \varepsilon r_a)f_A(P, H)A \\ - m_L\mu_L L - m_J\widehat{\mu}_J J - m_A\widehat{\mu}_A A$$

As we have $r_a, \varepsilon \in]0, 1]$, we get

$$\dot{Q} \leq \varphi(P) - m_L\mu_L L - m_J\widehat{\mu}_J J - m_A\widehat{\mu}_A A \leq \varphi_{max} - \mu_{min}Q + \mu_{min}P$$

where $\mu_{min} = min_{i\in\{L,J,A\}}\{\mu_i\}$ and $\varphi_{max} = sup_{[0,K_p]}\varphi$.
Finally, knowing that $\mu_{min}P \leq \mu_{min}K_p$ we deduce:

$$Q(t) \leq Q(0)e^{-\mu_{min}t} + \left(\frac{\varphi_{max}}{\mu_{min}} + K_p\right)\left(1 - e^{-\mu_{min}t}\right).$$

Then

$$\lim_{t\to\infty} Sup\ Q(t) \leq \frac{\varphi_{max}}{\mu_{min}} + K_p$$

and so

$$\lim_{t\to\infty} Sup\ H_i(t) \leq \frac{1}{m_i}\left(\frac{\varphi_{max}}{\mu_{min}} + K_p\right).$$

□

Corollary 3.3. *The compact domain*

$$\Omega_p := \left\{(P, H) \in \Omega : O \leq P + m_L L + m_J J + m_A A \leq \frac{\varphi_{max}}{\mu_{min}} + K_p\right\}$$

is positively invariant and attracts all trajectories from Ω.

Proposition 3.4. *The model* (1) *admits two trivial equilibria $E_{0i} := (P = 0, H = 0)$ and $E_0 := (P = K_p, H = 0)$. Additionally E_{0i} is locally unstable.*

Proof. The resolution of the following system

$$\begin{cases} \varphi(P) - f_{L,P}(P, H)L - f_{J,P}(P, H)J - f_{A,P}(P, H)A = 0 \\ bf_A(P, H)A - \tau_1 f_L(P, H)L - \mu_L L - \mu_L(P, H) = 0 \\ \tau_1 f_L(P, H)L - \tau_2 f_J(P, H)J - \mu_J J - \beta_J J - \mu_J(P, H) = 0 \\ \tau_2 f_J(P, H)J - \mu_A A - \beta_A A - \mu_A(P, H) = 0 \end{cases} \tag{5}$$

shows that trivial states are E_{0i} and E_0. To establish the local instability of the state E_{0i}, we consider the Jacobian matrix at E_{0i}. The Jacobian matrix at E_{0i} is

$$J(E_{0i}) = \begin{pmatrix} -\varphi'(0) & 0 & 0 & 0 \\ 0 & -\mu_L & 0 & 0 \\ 0 & 0 & -\widehat{\mu}_J & 0 \\ 0 & 0 & 0 & -\widehat{\mu}_A \end{pmatrix}$$

$Spect(E_{0i}) = \{\varphi'(0), -\mu_L, -\widehat{\mu}_J, -\widehat{\mu}_A\}$. The eigenvalues are strictly negative except $\varphi'(0)$ which is positive according to the previous hypotheses. So, this equilibrium state is locally unstable. □

Proposition 3.5. *There is existence and uniqueness of a solution of the system* (1) *on an horizon of infinite time.*

Proof. The existence and uniqueness of solution are obtained thanks to the theorem of Cauchy-Lipschitz [4, 5, 6, 19, 20] given the regularity of the functions involved in the model. Furthermore, this solution is defined for any positive t, given the fact that the trajectories remain in a bounded compact according to proposition 3.1. ☐

4. The model with a trophic chain predation for the fish population

In this section we are interested on the particular case of model (1) regarding the choice of functions involved in the model and also the type of predation for the fish population. The predation will be assumed linear and will be done in trophic cascade : the larvae will feed on the plankton, the juvenile will feed on the larvae and the adult will feed on the juvenile, [9, 13, 14, 15, 7, 15].

4.1. Equations of the model. The choice of predation functions is the following:

$$f_L(P,H) = v_1 P,\ \ f_J(P,H) = v_2 m_L L,\ \ f_A(P,H) = v_3 m_J J \ \text{ and } \ \varphi(P) = e\left(1-\frac{P}{K_p}\right)P$$

where

- e is the natural growth-rate of the plankton,
- v_1 is the annual individual intake of plankton per individual larva,
- v_2 is the annual individual intake of larvae per individual juvenile,
- v_3 is the annual individual intake of juvenile per individual fish adult.

Thus, the model (1) can be rewritten:

$$\begin{cases} \dfrac{dP}{dt} = e\left(1-\dfrac{P}{K_p}\right)P - v_1 PL, \quad P(0) = P_0 > 0 \\ \dfrac{dL}{dt} = b v_3 m_J JA - \tau_1 v_1 PL - \mu_L L - v_2 LJ, \quad L(0) = L_0 > 0 \\ \dfrac{dJ}{dt} = \tau_1 v_1 PL - \tau_2 v_2 m_L JL - \mu_J J - \beta_J J - v_3 JA, \quad J(0) = J_0 > 0 \\ \dfrac{dA}{dt} = \tau_2 v_2 m_L JL - \mu_A A - \beta_A A, \quad A(0) = A_0 > 0 \end{cases} \tag{6}$$

4.2. The equilibria of the model.

Proposition 4.1. *The model* (6) *admits two trivial equilibria* $E_{oi} = (0,0)$ *and* $E_0 = (K_p, 0)$. *The first one is always locally asymptotically unstable and the second is locally asymptotically stable.*

Proof. As we are in a particular case of model (1), the proposition 3.4 ensures that there are two trivial equilibria: $E_{0i} = (0,0)$ locally unstable and $E_0 = (K_p, 0)$. To obtain the local stability results of the latter, we write the Jacobian matrix:

$$J_p(E_0) = \begin{pmatrix} -e & -v_1 K_p & 0 & 0 \\ 0 & -\mu_L - \tau v_1 K_p & 0 & 0 \\ 0 & \tau v_1 K_p & -\mu_J - \beta_J & 0 \\ 0 & 0 & 0 & -\mu_A - \beta_A \end{pmatrix}$$

The characteristic polynomial of this matrix is:

$$P_{(K_p,0)}(X) = (X+e)(X+\mu_L+\tau_1 v_1 K_p)(X+\mu_J+\beta_J)(X+\mu_A+\beta_A).$$

All its eigenvalues are clearly strictly negative, which ensures the local stability of equilibrium E_0. ☐

Proposition 4.2. *The equilibrium E_0 is globally stable under the following sufficient condition : $bm_J \leq 1$.*

Proof. Let us consider the Lyapunov function (7) defined by

$$Q(t) = L(t) + J(t) + A(t). \tag{7}$$

The derivative of Q with respect to time is:

$$\dot{Q} = \big(bm_J v_3 - v_3\big)JA - \mu_L L - \widehat{\mu}_J J - \widehat{\mu}_A A - v_2 LJ \tag{8}$$

Q is positive on $\overline{\Omega}$, which is invariant and under $bm_J - 1 \leq 0$, $\dot{Q}$ is strictly negative on

$$\overline{\Omega} \setminus \{(P,0) : P \geq 0\}$$

So H tends to 0.
Moreover, as the system is bounded and $(0,0)$ is unstable, it follows from the equation on plankton that for any initial condition in Ω, P converges to K_p. ☐

Now, we consider the case of the positive components equilibria.
Let's consider the function π defined for all $\phi \in \mathbb{R}_+$ by:

$$\pi(\phi) = a_4\phi^4 + a_3\phi^3 + a_2\phi^2 + a_1\phi + a_0$$

where

$a_4 = eb\alpha v_3 m_J \tau_2 v_2 m_L (bv_3 m_J - v_3)$,

$a_3 = eb\alpha v_3 m_J \widehat{\mu}_A(\tau_2 v_2 m_L + v_2) + ev_2^2 m_L(bv_3 m_J - v_3)$,

$a_2 = eb\alpha v_3 m_J \widehat{\mu}_A \mu_L - ev_2\widehat{\mu}_A(\tau_2 v_2 m_L + v_2) + (ev_1\tau_1 + e\mu_L)(b\tau_2 v_2 m_J v_3 m_L - \tau_2 v_2 m_L v_3)$,

$a_1 = -ev_2\widehat{\mu}_A\mu_L - (e\tau_1 v_1 + e\mu_L)(\widehat{\mu}_A\tau_2 v_2 m_L + v_2) - \widehat{\mu}_A\widehat{\mu}_J\tau_1 K_p v_1^2$,

$a_0 = -\widehat{\mu}_A\mu_L$, and $\alpha = \dfrac{\tau_2 v_2 m_L}{\widehat{\mu}_A}$,

and the functions h and l defined from $\mathbb{R}_+$ to $\mathbb{R}$ by:

$$l(J) := \frac{e}{\tau_1 K_p v_1^2}\left[\frac{-b\tau_2 v_2 v_3 m_J m_L}{\widehat{\mu}_A}J^2 + v_2 J + (\tau_1 v_1 + \mu_L)\right] \tag{9}$$

$$h(J) := \frac{\widehat{\mu}_A\widehat{\mu}_J J}{\tau_2 v_2 m_L(bv_3 m_J - v_3)J^2 - \widehat{\mu}_A(\tau_2 v_2 m_L + v_2)J - \mu_L\widehat{\mu}_A} \tag{10}$$

We have the following proposition which gives the necessary and sufficient conditions of the existence of positive equilibrium.

Proposition 4.3. *The positive component equilibrium $\big(P > 0,\ L > 0,\ J > 0,\ A > 0\big)$ of system (6) exists if and only if there exists the solution of the equation $\pi(J) = 0$.*

Proof. In order to obtain the positive equilibria of this model, we must cancel the dynamics of (6). Indeed, equivalently, we cancel the dynamics of $(P, L, L+J, A)$. Then, on Ω we have:

$$g_1(P,L,J,A) = 0 \Longleftrightarrow P = K_p\left(1 - \frac{v_1}{e}L\right) \tag{11}$$

$$g_4(P,L,J,A)=0 \Longleftrightarrow A=\frac{\tau_2 v_2 m_L}{\widehat{\mu}_A}LJ \tag{12}$$

We cancel the dynamics of L by using those two last relations:

$$g_2(P,L,J,A)=0 \Longleftrightarrow L=l(J):=\frac{e}{\tau_1 K_p v_1^2}\left[\frac{-b\tau_2 v_2 v_3 m_J m_L}{\widehat{\mu}_A}J^2+v_2 J+(\tau_1 v_1+\mu_L)\right] \tag{13}$$

Finally, we have:

$$\begin{gathered}(g_2+g_3)(P,L,J,A)=0 \Longleftrightarrow \\ L=h(J):=\frac{\widehat{\mu}_A\widehat{\mu}_J J}{\tau_2 v_2 m_L(bv_3 m_J-v_3)J^2-\widehat{\mu}_A(\tau_2 v_2 m_L+v_2)J-\mu_L\widehat{\mu}_A}\end{gathered} \tag{14}$$

So, we get the functions l and h defined in (9) and (10).
Let us note $\mathcal{S}=\{J\in\mathbb{R}_+^*; l(J)=h(J)\}$

We notice that $\pi(J)=0 \Leftrightarrow J\in\mathcal{S}$. Indeed, let $J\in\mathcal{S}$, then

$$\begin{aligned}l(J)=&h(J)\Leftrightarrow \frac{\widehat{\mu}_A\widehat{\mu}_J J}{\tau_2 v_2 m_L(bv_3 m_J-v_3)J^2-\widehat{\mu}_A(\tau_2 v_2 m_L+v_2)J-\mu_L\widehat{\mu}_A}\\ &=\frac{e}{\tau_1 K_p v_1^2}(-b\alpha v_3 m_J J^2+v_2 J+(\tau_1 v_1+\mu_L))\\ &\Leftrightarrow \tau_1 K_p v_1^2\widehat{\mu}_A\widehat{\mu}_J J=\big[\tau_2 v_2 m_L(bv_3 m_J-v_3)J^2-\widehat{\mu}_A(\tau_2 v_2 m_L+v_2)J-\mu_L\widehat{\mu}_A\big]\\ &\quad\times\big[-eb\alpha v_3 m_J J^2+ev_2 J+e(\tau_1 v_1+\mu_L)\big]\\ &\Leftrightarrow \tau_1 K_p v_1^2\widehat{\mu}_A\widehat{\mu}_J J-\big[\tau_2 v_2 m_L(bv_3 m_J-v_3)J^2-\widehat{\mu}_A(\tau_2 v_2 m_L+v_2)J-\mu_L\widehat{\mu}_A\big]\\ &\quad\times\big[-eb\alpha v_3 m_J J^2+ev_2 J+e(\tau_1 v_1+\mu_L)\big]=0\end{aligned}$$

Let

$$\begin{aligned}\theta(J)=&\tau_1 K_p v_1^2\widehat{\mu}_A\widehat{\mu}_J J-\big[\tau_2 v_2 m_L(bv_3 m_J-v_3)J^2-\widehat{\mu}_A(\tau_2 v_2 m_L+v_2)J-\mu_L\widehat{\mu}_A\big]\\ &\times\big[-eb\alpha v_3 m_J J^2+ev_2 J+e(\tau_1 v_1+\mu_L)\big]\end{aligned}$$

and by developing $\theta(J)$, we recognize the polynomial function π.

Now, we are interested in the points belonging to $\mathcal{S}$. Let us suppose that $bm_J>1$. The signs of the coefficients involved in h and l allow to infer the existence of $J^{max}>0$ and $J^{min}>0$ such that:

$$l(J)>0,\quad \forall\ J\in]0,J^{max}[\quad\text{and } l(J)<0,\quad \forall\ J>J^{max}$$

and

$$h(J)<0,\quad \forall J\in]0,J^{min}[\quad\text{and } h(J)>0,\quad \forall J>J^{min}$$

A necessary condition for $\mathcal{S}$ to be non empty is $J^{min}<J^{max}$. The possible solutions verify $J^{min}<J<J^{max}$.
Additionally, function l is concave and h is convex for $J>J^{min}$. $\mathcal{S}$ contains at most two elements. For each element of $\mathcal{S}$, we can cancel the dynamics.

It remains to be proven that the equilibria obtained are in Ω.
Let $J^*>0$ such that $h(J^*)=l(J^*)>0$, then $L^*=l(J^*)$ is positive. The same goes for

$$A^*=\frac{\tau_2 v_2 m_L}{\widehat{\mu}_A}L^*J^*$$

According to (11), $P^* < K_p$. Finally $P^* > 0$ because it is impossible to cancel the dynamics of J if $H^* > 0$ and $P^* \leq 0$.

To finish, we show that it is possible to have positive equilibria. Indeed, it can be noted that function h does not depend on parameters e and K_p. Obviously when K_p tends to infinity, J^{max} also tends to infinity. So for K_p sufficiently high, we have $J^{min} < J^{max}$. Those two last quantities do not depend on e. Given the fact that l is proportional to e, we get that $\mathcal{S}$ contains two elements for e sufficiently high. So there exists a solution to the equation $\pi(J) = 0$, [9, 11, 13, 14, 15]. $\square$

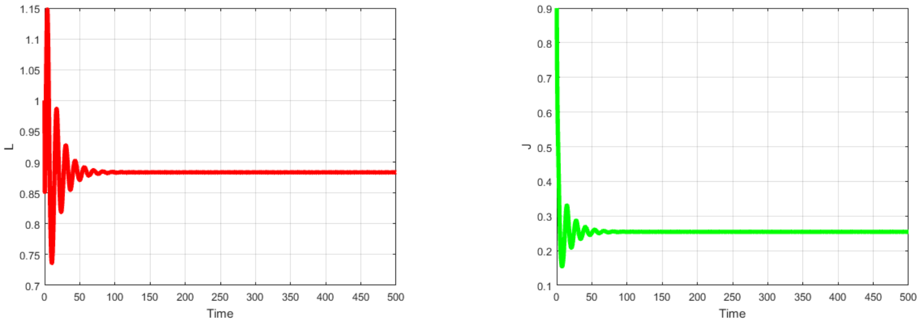
5. Numerical experiments

In this section we examine the synthesis of the results arrived at throughout the numerical simulations of the system. The purpose of this part is to show that, through the numerical simulations we can obtain the positive component equilibria even if there is the trophic chain predation. We aim at showing that despite predation, the only parameter that can disrupt the dynamics of the fish population is the mortality due to fishing, [20, 7, 10]. To illustrate the different behaviors of the model, we use the parameters given in Table 1.

Param.	Description	Value
e	the natural growth-rate of the plankton	1
K_p	the maximal carrying capacity of the plankton	50
ν_1	the annual individual intake of plankton per individual larva	1
b	the fertility-rate of the adult population	0.56
m_J	the weight for the juvenile class.	5
ν_3	the annual individual intake of juvenile per individual fish adult	11.1803
τ_1	the transfer-rate of the larva in juvenile population	0.175
τ_2	the transfer-rate of the juvenile in adult population	0.035
ν_2	the annual individual intake of larvae per individual juvenile	3.3437
m_L	the weight for the larva class.	1
μ_A	the weight for the adult class	25
ε	the fraction of hatched eggs.	0.8
r_a	the energy ra used to grow up and to reproduce	0.7
μ_L	external mortality for larva	0.5
μ_J	external mortality for juvenile	0.05
μ_A	external mortality for adult	0.05

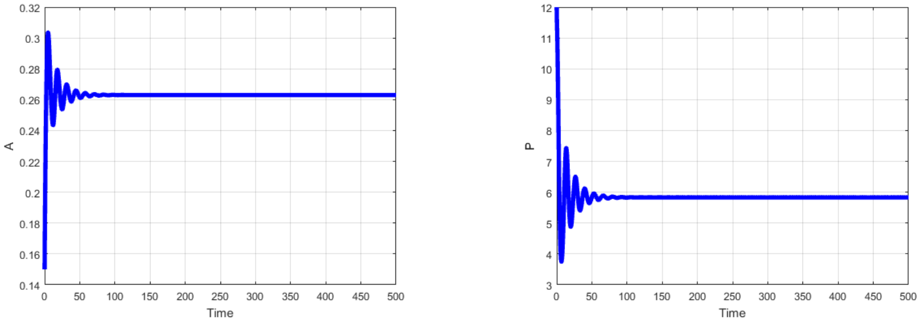
Table 1. Values used for the simulation.

Concerning the other parameters such as β_J, β_A, which designate mortality of the fish population due to the fishing, we bring out the numerical experiments by making them vary. Firstly, we confirm the existence of positive component equilibria. We examine the behavior of the system by varying the mortality values due to fishing. We consider $q_J = q_A = q$ and $E_J = E_A = E$. But as the mortality due to fishing depends on q and E, we will consider the catchability coefficient $q = 1$ and making vary the value of the fishing effort. Our aim of this experiment is to understand



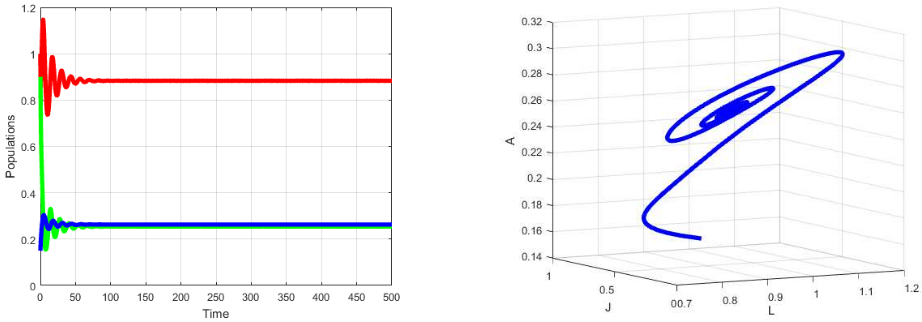
Larva trajectories.

Juvenile trajectories.



Adult trajectories.

Plankton trajectories.



Larva-Juvenile-Adult model trajectories.

Phase portrait of Larva-Juvenile-Adult system.

FIGURE 2. Global asymptotic stability of the coexisting equilibrium of the system with $\beta_J = 0.205$ and $\beta_A = 0.205$.

how the plankton-larva-juvenile-adult dynamics system behaves. Figures 2 shows the stability of these populations. The existence of centers confirms the existence of various classes of fish despite predation. Those results bring us to say that even if there is the trophic chain predation, we can still have an equilibrium. Thus, we talk about the phenomenon of subsistence.

Now the question that arises is: what can unbalanced the system? To answer this question, we continue our simulation by considering areas in which we allocate values related to the fishing effort.

Here, we consider that the plankton-larva-juvenile-adult system lives in an exploited area i.e $E = 0.245$. The numerical simulations show that, after a transitional phase, the equilibrium settles in with coexistence of the three populations.

Figure 3 shows the behavior of the dynamics. These figures show the existence of the centers between the different fish populations. Those results allow us to say that even if the area is exploited with such a value for the fishing effort, despite predation, we can see that the observed dynamics is very close to the one encountered in the case of a fish population living in an pristine area. It allows us to say that if the zone is exploited with such a value of fishing effort, the fish populations are not at risk.

The persistence of the convergence towards a center of those dynamics, despite fishing shows that the area is normally exploited.

Remark 5.1. Even if there is trophic chain predation and if the area is exploited with a value of fishing effort less than or equal to $E \leq 0.5$, there is no risk for the fish population. We are talking about a normally exploited area.

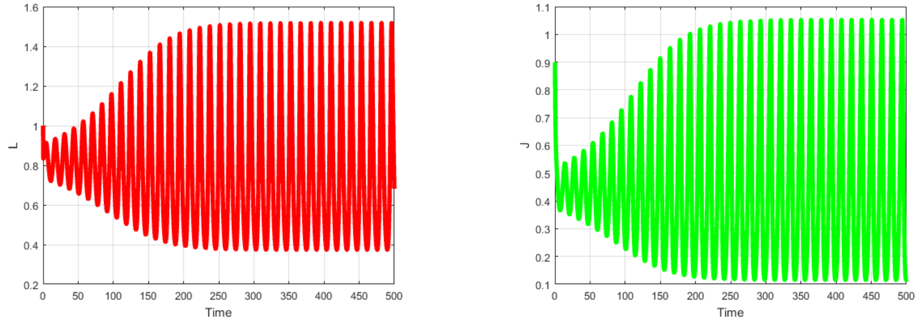
We end our numerical analysis by observing the behavior of the dynamics beyond the previously assigned values to fishing effort. The numerical study considers here a population of fish living in an area exploited with $E = 0.96$. Figure 4 shows the behavior of the dynamics of adult-larva system. We notice from this figure the extinction at long time for fish population. This is explained by the fact that adult females reach with difficulty their mature phase because of the excessive levies by fishing. This situation explains the non regeneration of the system. For those types of area, we talk of a severely-exploited-area.

Remark 5.2. In such cases of the area, exploited with $E > 0.5$ (figure 4), an efficient management policy of fishing must be urgently adopted, otherwise there is a real risk for the fish population.

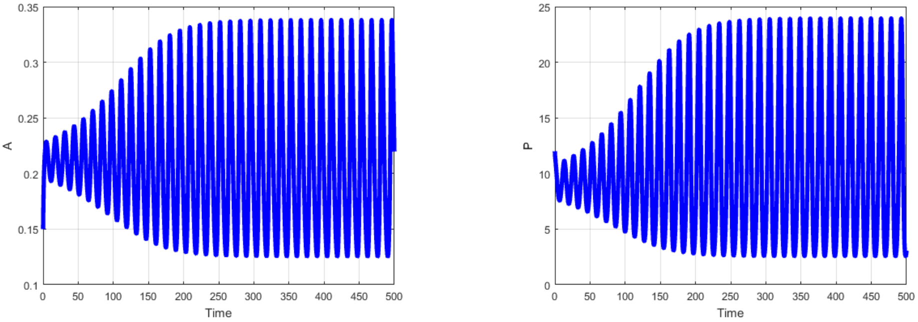
6. Conclusion

In this paper we dealt with the fish population dynamics under a diet on a plankton base. The mathematical model associated with this dynamics is based on ODEs systems. The mathematical study allowed us to show that despite predation on fish population, positive component equilibria exist. We can say that predation does not negatively influence the aquatic ecosystem. Additionally, the simulations allowed us to have an idea about the behavior of the dynamics based on different values of the fishing effort. We could observe through those numerical results that when the area is exploited with some fishing effort values, an efficient management policy must be adopted otherwise it is likely to be catastrophic for the fish population.

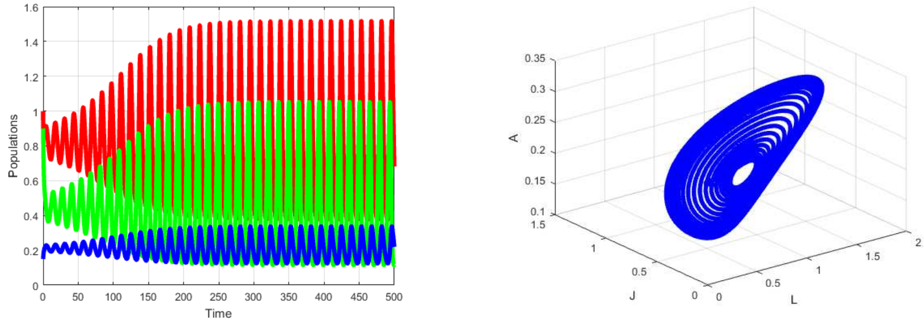
In our future works, we will continue our study, focusing on the impact of toxin produced by the phytoplankton on the fish population.



Larva trajectories. Juvenile trajectories.



Adult trajectories. Plankton trajectories.



Larva-Juvenile-Adult model trajectories. Phase portrait of Larva-Juvenile-Adult system.

FIGURE 3. Dynamics of the trajectories of the system with periodic solutions (stable limit cycle) and $\beta_J = \beta_A = 0.205$.

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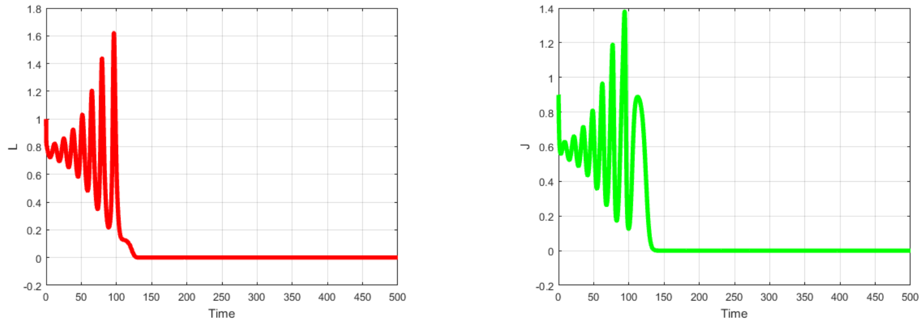
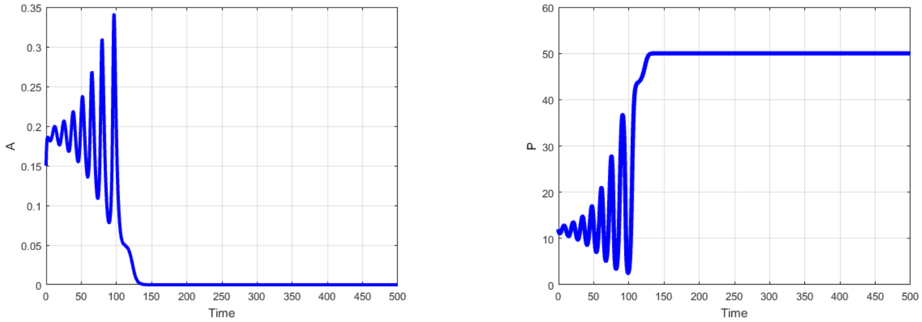
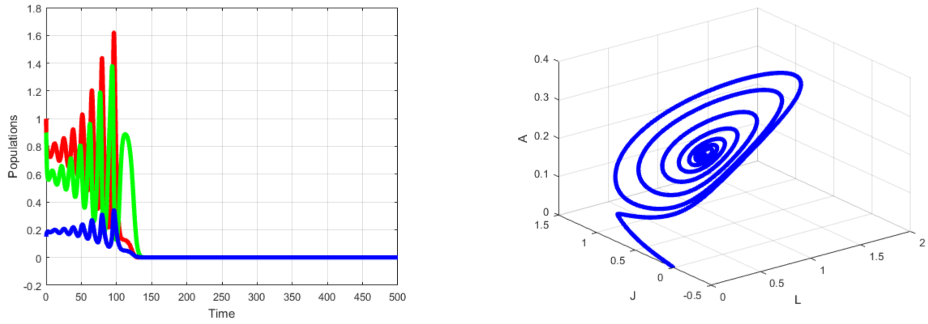


Larva trajectories. Juvenile trajectories.



Adult trajectories. Plankton trajectories.

Adult trajectories. Plankton trajectories



Larva-Juvenile-Adult model trajectories. Phase portrait of Larva-Juvenile-Adult system.

FIGURE 4. Oscillations of the Fish and Plankton trajectories of the System with $\beta_J = \beta_A = 0.960$.

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