Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 46(2), 2019, Pages 236–253
ISSN: 1223-6934

Strong convergence result for Meir-Keeler contractions and a countable family of accretive operators in Banach spaces with applications

C.C. Okeke, F.U. Ogbuisi, and O.T. Mewomo

Abstract. In this paper we introduce an iterative algorithm with Meir-Keeler contractions for finding zeros of the sum of finite families of m-accretive operators and finite family of α-inverse strongly accretive operators in a real smooth and uniformly convex Banach spaces. We also discuss application of this method to the approximation of solution to certain integro-differential equation with generalized p-Laplacian operators. Our results improves and compliments many recent and important results in the literature.

2010 Mathematics Subject Classification. 47H09; 47H10; 49J20; 49J40.
Key words and phrases. Accretive operators; resolvent; Meir-Keeler contraction; Banach space.

1. Introduction

Let E be a real Banach space and C nonempty, closed and convex subset of E. The modulus of convexity $\delta_E : [0,2] \to [0,1]$ is defined as

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = 1 = \|y\|,\ \|x-y\| \geq \epsilon\right\}.$$

E is called uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0,2]$; p-uniformly convex if there is $c_p > 0$ so that $\delta_E(\epsilon) > c_p\epsilon^p$ for any $\epsilon \in (0,2]$. The modulus of smoothness $\rho_E : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(\tau) = \left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

E is called uniformly smooth if $\lim_{\tau\to\infty} \frac{\rho_E(\tau)}{\tau} = 0$; q-uniformly smooth if there is $c_q > 0$ so that $\rho_E(\tau) \leq c_q\tau^q$ for any $\tau > 0$. Hilbert spaces, L_p (or l_p) spaces $(1 < p < \infty)$, and the sobolev spaces $(W^p_m,\ 1 < p < \infty)$, are q-uniformly smooth Banach spaces [4]. It is shown in [23] that there is no Banach space which is q-uniformly smooth with $q > 2$. It is obvious that every q-uniformly smooth Banach space is uniformly smooth.

The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},\quad x \in E.$$

Received December 17, 2017. Accepted June 25, 2019.
The first and second author acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DST-NRF COE-MaSS) Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS..

It is well known that J is single-valued and norm-to-norm uniformly continuous on each bounded subsets of E if E is a real smooth and uniformly convex Banach space (see [19]). In the sequel, we shall denote by j the single-valued normalized duality mapping. If E is a Hilbert space H, then j becomes the identity mapping on H.

Let $T : C \to E$ be a mapping. Then T is said to be

(i) k-Lipschitz if there exists $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $0 < k < 1$, then T is called a contraction and if $k = 1$, then T is said to be a nonexpansive mapping;

(ii) accretive if for all $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq 0,$$

where J is the normalized duality mapping;

(iii) α- inverse strongly accretive if for all $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq \alpha\|Tx - Ty\|^2,$$

for some $\alpha > 0$;

(iv) m-accretive if T is accretive and $R(I + \lambda T) = E$, $\forall\ \lambda > 0$;

(v) strongly positive if E is a real Banach space and there exists $\bar{\gamma} > 0$ such that

$$\langle Tx, jx\rangle \geq \bar{\gamma}\|x\|^2, \forall\ x \in C.$$

We denote by J_r^A (for $r > 0$) the resolvent of an accretive operator A; that is $J_r^A := (I + rA)^{-1}$. It is well known that J_r^A is nonexpansive and $F(J_r^A) = A^{-1}0$ (see, for example, [9]).

Let C be a convex subset of E, let K be a nonempty subset of C and let p be a retraction from C onto K, i.e, $Px = x$ for each $x \in K$. P is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$. and $t \geq 0$ with $Px + t(x - Px) \in C$. If there is a sunny nonexpansive retraction from C onto K, K is said to be a sunny nonexpansive retract of C.

Let $A : E \to E$ be a single-valued nonlinear mapping and $B : E \to 2^E$ be a set-valued mapping. We consider the following inclusion problem: find $u \in E$ such that

$$0 \in (A + B)x. \tag{1}$$

Many practical problems can be reduced to the Problem (1) and it is well known that this problem provides a convenient framework for the unified study of optimal solution in many optimization related areas including variational inequalities, complementarity, mathematical programming, mathematical economics, optimal control, equilibria, game theory, etc (see [11, 12] and reference therein).

The classical method for solving Problem (1) is the forward-backward splitting algorithm, which were proposed by Lions and Mercier [8], Passty [13] and in a dual form for convex programming, Han and Lou [6]. The classical forward-backward splitting algorithm is given by: $x_1 \in E$ and

$$x_{n+1} = (I + r_nB)^{-1}(I - r_nA)x_n, \quad n \geq 1. \tag{2}$$

We see that for each step of iterate involves only with A as the forward step and B as the Backward step, but not the sum of $A+B$ and the based on the iterative algorithm

(2) much work has been done for finding $x \in H$ such that $x \in (A+B)^{-1}0$, where A and B are α-inversely strong monotone mapping and maximal monotone operator defined on the Hilbert space H, respectively.

In 2014, Qin et al., [14] introduced the iterative algorithm in q-uniformly smooth Banach spaces $x_0 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n (I + r_n B)^{-1}[(I - r_n A)x_n + e_n] + \gamma_n f_n, \quad n \geq 0, \tag{3}$$

where C is a closed convex subset of E, $\{e_n\}$ is the error sequence, f is a contraction, A and B are α-inversely strongly accretive operator and m-accretive operator respectively. If $(A+B)^{-1}0 \neq \emptyset$, they proved that $\{x_n\}$ converges strongly to $x = Q_{(A+B)^{-1}0} f(x)$, where $Q_{(A+B)^{-1}0}$ is the unique sunny nonexpansive retraction of E onto $(A+B)^{-1}0$ under some conditions.

Recently, Wei and Duan [21] presented the following iterative algorithm with errors in a real smooth and uniformly convex Banach space:

$$\begin{cases} x_0 \in C, \\ y_n = Q_C[(1-\alpha_n)(x_n + e_n)], \\ z_n = (1-\beta_n)x_n + \beta_n[a_0 y_n + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(y_n - r_{n,i} B_i y_n)], \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T) z_n, \quad n \geq 0, \end{cases} \tag{4}$$

where C is a nonempty, closed and convex sunny nonexpansive retract of E, Q_C is the sunny nonexpansive retraction of E onto C, $\{e_n\} \subset E$ is the error sequence, $\{A_i\}_{i=1}^N$ is finite family of m-accretive operators and $\{B_i\}_{i=1}^N$ is a finite family of α-inverse strongly accretive operators. $T : E \to E$ is a strongly positive bounded linear operator with coefficient $\bar{\gamma}$ and $f : E \to E$ is a contraction with coefficient $k \in (0,1)$. $J_{r_{n,i}}^{A_i} = (I + r_{n,i} A_i)^{-1}$, for $i = 1, 2, ..., N$, $\sum_{i=0}^N a_i = 1$, $0 < a_i < 1$, for $i = 0, 1, 2, ..., N$. Then $\{x_n\}$ converges strongly to $p_0 \in \cap_{i=1}^N (A_i + B_i)^{-1}0$, which is also a solution of some variational inequality problem.

Motivated by the works of Song *et al.* [16], Wei and Duan [21] and Shehu and Cai [18], we study and prove strong convergence results, under some mild conditions, using generalized forward-backward method which involve viscosity approximation method with Meir-Keeler contractions for solving the inclusion problem (1) for a finite family of m-accretive and α- inverse strongly accretive operators in the framework of uniformly convex and uniformly smooth Banach spaces. Finally we provide some applications of our result to certain integro-differential equation with generalized p-Laplacian operator. Our results is interesting and it also improves and compliments the result of Song *et al.* [16] and Wei and Duan [21] (see Remark 3.1 for details).

2. Preliminaries

Theorem 2.1. *(Banach contraction mapping principle* [1]*). Let (X,d) be a complete metric space and let f be a contraction on X. Then f has a unique fixed point.*

Theorem 2.2. *(Meir and Keeler* [10]*). Let (X,d) be a complete metric space and let f be a Meir-Keeler contraction (MKC, for short) on X, that is, for every $\epsilon > 0$, the exists $\delta > 0$ such that $d(x,y) < \epsilon + \delta$ implies $d(f(x), f(y)) < \epsilon$ for all $x, y \in X$. Then f has a unique fixed point.*

Remark 2.1. It is well known that Theorem 2.2 is a generalization of Theorem 2.1 since contractions are proper subclass of Meir-Keeler contractions.

We now state some important lemmas that will be needed in our main results.

Lemma 2.3. *(see [3]) Assume A is a strongly positive bounded operator with coefficient $\bar{\gamma} > 0$ on a real smooth Banach space E and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq I - \rho\bar{\gamma}$.*

Lemma 2.4. *(see [15] Lemma 2.3) Let f be an MKC on a convex subset of a Banach space E. Then for each $\epsilon > 0$, there exists $r_\epsilon \in (0, 1)$ such that*

$$\|x - y\| \geq \epsilon \implies \|f(x) - f(y)\| \leq r_\epsilon \|x - y\| \quad \forall x, y \in C. \tag{5}$$

Lemma 2.5. *(see [2]) Let E be a Banach space and let A be an m-accretive operator. For $\lambda > 0$, $\mu > 0$ and $x \in E$, we have*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x\right),$$

where $J_\lambda^A = (I + \lambda A)^{-1}$ and $J_\mu^A = (I + \mu A)^{-1}$.

Lemma 2.6. *(see [17]) Let $\{x_n\}$, $\{z_n\}$ be bounded sequences in E and $\{\beta_n\}$ be a sequence in $[0, 1]$ which satisfied the following condition: $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.7. *(see [7]) Let C be a nonempty closed and convex subset of a reflexive Banach space E which satisfies the Opial condition, and suppose $T : C \to E$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ implies $x = Tx$.*

Lemma 2.8. *(see [22]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n \gamma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9. *(see [5] Let E be a real Banach space with Fréchet differentiable norm. For $x \in E$, let $\beta^*(t)$ be defined for $0 < t < \infty$ by*

$$\beta^*(t) = \sup\left\{\left|\frac{\|x + ty\|^2 - \|x\|^2}{t} - 2\langle y, j(x)\rangle\right| : \|y\| = 1\right\}. \tag{6}$$

Then, $\lim_{t\to 0^+} \beta^*(t) = 0$ and

$$\|x + h\|^2 \leq \|x\|^2 + 2\langle h, j(x)\rangle + \|h\|\beta^*(\|h\|)$$

for all $h \in E \setminus \{0\}$.

In the result of Cholamjik and Suantai [5], the authors assumed that $\beta^*(t) \leq 2t$ for $t > 0$. In our more general setting, throughout this paper, we will assume that

$$\beta^*(t) \leq ct, \quad t > 0 \ \text{ and for some } \ c > 1,$$

where β^* is the function appearing in (6).

3. Main result

Lemma 3.1. *Let E be a real smooth and uniformly convex Banach space and C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a nonexpansive mapping and $f : C \to C$ be MKC, $M : E \to E$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous at zero, $0 \leq \eta < \frac{\bar{\gamma}}{2}$ and $F(T) \neq \emptyset$. If for each $t \in (0,1)$, define $S_t : E \to E$ by*

$$S_t x := t\eta f(x) + (I - tM)Tx, \tag{7}$$

then S_t has a fixed point x_t, for each $0 < t \leq \|M\|^{-1}$, which converges strongly to the fixed point of T, as $t \to 0$. That is $\lim_{t\to 0} x_t = x_0 \in F(T)$. Moreover, x_0 satisfies the following variational inequality

$$\langle (M - \eta f)x_0, j(x_0 - z)\rangle \leq 0, \quad \forall z \in F(T). \tag{8}$$

Proof. From the definition of MKC, we can see that MKC is also a nonexpansive mapping. Hence we obtain

$$\begin{aligned} \|S_t x - S_t y\| &\leq t\eta\|f(x) - f(y)\| + \|(1 - tM)(Tx - Ty)\| \\ &\leq t\eta\|f(x) - f(y)\| + (1 - t\bar{\gamma})\|x - y\| \\ &\leq t\eta\|x - y\| + (1 - t\bar{\gamma})\|x - y\| \\ &\leq [1 - t(\bar{\gamma} - k\eta)]\,\|x - y\|, \end{aligned}$$

which implies that S_t is a contraction since $0 < \eta < \frac{\bar{\gamma}}{2}$. Then Theorem 2.1 implies that S_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation

$$x_t = t\eta f(x_t) + (I - tM)Tx_t. \tag{9}$$

Next we show that the solution to the variational inequality (8) is unique. Suppose both $x_0 \in F(T)$ and $\hat{x}$ are solutions of (8), without lost of generalities, we may assume that there is a number ϵ such that $\|x_0 - \hat{x}\| \geq \epsilon$. Then by Lemma 2.4, there exists a number $k > 0$ such that $\|f(x_0) - f(\hat{x})\| \leq k_\epsilon\|x_0 - \hat{x}\|$. From (8) we obtain

$$\begin{cases} \langle (M - \eta f)x_0, j(x_0 - \hat{x})\rangle \leq 0, \\ \langle (M - \eta f)\hat{x}, j(\hat{x} - x_0)\rangle \leq 0. \end{cases} \tag{10}$$

Adding up (10), we obtain

$$\begin{aligned} &\langle (M - \eta f)\hat{x} - (M - \eta f)x_0, j(\hat{x} - x_0)\rangle = \\ &\qquad = \langle M(\hat{x} - x_0), j(\hat{x} - x_0)\rangle - \eta\langle f(\hat{x}) - f(x_0), j(\hat{x} - x_0)\rangle \\ &\qquad \geq \hat{\gamma}\|\hat{x} - x_0\|^2 - k\eta\|\hat{x} - x_0\|^2 = (\hat{\gamma} - k\eta)\|\hat{x} - x_0\|^2 \\ &\qquad \geq (\hat{\gamma} - k\eta)\epsilon^2 > 0. \end{aligned}$$

Therefore $x_0 = \bar{x}$ and the uniqueness is proved. Hence x_0 is a unique solution of (8).

Now we show that $\{x_t\}$ is bounded. Indeed, we may assume with no loss of generality, $t < \|M\|^{-1}$, for all $p \in F(T)$, fixed ϵ_1, for each $t \in (0,1)$.

Case 1 ($\|x_t - p\| < \epsilon_1$): In this case, $\{x_t\}$ is bounded.

Case 2 ($\|x_t - p\| \geq \epsilon_1$): In this case, we obtain by Lemma 2.3 and 2.4 that there is a number r_1 such that

$$\|f(x_t) - f(p)\| < r_1 \|x_t - p\|. \tag{11}$$

Hence we obtain

$$\begin{aligned}
\|x_t - p\| &= \|t\eta f(x_t) + (I - tM)Tx_t - p\| \\
&= \|t(\eta f(x_t) - Mp) + (I - tM)(Tx_t - p)\| \\
&\leq t\|\eta f(x_t) - Mp)\| + (1 - t\bar{\gamma})\|x_t - p\| \\
&\leq t\|\eta f(x_t) - \eta f(p)\| + t\|\eta f(p) - Mp\| + (1 - t\bar{\gamma})\|x_t - p\| \\
&\leq t\eta r_1 \|x_t - p\| + t\|\eta f(p) - Mp\| + (1 - t\bar{\gamma})\|x_t - p\|.
\end{aligned}$$

Therefore

$$\|x_t - p\| \quad \leq \quad \frac{\|\eta f(p) - Mp\|}{\bar{\gamma} - \gamma r_1}. \tag{12}$$

This implies that $\{x_t\}$ is bounded. Consequently $\{f(x_t)\}$ and $\{Tx_t\}$ are bounded.

Since $\{f(x_t)\}$ and $\{Tx_t\}$ are bounded, we obtain from (9) that

$$\|x_t - Tx_t\| = t\|\eta f(x_t) - MTx_t\| \to 0, \quad as \quad t \to 0. \tag{13}$$

To prove that $x_t \to x_0$ ($x_0 \in F(T)$) as $t \to 0$.

Since $\{x_t\}$ is bounded and E uniformly convex by Milman Pettis Theorem we have E is reflexive. Hence there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n} \rightharpoonup x^*$. By (12) we have that $x_{t_n} - Tx_{t_n} \to 0$, as $t_n \to 0$. Since E satisfies Opial's condition, it follows from Lemma 2.6 that $x^* \in F(T)$. Claim

$$\|x_{t_n} - x^*\| \to 0. \tag{14}$$

Suppose by contradiction, there is a number ϵ_0 and a subsequence $\{x_{t_m}\}$ of $\{x_{t_n}\}$ such that $\|x_{t_m} - x^*\| \geq \epsilon_0$. From Lemma 2.4, there is a number $r_{\epsilon_0} > 0$ such that $\|f(x_{t_m}) - f(x^*)\| \leq r_{\epsilon_0} \|x_{t_m} - x^*\|$, we have

$$\begin{aligned}
\|x_{t_m} - x^*\|^2 &= t_m \langle \eta f(x_{t_m}) - Mx^*, j(x_{t_m} - x^*)\rangle + \langle (1 - t_m)(Tx_{t_m} - x^*), j(x_{t_m} - x^*)\rangle \\
&\leq t_m \langle \eta f(x_{t_m}) - Ax^*, j(x_{t_m} - x^*)\rangle + (1 - t_m\bar{\gamma})\|x_{t_m} - x^*\|^2.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\|x_{t_m} - x^*\|^2 &\leq \frac{1}{\bar{\gamma}} \langle \eta f(x_{t_m}) - Mx^*, j(x_{t_m} - x^*)\rangle \\
&\leq \frac{1}{\bar{\gamma}} [\langle \eta f(x_{t_m}) - \eta f(x^*), j(x_{t_m} - x^*)\rangle + \langle \eta f(x^*) - Mx^*, j(x_{t_m} - x^*)\rangle] \\
&\leq \frac{1}{\bar{\gamma}} [\eta r_{\epsilon_0} \|x_{t_m} - x^*\|^2 + \langle \eta f(x^*) - Mx^*, j(x_{t_m} - x^*)\rangle].
\end{aligned}$$

Therefore

$$\|x_{t_m} - x^*\|^2 \quad \leq \quad \frac{\langle \eta f(x^*) - Mx^*, j(x_{t_m} - x^*)\rangle}{\bar{\gamma} - \eta r_{\epsilon_0}}. \tag{15}$$

Using the fact the duality map j is single valued and weakly sequentially continuous at zero by (15), we get that $x_{t_m} \to x^*$. It is a contradiction. Hence, we have $x_{t_n} \to x^*$.

Finally, we show that x^* solves the variational inequality (8). Since

$$x_t = t\eta f(x_t) + (I - tM)Tx_t,$$

we obtain

$$(M - \eta f)x_t = -\frac{1}{t}(I - tM)(1 - T)x_t. \tag{16}$$

Notice

$$\begin{aligned}\langle (I-T)x_t - (I-T)z, j(x_t - z)\rangle &\geq \|x_t - z\|^2 - \|Tx_t - Tz\|\|x_t - z\| \\ &\geq \|x_t - z\|^2 - \|x_t - z\|^2 \\ &= 0.\end{aligned}$$

It follows that, for $z \in F(T)$,

$$\begin{aligned}\langle (M-\eta f)x_t, j(x_t - z)\rangle &= -\frac{1}{t}\langle (I - tM)(I-T)x_t, j(x_t - z)\rangle \\ &= -\frac{1}{t}\langle (I-T)x_t - (I-T)z, j(x_t - z)\rangle + \langle M(I-T)x_t, j(x_t - z)\rangle \\ &\leq \langle M(I-T)x_t, j(x_t - z)\rangle. \end{aligned} \tag{17}$$

Now, replacing t in (17) with t_n and letting $n \to \infty$, noticing that $(I-T)x_{t_n} \to (I-T)x^* = 0$ for $x^* \in F(T)$, we obtain $\langle (M - \eta f)x_t, j(x_t - z)\rangle \leq 0$. That is $x^* \in F(T)$ is a solution of (8). Hence $x_0 = x^*$ by uniqueness. Hence, we have show that each cluster point of $\{x_t\}$ as $t \to 0$ equals $\hat{x}$, therefore, $x_t \to \hat{x}$ as $t \to 0$. □

Lemma 3.2. *Let E be a real smooth and uniformly convex Banach space. Let C be a nonempty convex and closed subset of E. Let $A_i : E \to 2^E$ $(i = 1, 2, ..., N)$ be m-accretive operators such that $\overline{D(A_i)} \subseteq C$ and let $B_i : C \to E$ be α_i-inverse strongly accretive operators such that $\cap_{i=1}^N (A_i + B_i)^{-1}0 \neq \emptyset$. Let $a_0, a_1, ..., a_N$ be real numbers in $(0, 1)$ such that $\sum_{i=0}^N a_i = 1$ and $P_n = a_0 I + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)$, where $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$ and $0 < r_{n,i} \leq \frac{2\alpha_i}{c}$ $\forall$ $i = 1, 2, ..., N$ and $n \geq 1$. Then $P_n : C \to C$ is nonexpansive and $F(P_n) = \cap_{i=1}^N (A_i + B_i)^{-1}0$, for all $n \geq 1$.*

Proof. First, we show that P_n is nonexpansive for all $n \geq 1$. Let $x, y \in C$. Then for $i = 1, 2, ..., N$, it follows that

$$\begin{aligned}\|(I - r_{n,i}B_i)x - (I - r_{n,i}B_i)y\|^2 &= \|x - y - r_{n,i}(B_i x - B_i y)\|^2 \\ &\leq \|x - y\|^2 - 2r_{n,i}\langle B_i x - B_i y, j(x - y)\rangle + cr_{n,i}^2\|B_i x - B_i y\|^2 \\ &\leq \|x - y\|^2 - 2r_{n,i}\alpha\|B_i x - B_i y\|^2 + cr_{n,i}^2\|B_i x - B_i y\|^2 \\ &= \|x - y\|^2 - (2\alpha - cr_{n,i})r_{n,i}\|B_i x - B_i y\|^2 \\ &\leq \|x - y\|^2.\end{aligned}$$

Thus $(I - r_{n,i}B_i)$ is nonexpansive for all $i = 1, 2, ..., N$.

Since $J_{r_{n,i}}^{A_i}$ and $(1 - r_{n,i}B_i)$ are nonexpansive for all $i = 1, 2, ..., N$, we get that

$$\begin{aligned} \|P_n x - P_n y\| &\le a_0\|x-y\| + \sum_{i=1}^{N} a_i \left\| J_{r_{n,i}}^{A_i}(1 - r_{n,i}B_i)x - J_{r_{n,i}}^{A_i}(1 - r_{n,i}B_i)y \right\| \\ &\le a_0\|x-y\| + \sum_{i=1}^{N} a_i \|(1 - r_{n,i}B_i)x - (1 - r_{n,i}B_i)y\| \\ &\le a_0\|x-y\| + \sum_{i=1}^{N} a_i\|x-y\| \\ &= \|x-y\|. \end{aligned}$$

Thus P_n is nonexpansive for all $n \ge 1$.

Next we show that $F(P_n) = \cap_{i=1}^{N}(A_i + B_i)^{-1}0$, for all $n \ge 1$. It is obvious that $\cap_{i=1}^{N}(A_i + B_i)^{-1}0 \subseteq F(P_n)$. So, we are left to show that $F(P_n) \subseteq \cap_{i=1}^{N}(A_i + B_i)^{-1}0$. Let $u \in F(P_n)$. Then $P_n u = u$ and for all $v \in \cap_{i=1}^{N}(A_i + B_i)^{-1}0 \subseteq F(P_n)$, we have

$$\begin{aligned} \|u - v\| &\le a_0\|u - v\| + a_1 \left\| J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)u - v \right\| + ... \\ &\quad + a_N \left\| J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v \right\| \\ &\le (a_0 + a_1 + ... + a_{N-1})\|u - v\| + a_N\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v\| \\ &\le (1 - a_N)\|u - v\| + a_N \left\| J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v \right\|. \end{aligned}$$

Therefore

$$\|u - v\| = (1 - a_N)\|u - v\| + a_N \left\| J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v \right\|,$$

which implies that

$$\|u - v\| = \left\| J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v \right\|.$$

Similarly,

$$\|u - v\| = \left\| J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)u - v \right\| = ... = \left\| J_{r_{n,N-1}}^{A_{N-1}}(I - r_{n,N-1}B_{N-1})u - v \right\|.$$

Then

$$\begin{aligned} \|u - v\| =& \frac{a_1}{\sum_{i=1}^{N} a_i}\| \big(J_{r_{n,1}}(I - r_{n,1}B_1)u - v\big) \| + \frac{a_2}{\sum_{i=1}^{N} a_i}\| \big(J_{r_{n,2}}(I - r_{n,2}B_2)u - v\big) \| \\ &+ ... + \frac{a_N}{\sum_{i=1}^{N} a_i}\| \big(J_{r_{n,N}}(I - r_{n,N}B_N)u - v\big) \|. \end{aligned}$$

By strict convexity of E, we have that

$$u - v = J_{r_{n,1}}(I - r_{n,1}B_1)u - v = J_{r_{n,2}}(I - r_{n,2}B_2)u - v = ... = J_{r_{n,N}}(I - r_{n,N}B_N)u - v.$$

Therefore, $J_{r_{n,i}}(I - r_{n,i}B_i)u = u$, for $i = 1, 2, ..., N$. Then $u \in \cap_{i=1}^{N}(A_i + B_i)^{-1}0$.

Thus $F(P_n) \subseteq \cap_{i=1}^{N}(A_i + B_i)^{-1}0$. □

Theorem 3.3. *Let E be a real smooth and uniformly convex Banach space and C be a nonempty, closed and convex subset of E, and let $f : C \to C$ be a MKC. Let $M : C \to C$ be a strong positive bounded linear operator, $\bar{\gamma} > 0$ such that $0 \le \eta < \frac{\bar{\gamma}}{2}$. Suppose that the duality mapping $j : E \to E^*$ is weakly sequentially continuous at* *zero. Let $A_i : C \to 2^E$ be m-accretive operators and $B_i : C \to E$ be α_i-inverse strongly accretive operators, for $i = 1, 2, ..., N$ such that $\cap_{i=1}^N (A_i + B_i)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be generated by $x_1 \in E$,*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)\left[a_0 x_n + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)x_n\right], \\ x_{n+1} = \alpha_n \eta f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n M)y_n, \ n \geq 1, \end{cases} \tag{18}$$

for all $n \geq 1$, where $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$ for $i = 1, 2, ..., N$, and $0 < a_i < 1$, for $i = 0, 1, 2, ..., N$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real number sequence in $(0, 1)$ and $\{r_{n,i}\} \subset (0, \infty)$. Suppose that the above sequence satisfy the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $0 < r_{n,i} < \frac{2\alpha}{c}$ *and* $\sum_{n=1}^\infty |r_{n+1,i} - r_{n,i}| < \infty$ *for* $n \geq 1$ *and* $i = 1, 2, ...N$, *where c is a constant;*
(iii) $\lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0$;
(iv) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.
Then $\{x_n\}$ converges strongly to a point $x_0 \in \cap_{i=1}^N (A_i + B_i)^{-1}0$, which is the unique solution of the variational inequality: $\forall z \in \cap_{i=1}^N (A_i + B_i)^{-1}0$.

$$\langle (M - \eta f)x_0, J(x_0 - z)\rangle \leq 0, \tag{19}$$

where $x_0 = Q_{\cap_{i=1}^N (A_i+B_i)^{-1}0} f(x_0)$, and $Q_{\cap_{i=1}^N (A_i+B_i)^{-1}0}$ is the unique sunny nonexpansive retraction of E onto $\cap_{i=1}^N (A_i + B_i)^{-1}0$.

Proof. Put $P_n = a_0 I + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)$ and $u_{n_{n,i}} = (I - r_{n,i}B_i)x_n$ for $i = 1, 2, 3, ..., N$ and $n \geq 1$. Then we obtain from (18) and Lemma 3.2 that

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)P_n x_n - p\| \\ &\leq \|\beta_n(x_n - p) + (1-\beta_n)(P_n x_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{20}$$

From the definition of MKC and Lemma 2.4, for each $\epsilon > 0$ there is a number $r_\epsilon \in (0, 1)$, if $\|x_n - z\| < \epsilon$ then $\|f(x_n) - f(z)\| \leq r_\epsilon\|x_n - z\|$. it follows from (18) and (20) that

$$\begin{aligned} \|x_{n+1} - p\| =& \|\alpha_n \eta f(x_n) + \gamma_n x_n + (1-\gamma_n)I - \alpha_n M)y_n - p\| \\ =& \|\alpha_n(\eta f(x_n) - Mp) + \gamma_n(x_n - p) + ((1-\gamma_n)I - \alpha_n M)(y_n - p)\| \\ \leq& \alpha_n\|\eta f(x_n) - Mp\| + \gamma_n\|x_n - p\| + (1 - \gamma_n - \alpha_n\bar{\gamma})\|x_n - p\| \\ \leq& \alpha_n\eta\max\{r_\epsilon\|x_n - p\|, \epsilon\} + \alpha_n\|\eta f(p) - Mp\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\ =& \max\{(1-\alpha_n\bar{\gamma})\|x_n - p\| + \alpha_n\eta r_\epsilon\|x_n - p\| + \alpha_n\|\eta f(p) - Mp\|, \\ & (1-\alpha_n\bar{\gamma})\|x_n - p\| + \alpha_n\eta\epsilon + \alpha_n\|\eta f(p) - Mp\|\} \\ =& \max\{(1 - \alpha_n\bar{\gamma} + \alpha_n\eta r_\epsilon)\|x_n - p\| + \alpha_n\|\eta f(p) - Mp\|, (1-\alpha_n\bar{\gamma})\|x_n - p\| \\ & + \alpha_n\eta\epsilon + \alpha_n\|\eta f(p) - Mp\|\} \\ =& \max\{[1 - (\alpha_n\bar{\gamma} - \alpha_n\eta r_\epsilon)]\|x_n - p\| + \alpha_n\|\eta f(p) - Mp\|, (1-\alpha_n\bar{\gamma})\|x_n - p\| \\ & + \alpha_n\eta\epsilon + \alpha_n\|\eta f(p) - Mp\|\}. \end{aligned}$$

Inductively, we obtain

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\eta f(p) - Mp\|}{\bar{\gamma} - \eta r_\epsilon}, \frac{\gamma\epsilon + \|\eta f(p) - Mp\|}{\bar{\gamma}}, \right\} \quad n \geq 1, \tag{21}$$

which implies that the sequence $\{x_n\}$ is bounded.

Next we show that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$.

First we consider $\|J^{A_i}_{r_{n+1,i}} u_{n+1,i} - J^{A_i}_{r_{n,i}} u_{n,i}\|$, if $r_{n,i} \leq r_{n+1,i}$ then it follows from Lemma 2.5 that

$$\begin{aligned}
\left\|J^{A_i}_{r_{n+1,i}} u_{n+1,i} - J^{A_i}_{r_{n,i}} u_{n,i}\right\| &= \\
&= \left\|J^{A_i}_{r_{n,i}}\left(\frac{r_{n,i}}{r_{n+1,i}} u_{n+1,i} + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right) J^{A_i}_{r_{n+1,i}} u_{n+1,i}\right) - J^{A_i}_{r_{n,i}} u_{n,i}\right\| \\
&\leq \left\|\frac{r_{n,i}}{r_{n+1,i}} u_{n+1,i} + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right) J^{A_i}_{r_{n+1,i}} u_{n+1,i} - u_{n,i}\right\| \\
&\leq \frac{r_{n,i}}{r_{n+1,i}}\|u_{n+1,i} - u_{n,i}\| + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right)\left\|J^{A_i}_{n+1,i} u_{n+1,i} - u_{n,i}\right\| \\
&\leq \|u_{n+1,i} - u_{n,i}\| + \frac{r_{n+1,i} - r_{n,i}}{b} 2M_1.
\end{aligned} \tag{22}$$

If $r_{n+1,i} \leq r_{n,i}$, using similar proof as in (22), we obtain

$$\left\|J^{A_i}_{r_{n+1,i}} u_{n+1,i} - J^{A_i}_{r_{n,i}} u_{n,i}\right\| \quad \leq \quad \|u_{n+1,i} - u_{n,i}\| + \frac{r_{n,i} - r_{n+,i}}{b} 2M_1. \tag{23}$$

Combining (22) and (23), we have, for $n \geq 1$,

$$\begin{aligned}
\left\|J^{A_i}_{r_{n+1,i}} u_{n+1,i} - J^{A_i}_{r_{n,i}} u_{n,i}\right\| &\leq \|u_{n+1,i} - u_{n,i}\| + \frac{2|r_{n,i} - r_{n+,i}|}{b} M_1 \\
&\leq \|(I - r_{n+1,i}B_i)(x_{n+1} - x_n)\| + |r_{n+1,i} - r_{n,i}|\|B_i x_n\| + \frac{2|r_{n+1,i} - r_{n,i}|}{b} M_1 \\
&\leq \|x_{n+1} - x_n\| + |r_{n+1,i} - r_{n,i}|\|B_i x_n\| + \frac{2|r_{n+1,i} - r_{n,i}|}{b} M_1.
\end{aligned} \tag{24}$$

Set $M_2 = \left(\frac{2}{b} + M_1\right)$ and using (24), we obtain

$$\begin{aligned}
\|P_{n+1}x_{n+1} - P_n x_n\| \quad &\leq \quad a_0\|x_{n+1} - x_n\| \\
&\quad + \sum_{i=1}^{N}\left\|a_i\left(J^{A_i}_{r_{n+1,i}}(I - r_{n+1,i}B_i)x_n - J^{A_i}_{r_{n,i}}(I - r_{n,i}B_i)x_n\right)\right\| \\
&\leq \quad \|x_{n+1} - x_n\| + M_2\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}|.
\end{aligned} \tag{25}$$

Next, from (18), we get that

$$x_{n+1} = \alpha_n \eta f(x_n) + \gamma_n x_n + [(1 - \gamma)I - \alpha_n M]Q_n x_n. \tag{26}$$

Now, define

$$z_n = \frac{x_{n+1} - \gamma_n x_n}{1 - \gamma_n}. \tag{27}$$

Hence, we obtain

$$\begin{aligned} z_{n+1} - z_n = \\ = \frac{\alpha_{n+1}\eta f(x_{n+1}) + \gamma_{n+1}x_{n+1} + \left[(1-\gamma_{n+1})I - \alpha_{n+1}M\right]Q_{n+1}x_{n+1} - \gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\ - \frac{\alpha_n f(x_n) + \gamma_n x_n + \left[(1-\gamma_n)I - \alpha_n M\right]Q_n x_n - \gamma_n x_n}{1-\gamma_n} \\ = \frac{\alpha_{n+1}\left[\eta f(x_{n+1}) - MQ_{n+1}x_{n+1}\right]}{1-\gamma_{n+1}} - \frac{\alpha_n\left[\eta f(x_n) - MQ_n x_n\right]}{1-\gamma_n} + Q_{n+1}x_{n+1} - Q_n x_n, \end{aligned} \tag{28}$$

which implies that

$$\begin{aligned} \|z_{n+1} - z_n\| \quad \leq \quad & \frac{\alpha_{n+1}\|\eta f(x_{n+1}) - MQ_{n+1}x_{n+1}\|}{1-\gamma_{n+1}} + \frac{\alpha_n\|\eta f(x_n) - MQ_n x_n\|}{1-\gamma_n} \\ & + \|Q_{n+1}x_{n+1} - Q_n x_n\|. \end{aligned} \tag{29}$$

Now, we estimate $\|Q_{n+1}x_{n+1} - Q_n x_n\|$.

$$\begin{aligned} \|Q_{n+1}x_{n+1} - Q_n x_n\| &= \|\left[\beta_{n+1}x_{n+1} + (1-\beta_{n+1})P_{n+1}x_{n+1}\right] - \left[\beta_n x_n + (1-\beta_n)P_n x_n\right]\| \\ &\leq (1-\beta_{n+1})\|P_{n+1}x_{n+1} - P_{n+1}x_n\| + |\beta_{n+1} - \beta_n|\|P_n x_n\| \\ &\quad + \beta_{n+1}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|x_n\| \\ &\leq (1-\beta_{n+1})\|x_{n+1} - x_n\| + M_2(1-\beta_{n+1})\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| + |\beta_{n+1} - \beta_n|\|P_n x_n\| \\ &\quad + \beta_{n+1}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|x_n\| \\ &\leq \|x_{n+1} - x_n\| + M_2(1-\beta_{n+1})\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| + |\beta_{n+1} - \beta_n|\|P_n x_n\| \\ &\quad + |\beta_{n+1} - \beta_n|\|x_n\|. \end{aligned} \tag{30}$$

From (29) and (30), we obtain

$$\begin{aligned} \|z_{n+1} - z_n\| \leq & \frac{\alpha_{n+1}\|\eta f(x_{n+1}) - MQ_{n+1}x_{n+1}\|}{1-\gamma_{n+1}} + \frac{\alpha_n\|\eta f(x_n) - MQ_n x_n\|}{1-\gamma_n} \\ & + \|x_{n+1} - x_n\| + M_2(1-\beta_{n+1})\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| + |\beta_{n+1} - \beta_n|\|P_n x_n\| \\ & + |\beta_{n+1} - \beta_n|\|x_n\|. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq & \frac{\alpha_{n+1}\|\eta f(x_{n+1}) - MQ_{n+1}x_{n+1}\|}{1-\gamma_{n+1}} \\ & + \frac{\alpha_n\|\eta f(x_n) - MQ_n x_n\|}{1-\gamma_n} + M_2(1-\beta_{n+1})\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| \\ & + |\beta_{n+1} - \beta_n|\|P_n x_n\| + |\beta_{n+1} - \beta_n|\|x_n\|. \end{aligned} \tag{31}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{P_nx_n\}$ and $\{Q_nx_n\}$ are bounded by conditions (i), (ii) and (iii), we have that

$$\limsup_{n\to\infty}\{\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\}\leq 0. \tag{32}$$

Thus by Lemma 2.6, we obtain

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{33}$$

Hence we obtain from (28) and (33) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{34}$$

Also from (18), we obtain

$$\begin{aligned}\|Q_nx_n-x_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-Q_nx_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_n\eta f(x_n)+\gamma_n(x_n-Q_nx_n)-\alpha_nMQ_nx_n\|\\ &\leq \|x_n-x_{n+1}\|+\alpha_n(\|\eta f(x_n)\|+\|MQ_nx_n\|)+\gamma_n\|x_n-Q_nx_n\|,\end{aligned}$$

which implies that

$$\|Q_nx_n-x_n\| \leq \frac{1}{1-\gamma_n}(\|x_n-x_{n+1}\|+\alpha_n(\|\eta f(x_n)\|+\|MQ_nx_n\|)). \tag{35}$$

Hence from condition (i), (34) and (35), we get that

$$\lim_{n\to\infty}\|Q_nx_n-x_n\|=0. \tag{36}$$

Next, we estimate $\|P_nx_n-x_n\|$

$$\begin{aligned}\|P_nx_n-x_n\| &\leq \|x_n-Q_nx_n\|+\|Q_nx_n-P_nx_n\|\\ &\leq \|x_n-Q_nx_n\|+\|\beta_nx_n+(1-\beta_n)P_nx_n-P_nx_n\|\\ &\leq \|x_n-Q_nx_n\|+\beta_n\|x_n-P_nx_n\|,\end{aligned}$$

which implies that

$$\|P_nx_n-x_n\| \leq \frac{1}{1-\beta_n}\|x_n-Q_nx_n\|\to 0,\quad n\to\quad\infty. \tag{37}$$

Also we have

$$\begin{aligned}\|y_n-x_n\| &= \|\beta_nx_n+(1-\beta_n)P_nx_n-x_n\|\\ &= \beta_n\|x_n-P_nx_n\|+\|P_nx_n-x_n\|\to 0,\quad n\to\infty.\end{aligned} \tag{38}$$

Also we can obtain that

$$\|y_n-P_nx_n\|\leq\|y_n-x_n\|+\|x_n-P_nx_n\|\to 0,\quad n\to\infty.$$

In similar way, we obtain

$$\|x_{n+1}-y_n\|\leq\|x_{n+1}-x_n\|+\|x_n-y_n\|\to 0,\quad n\to\infty.$$

From (13) and Lemma 7, we know that there exists z_t such that $z_t=t\eta f(x_t)+(1-tM)P_nTx_t$ for $t\in(0,1)$. Moreover, $z_t\to x_0\in F(P_n)=\cap_{n=1}^{N}(A_i+B_i)^{-1}0,\ as\ t\to 0,$ and x_0 is the unique solution of the variational inequality (3.2).

Next we show that

$$\limsup_{n\to\infty}\langle\eta f(\eta)-M\hat{x},j(x_n-\hat{x})\rangle\leq 0, \tag{39}$$

where $\hat{x} = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \longmapsto t\eta f(x) + (1 - tM)P_n T x. \tag{40}$$

Now, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \eta f(\hat{x}) - M\hat{x}, j(x_n - \hat{x})\rangle = \lim_{k\to\infty} \langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_k} - \hat{x})\rangle. \tag{41}$$

We may also assume that $x_{n_k} \rightharpoonup q$. Note that $q \in F(P_n)$ by Lemma 2.7 and (39). Since j is weakly sequentially continuous duality mapping, we obtain from Lemma 7 that

$$\begin{aligned}\limsup_{n\to\infty} \langle \eta f(\hat{x}) - M\hat{x}, j(x_n - \hat{x})\rangle &= \lim_{k\to\infty} \langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_k} - \hat{x})\rangle \\ &= \langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_k} - \hat{x})\rangle \le 0. \end{aligned} \tag{42}$$

Hence, we obtain

$$\limsup_{n\to\infty} \langle \eta f(\hat{x}) - M\hat{x}, j(x_n - \hat{x})\rangle \le 0.$$

Finally, we show that $\|x_n - \hat{x}\| \to 0, \quad n \to \infty$. To do this, we divide the rest of the proof into two cases.
By contradiction, there is number ϵ_0 such that

$$\limsup_{n\to\infty} \|x_n - \hat{x}\| \ge \epsilon_0. \tag{43}$$

Case 1. Fixed ϵ_1 ($\epsilon_1 < \epsilon_0$), if for some $n \ge N \in \mathbb{N}$ such that $\|x_n - \hat{x}\| \ge \epsilon_0 - \epsilon_1$, and for the other $n \ge N \in \mathbb{N}$ such that $\|x_n - \hat{x}\| < \epsilon_0 - \epsilon_1$. Let

$$M_n = \frac{2\langle \eta f(\hat{x}) - M\hat{x}, j(x_{n+1} - \hat{x})\rangle}{(\epsilon_0 - \epsilon_1)^2}. \tag{44}$$

From (39), we know that $\limsup_{n\to\infty} M_n \le 0$. Hence, there is a number N, when $n > N$, we have $M_n \le \bar{\gamma} - \eta$. There exists $n_0 \ge N$ such that $\|x_{n_0} - \hat{x}\| < \epsilon_0 - \epsilon_1$, then we have

$$\begin{aligned}
&\|x_{n_0+1} - \hat{x}\|^2 = \\
&= \|\alpha_{n_0} f(x_{n_0}) + \gamma_{n_0} x_{n_0} + [(1-\gamma_{n_0})I - \alpha_{n_0} M]y_{n_0} - \hat{x}\|^2 \\
&= \|[(1-\gamma_{n_0})I - \alpha_{n_0} M](y_{n_0} - \hat{x}) + \alpha_{n_0}(\eta f(x_{n_0}) - M\hat{x}) + \gamma_{n_0}(x_{n_0} - \hat{x})\|^2 \\
&= \langle [(1-\gamma_{n_0})I - \alpha_{n_0} M]y_{n_0} - \hat{x}) + \alpha_{n_0}(\eta f(x_{n_0}) - M\hat{x}) + \gamma_{n_0}(x_{n_0} - \hat{x}), j(x_{n_0+1} - \hat{x})\rangle \\
&= \langle [(1-\gamma_{n_0})I - \alpha_{n_0} M](y_{n_0} - \hat{x}), j(x_{n_0+1} - \hat{x})\rangle + \langle \alpha_{n_0}(\eta f(x_{n_0}) - M\hat{x}), j(x_{n_0+1} - \hat{x})\rangle \\
&+ \langle \gamma_{n_0}(x_{n_0} - \hat{x}), j(x_{n_0+1} - \hat{x})\rangle \\
&= \langle [(1-\gamma_{n_0})I - \alpha_{n_0} M](y_{n_0} - \hat{x}), j(x_{n_0+1} - \hat{x})\rangle + \alpha_{n_0}\eta\langle f(x_{n_0}) - f(\hat{x}), j(x_{n_0+1} - \hat{x})\rangle \\
&+ \alpha_{n_0}\langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_0+1} - \hat{x})\rangle + \langle \gamma_{n_0}(x_{n_0} - \hat{x}), j(x_{n_0+1} - \hat{x})\rangle \\
&\le (1 - \gamma_{n_0} - \alpha_{n_0}\bar{\gamma})\|x_{n_0} - \hat{x}\|\|x_{n_0+1} - \hat{x}\| + \alpha_{n_0}\eta\|f(x_{n_0}) - f(\hat{x})\|\|x_{n_0+1} - \hat{x}\| \\
&+ \alpha_{n_0}\langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_0+1} - \hat{x})\rangle + \gamma_{n_0}\|x_{n_0} - \hat{x}\|\|x_{n_0+1} - \hat{x}\| \\
&< [1 - \alpha_{n_0}(\bar{\gamma} - \eta)](\epsilon_0 - \epsilon_1)\|x_{n_0+1} - \bar{x}\| + \alpha_{n_0}\langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_0+1} - \hat{x})\rangle \\
&\le \frac{1}{2}[1 - \alpha_{n_0}(\bar{\gamma} - \eta)]^2(\epsilon_0 - \epsilon_1)^2 + \frac{1}{2}\|x_{n_0+1} - \hat{x}\|^2 + \alpha_{n_0}\langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_0+1} - \hat{x})\rangle,
\end{aligned} \tag{45}$$

which implies from (45) that

$$\begin{aligned}\|x_{n_0+1}-\hat{x}\|^2 &\leq [1-\alpha_{n_0}(\bar{\gamma}-\eta)]^2(\epsilon_0-\epsilon_1)^2+2\alpha_{n_0}\langle \eta f(\hat{x})-M\hat{x}, j(x_{n_0+1}-\hat{x})\rangle \\ &\leq [1-\alpha_{n_0}(\bar{\gamma}-\eta)](\epsilon_0-\epsilon_1)^2+2\alpha_{n_0}\langle \eta f(\hat{x})-M\hat{x}, j(x_{n_0+1}-\hat{x})\rangle \\ &= [1-\alpha_{n_0}(\bar{\gamma}-\eta-M_n)](\epsilon_0-\epsilon_1)^2 \\ &\leq (\epsilon_0-\epsilon_1)^2. \end{aligned} \tag{46}$$

Hence, we have

$$\|x_{n_0+1}-\hat{x}\| < \epsilon_0-\epsilon_1, \quad for\ \epsilon_0>\epsilon_1.$$

In similar manner, we obtain

$$\|x_n-\hat{x}\| < \epsilon_0-\epsilon_1, \ \ \forall\ n\geq n_0,$$

which contradicts the fact that $\limsup_{n\to\infty}\|x_n-\hat{x}\|\geq\epsilon_0$.
Case 2. Fixed ϵ_1 ($\epsilon_1<\epsilon_0$), if $\|x_n-\hat{x}\|\geq\epsilon_0-\epsilon_1$ for all $n\geq N\in\mathbb{N}$, from Lemma 2.4, there is a number r_ϵ, $(0<r_\epsilon<1)$ such that

$$\|f(x_n)-f(\hat{x})\|\leq r\|x_n-\hat{x}\|, \ \ n\geq N. \tag{47}$$

From (18) and (47), we obtain

$$\begin{aligned} &\|x_{n_0+1}-\hat{x}\|^2 = \\ =&\|\alpha_n\eta f(x_n)+\gamma_n x_n+[(1-\gamma_n)I-\alpha_n M]y_n-\hat{x}\|^2 \\ =&\|[(1-\gamma_n)I-\alpha_n M](y_n-\hat{x})+\alpha_n(\eta f(x_n)-M\hat{x})+\gamma_n(x_n-\hat{x})\|^2 \\ =&\langle[(1-\gamma_n)I-\alpha_n M](y_n-\hat{x})+\alpha_n(\eta f(x_n)-M\hat{x})+\gamma_n(x_n-\hat{x}), j(x_{n_0+1}-\hat{x})\rangle \\ =&\langle[(1-\gamma_n)I-\alpha_n M](y_n-\hat{x}), j(x_{n+1}-\hat{x})\rangle+\langle\alpha_n(\eta f(x_n)-M\hat{x}), j(x_{n+1}-\hat{x})\rangle \\ &+\langle\gamma_n(x_n-\hat{x}), j(x_{n+1}-\hat{x})\rangle \\ \leq&\langle[(1-\gamma_n)I-\alpha_n M](y_n-\hat{x}), j(x_{n+1}-\hat{x})\rangle+\langle\alpha_n(\eta f(x_n)-f(\hat{x})), j(x_{n+1}-\hat{x})\rangle \\ &+\langle\alpha_n\eta f(\hat{x}-M\hat{x}), j(x_{n+1}-\hat{x})\rangle+\langle\gamma_n(x_n-\hat{x}), j(x_{n+1}-\hat{x})\rangle \\ \leq&(1-\gamma_n-\alpha_n\hat{\gamma})\|x_n-\hat{x}\|\|x_{n+1}-\hat{x}\|+\alpha_n\eta r\|x_n-\hat{x}\|\|x_{n+1}-\hat{x}\| \\ &+\langle\alpha_n\eta f(\hat{x}-M\hat{x}), j(x_{n+1}-\hat{x})\rangle+\gamma_n\|x_n-\hat{x}\|\|x_{n+1}-\hat{x}\| \\ \leq&[1-\alpha_n(\hat{\gamma}-\eta r)]\|x_n-x_{n+1}\|\|x_{n+1}-\hat{x}\|+\langle\alpha_n\eta f(\hat{x}-M\hat{x}), j(x_{n+1}-\hat{x})\rangle \\ \leq&[1-\alpha_n(\hat{\gamma}-\eta r)]\frac{1}{2}\|x_n-\hat{x}\|^2+\frac{1}{2}\|x_{n+1}-\hat{x}\|^2+\langle\alpha_n\eta f(\hat{x}-M\hat{x}), j(x_{n+1}-\hat{x})\rangle, \end{aligned}$$

which implies that

$$\|x_{n+1}-\hat{x}\|^2\leq[1-\alpha_n(\bar{\gamma}-\eta r)]\|x_n-\hat{x}\|+2\alpha_n\langle\eta f(\hat{x}-M\hat{x}), j(x_{n+1}-\hat{x})\rangle. \tag{48}$$

Hence from Lemma 2.8 and (48), we conclude that $x_n\to\hat{x}$ as $n\to\infty$, which contradict the fact that $\|x_n-\hat{x}\|\geq\epsilon_0-\epsilon_1$. This complete the proof. □

Remark 3.1. We make the following comments which highlight our contribution in this paper.

(i) We know that the Meir-Keeler contraction is a generalization of the contraction mapping and also the condition

$$\langle Bx-By, j((I-rB)x-(I-rB)y\rangle\geq 0$$

for all $x, y \in E$ and for all $r > 0$ assumed in the result of Wei and Duan [21] is dispensed in our result. Hence, our results improves the results of Wei and Duan [21].

(ii) It is well known that real smooth and uniformly convex Banach space are more general than Hilbert space or q-uniformly smooth Banach space and also our normalized duality mapping j is weakly sequentially continuous in most of the existing related work is weaken to j weakly sequentially continuous at zero. Hence our result extends the results of Song *et al.* [16].

If $i = 1$ and f is a contraction, then from Theorem 3.3 we obtain the following:

Corollary 3.4. *Let E be a real smooth and uniformly convex Banach space and C be a nonempty, closed and convex subset of E, and let $f : C \to C$ be a contraction mapping with $k \in (0, 1)$. Let $M : C \to C$ be a strong positive bounded linear operator $\bar{\gamma} > 0$ such that $0 \leq \eta < \frac{2\bar{\gamma}}{k}$. Suppose that the duality mapping $j : E \to E^*$ is weakly sequentially continuous at zero. Let $A : C \to 2^E$ be m-accretive operator and $B : C \to E$ be α-inversely strongly accretive operator, such that $(A + B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be generated by the following algorithm:*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) J^A_{r_n}(I - r_n B)x_n, \\ x_{n+1} = \alpha_n \eta f(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \alpha_n M)y_n, \ n \geq 1, \end{cases} \tag{49}$$

for all $n \geq 1$, where $J^A_{r_n} = (I + r_n A)^{-1}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real number sequence in $(0, 1)$ and $\{r_n\} \subset (0, \infty)$. Suppose that the above sequence satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < r_n < \frac{2\alpha}{c}$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ *for* $n \geq 1$ *and c is a constant;*
(iii) $\lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0$;
(iv) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Then $\{x_n\}$ converges strongly to a point $x_0 \in (A+B)^{-1}0$, which is the unique solution of the variational inequality: $\forall z \in (A + B)^{-1}0$.

$$\langle (M - \eta f)x_0, J(x_0 - z)\rangle \leq 0. \tag{50}$$

where $x_0 = Q_{(A+B)^{-1}(0)} f(x_0)$, and $Q_{(A+B)^{-1}(0)}$ is the unique sunny nonexpansive retraction of E onto $(A + B)^{-1}(0)$.

4. Applications

In this section, we give an application of our Corollary 3.4 to approximation of solution of certain nonlinear integro-differential equation involving the generalized p-Laplacian. Throughout this section, we shall assume $N \geq 1$, $\frac{2N}{N+1} < r \leq \min\{p, p'\} < +\infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, and $\frac{1}{r} + \frac{1}{r'} = 1$.

Let $V = L^p(0, T; W^{1,p}(\Omega))$ and V^* be the dual space of V. The norm in V will be denoted by $\|.\|_v$, which is defined by

$$\|u(x, t)\|_v := \left(\int_0^T \|u(x, t)\|^p_{W^{1,p(\Omega)}} dt \right)^{\frac{1}{p}}, \quad u(x, t) \in V.$$

Also, let $W = L^{\max\{p,p'\}}(0, T; L^{\max\{p,p'\}}(\Omega))$.

Now, using the result obtained in Corollary 3.4, we shall study the existence and uniqueness of the solution and iterative approximation of the unique solution of the following nonlinear integro-differential equation.

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left[a(x)\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right] + b(x)|u|^{q-2}u + c(x)|u|^{r-2}u \\ +g(x,u,\nabla u) + a_1\frac{\partial}{\partial t}\int_\Omega u dx = f(x,t) \quad a.e. \text{ in } \Omega\times(0,T) \\ -\left\langle \vartheta, a(x)\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right\rangle \in \beta_x(u(x)) \quad a.e \text{ on } \Gamma\times(0,T) \\ u(x,0) = u(x,T), \end{cases} \tag{51}$$

where Ω is a bounded conical domain of the Euclidean space $\mathbb{R}^N$, Γ is the boundary Ω with $\Gamma \in C^1$ and ϑ denotes the exterior normal derivatives to Γ. Also $f(x,t) \in W$, a, b and c are strictly positive bounded and continuous functions on Ω such that

$$\begin{aligned} 0 < a^- &= \inf_{x\in\Omega} a(x) \le a^+ = \sup_{x\in\Omega} a(x) < \infty \\ 0 < b^- &= \inf_{x\in\Omega} b(x) \le b^+ = \sup_{x\in\Omega} b(x) < \infty \\ 0 < c^- &= \inf_{x\in\Omega} c(x) \le c^+ = \sup_{x\in\Omega} c(x) < \infty. \end{aligned}$$

Moreover, a_1 is a positive constant and β_x is the subdifferential of ϑ_x, where $\vartheta_x = \vartheta(x,.) : \mathbb{R} \to \mathbb{R}$ for $x \in \Gamma$ and $\vartheta : \Gamma \times \mathbb{R} \to \mathbb{R}$ is the given function.

Lemma 4.1. [20] *The mapping $A : W \to 2^W$ is m-accretive.*

Lemma 4.2. [20] *Define $B : D(B) = L^{\max\{p,p'\}}(0,T; W^{1,\max\{p,p'\}}(\Omega)) \subset W \to W$ by*

$$(Bu)(x,t) = g(x,u,\nabla u) - f(x,t),$$

for $u(x,t) \in D(B)$. Then B is inversely strongly accretive.

Recently, Y. Shehu and G. Cai [18] proved the following theorem

Theorem 4.3. [18] *$u(x,t) \in W$ is the unique solution of the nonlinear boundary value problem* (51) *if and only if $u(x,t) \in (A+B)^{-1}(0)$.*

Now, using Theorem 4.3, Lemma 4.1 and 4.2 we obtain the following result.

Theorem 4.4. *Let $2 \le p < \infty$. Suppose A and B are the same as those in Lemma 4.1 and 4.2 respectively. Let*

$$f : W = L^{\max\{p,p'\}}(0,T; L^{\max\{p,p'\}}(\Omega)) \to L^{\max\{p,p'\}}(0,T; L^{\max\{p,p'\}}(\Omega))$$

be a fixed contraction with coefficient $k \in (0,1)$. Let $M : L^{\max\{p,p'\}}(0,T; L^{\max\{p,p'\}}(\Omega)) \to L^{\max\{p,p'\}}(0,T; L^{\max\{p,p'\}}(\Omega))$ be a strong positive bounded linear operator $\bar{\gamma} > 0$ such that $0 \le \eta < \frac{2\bar{\gamma}}{k}$. Suppose that the duality mapping $j_{\max\{p,p'\}} : E \to E^*$ is weakly sequentially continuous at zero such that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < r_n < \frac{2\alpha}{c}$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ *for* $n \ge 1$ *and* c *is a constant;*
(iii) $\lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0$;
(iv) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$.

Let the sequence $\{u_n(x,t)\}_{n=1}^{\infty}$ *be generated by* $u_1(x,t) \in W$,

$$\begin{cases} y_n = \beta_n u_n(x,t) + (1-\beta_n)J_{r_n}^A(I - r_nB)u_n(x,t), \\ u_{n+1}(x,t) = \alpha_n \eta f(u_n(x,t)) + \gamma_n u_n(x,t) + ((1-\gamma_n)I - \alpha_n M)y_n, \ n \geq 1. \end{cases} \tag{52}$$

Then $\{u_n(x,t)\}_{n=1}^{\infty}$ *converges strongly to* $u(x,t) \in (A+B)^{-1}(0)$, *which is the unique solution of the variational inequality:* $\forall z(x,t) \in (A+B)^{-1}0$.

$$\langle (M - \eta f)u(x,t), j_{\max\{p,p'\}}(u(x,t) - z(x,t))\rangle \leq 0. \tag{53}$$

where $u(x,t) = Q_{(A+B)^{-1}(0)}f(u(x,t))$, *and* $Q_{(A+B)^{-1}(0)}$ *is the unique sunny nonexpansive retraction of* E *onto* $(A+B)^{-1}(0)$.

Acknowledgement: The first and second author acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DST-NRF COE-MaSS) Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

References

[1] S. Banach, Surles ope'rations dans les ensembles abstraits et leur application aux e'quations inte'grales, *Fund. Math.* **3** (1922), 133–181.

[2] V. Barbu, *Nonlinear convergence semigroups and differential equations in Banach spaces*, Noordhoff, Groningen, (1991).

[3] G. Cai, C.S. Hu, Strong convergence theorems of a general iterative process for a finite family of λ_i-strict pseudo-contractions in q-uniformly smooth Banach spaces, *Comput. Math. Appl.* **59** (2010), no. 1, 149–160.

[4] C. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations*, vol. 1965 of Lectures Notes in Mathematics, Springer, London, UK, 2009.

[5] P. Cholamijiak, S. Suantai, Weak convergence theorems for a countable family of strictly pseudocontractions in Banach spaces, *Fixed Point Theory Appl.* **2010** (2010), Art. ID 632137.

[6] S.P. Han, G. Lou, A parallel for a class of convex programs, *SIAM J. Control Optim.* **26** (1988), 345–355.

[7] J.S. Jung, Iterative approaches to common fixed fixed points of nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **302** (2005), no. 2, 509–520.

[8] P. L. Lions, B. Mercier, Splitting algorithm for the sum of two nonlinear operators, *SIAM J. Numer. Anal.* **16** (1979), 964–979.

[9] B. Lemaire, Which fixed point does the iteration method select?: In *Recent Advances in optimization,* Springer, Berlin, Germany, **452** (1997), 154–157.

[10] A. Meir, E. Keeler, A theorem on contraction mappings, *J. Math. Anal. and Appl.* **28** (1969), 326–329.

[11] F. U. Ogbuisi, O. T. Mewomo, Iterative solution of split variational inclusion problem in real Banach space, *Afr. Mat.* **28** (2017), no. 1-2, 295–309.

[12] C.C. Okeke, A.U. Bello, C. Izuchukwu, O.T. Mewomo, Split equality for monotone inclusion problem and fixed point problem in real Banach spaces, *Aust. J. Math. Anal. Appl.* **14** (2017), no. 2, Art. 13.

[13] G.B. Passty, Ergodic convergence to a zero of the sum of monotone operators in Hilbert spaces, *J. Math. Anal. Appl.* **72** (1979), 383–390.

[14] X.L. Qin, S.Y. Cho, L. Wang, Convergence of a splitting algorithm for sum of two accretive operators with applications, *Fixed point Theory Appl.* **2014** (2014), 166.

[15] T. Suziki, Moudafi's viscosity approximations with Meir-Keeler contractions, *J. Math. Anal. Appl.* **325** (2007), no. 1, 342–352.

[16] Y. Song, C. Hu, Strong convergence theorems of a new general iterative process with Meir-Keeler contractions for a countable family of λ_i-strictly pseudocontractions in q-uniformly smooth Banach spaces, *Fixed Point Theory Appl.* **2010** (2010), Art. ID 354202, 19 pages.
[17] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.* **305** (2005), no. 1, 227–239.
[18] Y. Shehu, G. Cai, Strong convergence result for forward-backward splitting methods for accretive operators in Banach spaces with applications, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A. Mat. RASCAM* **112** (2018), no. 1, 71–87.
[19] W. Takahashi, *Nonlinear functional analysis-fixed point theory and applications*, Yokohama Publisher Inc., Yokohama (2000).
[20] L. Wei, R.P. Agrawal, P.J.Y. Wong, Discussion on the existence and uniqueness of solution to nonlinear integro-differential systems, *Comput. Math. Appl.* **69**, (2015), no. 5, 374–389.
[21] L. Wei, L. Duan, A new iterative algorithm for the sum of two different types of finitely many accretive operators in Banach spaces and its connection with capillarity equation, *Fixed Point Theory Appl.* **2015** (2015), 25.
[22] H.K. Xu, Iterative algorithms for nonlinear operators, *J. Lond. Math. Soc.* **66**, (2002), 1–17.
[23] Z.B. Xu, G.F. Roach, Characteristic inequalities of uniformly smooth Banach spaces, *J. Math. Anal. Appl.* **157**, (1991), 189–210.

(C.C. Okeke) School of Mathematics, Statistics and Computer Science, University of Kwazulu-Natal, Durban, South Africa
DST-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS)
E-mail address: 215082178@stu.ukzn.ac.za

(F.U. Ogbuisi) School of Mathematics, Statistics and Computer Science, University of Kwazulu-Natal, Durban, South Africa
DST-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS)
E-mail address: 215082189@stu.ukzn.ac.za

(O.T. Mewomo) School of Mathematics, Statistics and Computer Science, University of Kwazulu-Natal, Durban, South Africa
E-mail address: mewomoo@ukzn.ac.za