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On the minimal solution of bi-Laplacian equation

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Abstract. We study the existence of positive solution to the problem

$$\Delta^2 u - q\Delta u - \mu\alpha(x)u = h(u) + \lambda\beta(x) \quad \text{in } \Omega,$$
$$u > 0 \quad \text{in } \Omega,$$
$$u = 0 = \Delta u \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. We verify the existence of a value $\lambda_0 > 0$ such that when $0 < \lambda < \lambda_0$, then one can find a positive solution in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. As $\lambda \uparrow \lambda_0$, then u_λ of minimal positive solutions converge to a solution of the main problem but for λ_0.

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1. Introduction

This note deals with the semilinear elliptic PDE's with biharmonic operator,

$$\begin{aligned} \Delta^2 u - q\Delta u - \mu\alpha(x)u &= h(u) + \lambda\beta(x) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 = \Delta u \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Delta^2 u = \Delta(\Delta u)$, Ω is an arbitrary bounded domain in $\mathbb{R}^N$, $N \geq 5$. α, β and h are non-negative functions. $\alpha \in L^1_{\text{loc}}(\Omega)$, $\beta \in L^2(\Omega)$ and $\beta \not\equiv 0$. q, μ, and λ are positive constants. We suppose that

$$h : [0, \infty) \to [0, \infty) \text{ is a convex } C^1\text{−function with } h(0) = 0 = h'(0);\; h(t) \neq 0 (\forall t > 0) \tag{2}$$

and satisfying the growth known property:

$$\lim_{t\to\infty} \frac{h(t)}{t} = \infty, \tag{3}$$

(as an example $h(t) = t^2$ satisfies). Morevere, assume that

$$\int_1^\infty g(s)ds < \infty \quad \text{and } sg(s) < 1 \quad \text{for } s > 1, \tag{4}$$

where for $s \geq 1$,

$$g(s) := \sup_{t>0} \frac{h(t)}{h(ts)}. \tag{5}$$

Direct computation shows that g is a non-increasing and non-negative function. It is clearly that $t \to \frac{h(t)}{t}$ is increasing since h is convex and so $s \to sg(s)$ is non-increasing.

The convenience is endowed with the $W^{k,p}(\Omega)$ norm $\big(\int_\Omega \sum_{0\leq|a|\leq k} |D^a u|^p dx\big)^{1/p}$. Then from

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\int_\Omega |u|^p dx + \int_\Omega |D^k u|^p dx\Big)^{1/p}, \tag{6}$$

defines a norm which is equivalent to the usual norm in $W^{k,p}(\Omega)$ (see [1]). Since it is clear that

$$\|u\|_{W_0^{k,p}(\Omega)} = \Big(\int_\Omega |D^k u|^p dx\Big)^{1/p}, \tag{7}$$

Ω is a smooth bounded domain and $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W^{k,p}(\Omega)$, invoking [11, Theorem 2.2].

It is an equivalent norm to (6). From now on we consider $W_0^{k,p}(\Omega)$ endowed with the norm defined in (7). (see [11, 12]).

For $\alpha \in L^1_{\text{loc}}(\Omega)$, there exists a positive constant $\kappa > 0$ such that

$$\int_\Omega (|\Delta u|^2 + q|\nabla u|^2 - \alpha(x)^2 u^2)dx \geq \kappa \int_\Omega u^2 \quad \forall u \in C_0^\infty(\Omega). \tag{8}$$

The a same arguments in [5] implies that

$$\int_\Omega (|\Delta u|^2 + q|\nabla u|^2 - \alpha(x)^2 u^2)dx \geq \kappa \int_\Omega u^2 \quad \forall u \in W^{2,2} \cap W_0^{1,2}(\Omega). \tag{9}$$

Morevere assume that

$$0 < \mu < \sqrt{\kappa}. \tag{10}$$

Both (8) and (10) implies that

$$\mu \int_\Omega \alpha(x) u^2 dx \leq \mu\Big(\int_\Omega \alpha(x)^2 u^2 dx\Big)^{1/2}\Big(\int_\Omega u^2 dx\Big)^{1/2} \leq \frac{\mu}{\sqrt{\kappa}} \int_\Omega (|\Delta u|^2 + q|\nabla u|^2 dx) \tag{11}$$

for all $u \in C_0^\infty(\Omega)$. Then

$$\|u\|_H^2 := \int_\Omega [|\Delta u|^2 + q|\nabla u|^2 - \mu\alpha(x)u^2]dx,$$

introduce a new norm in $C_0^\infty(\Omega)$ and completion of $C_0^\infty(\Omega)$ with respect to this norm yields the Hilbert space H. This norm $\|u\|_H$ is equivalent to $\|u\|_{W_0^{2,2}(\Omega)}$ by (11), (10) and (7). Morevere, from (11), the equivalence of this norm and Poincare inequality implies the existence $\tilde{\kappa} > 0$ in which

$$\int_\Omega (|\Delta u|^2 + q|\nabla u|^2 - \mu\alpha(x)u^2)dx \geq \tilde{\kappa} \int_\Omega u^2 dx \quad \forall \ \ (u \in)C_0^\infty(\Omega).$$

The standard density argument and Fatou's lemma implies that

$$\int_\Omega (|\Delta u|^2 + q|\nabla u|^2 - \mu\alpha(x)u^2)dx \geq \tilde{\kappa} \int_\Omega u^2 dx \quad \forall (u \in W^{2,2} \cap W_0^{1,2}(\Omega)). \tag{12}$$

This inequality shows that the first eigenvalue of $\Delta^2 - q\Delta - \mu\alpha(x)$ is strictly positive.

Definition 1.1. $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is called a positive solution of (1) if $u > 0$ a.e., $h(u) \in L^2(\Omega)$ and u satisfies

$$\int_\Omega (\Delta u \Delta \psi + q \nabla u \nabla \psi - \mu\alpha(x) u \psi) dx = \int_\Omega (h(u) + \lambda\beta(x))\psi \, dx \quad \forall \psi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Similarly $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is called a supersolution (subsolution) if $h(u) \in L^2(\Omega)$ and for all positive $\psi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$,

$$\int_\Omega (\Delta u \Delta \psi + q \nabla u \nabla \psi - \mu\alpha(x) u \psi) dx \geq (\leq) \int_\Omega (h(u) + \lambda\beta(x))\psi \; dx.$$

Definition 1.2. $u \in L^1(\Omega)$ is called positive distributional solution or very weak solution of (1) if $u > 0$ a.e., $\mu\alpha(x)u + h(u) \in L^1_{\text{loc}}(\Omega)$ and u satisfies (1) in the distributional sense, i.e.,

$$\int_\Omega u(\Delta^2 \psi + q \nabla^2 \psi - \mu\alpha(x)\psi) dx = \int_\Omega (h(u) + \lambda\beta(x))\psi \; dx \quad \forall \psi \in C_0^\infty(\Omega). \tag{13}$$

Definition 1.3. $u \in L^1(\Omega)$ is called weak supersolution (subsolution) for

$$\Delta^2 u - q\Delta u = g(x, u) \quad \text{in } \Omega,$$

in the sense of distribution if $g(x, u) \in L^1(\Omega)$ and for all positive $\psi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega u(\Delta^2 \psi + q \nabla^2 \psi) dx \geq (\leq) \int_\Omega g(x, u)\psi dx.$$

u is called a distributional solution if weak supersolution and as well a weak subsolution in the sense of distribution.

Similar type of this problem was studied by Bhakta in [5] with lack of second sentence i.e. $-q\Delta u$. Problems like this have been studied by many researchers while in its general form it has widely been studied by Dupaigne and Nedev in [9]. In [9], the authors have proved a mandatory and adequate condition for the existence of L^1 solution and they have also established an estimate from above and below for the solution. We also refer [**?**, 6, 8] (and the references therein) for the related problems in the second order case.

Problems of higher order are relatively different from those of second order case. In this case several technical difficulties occurred due to lack of the maximum principle. So, till date the knowledge on higher order nonlinear problems is incomplete, in contrast with the second-order case. In the case of fourth-order problem Navier boundary conditions have the key role to prove existence results as under this boundary condition, equation with biLaplacian operator can be rewritten as a second order system with Dirichlet boundary value problems. Then the Maximum Principle can be easily proved by using classical elliptic theory. As a result, a Comparison Principle which plays as one of the most important parameters in proving existence results can be deduced.

Literature survey revealed that many research groups in recent years have deal with $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ solution of semilinear elliptic and parabolic problem with biLaplacian operator and some specific nonlinearities. In this text we have referred to some of them [2, 4, 7, 10] (also see the references therein). Semilinear elliptic equations with biharmonic operator can be applied in continuum mechanics, bio- physics and differential geometry. Particularly, in the modeling of thin elastic plates, clamped plates and in the study of the Paneitz-Branson equation and the Willmore equation (see[11]).

2. Preliminary lemmas

Lemma 2.1 (Strong Maximum Principle)**.** *Suppose that u is a nontrivial supersolution of*

$$\begin{aligned} \Delta^2 u - q\Delta u = 0 \quad &\text{in } \Omega, \\ u = 0 = \Delta u \quad &\text{on } \partial\Omega. \end{aligned} \tag{14}$$

Then $-\Delta u > 0$ and $u > 0$ in Ω.

Considering the change of variables $-\Delta u = v$, if u is a supersolution to above problem(14), then v is a supersolution to

$$\begin{aligned} -\Delta v + qv = 0 \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{15}$$

[see [3] Theorem 1.7.9]. Applying the known Strong Maximum Principle to the Laplacian operator, it immediately follows that $v > 0$ in Ω and then $u > 0$ inΩ.

Lemma 2.2 (Comparison Principle)**.** *Assume that u and v satisfy the following:*

$$\begin{aligned} \Delta^2 u - q\Delta u \geq \Delta^2 v - q\Delta v \quad &\text{in } \Omega, \\ u \geq v \quad &\text{on } \partial\Omega, \\ -\Delta u \geq -\Delta v \quad &\text{on } \partial\Omega. \end{aligned} \tag{16}$$

Then, $-\Delta u \geq -\Delta v$ and $u \geq v$ in Ω.

It is sufficient to apply to $w = u - v$, a supersolution to (15), the previous Strong Maximum Principle. see [12, Lemma 3.3].

Lemma 2.3 (Weak Harnack Principle [12, Lemma 3.4])**.** *Suppose that u a positive distributional supersolution to* (15)*. Then for any $B_R(x_0) \Subset \Omega$, there exists a positive constant $C = C(\theta, \rho, m, R)$,*

$$\|u\|_{L^m(B_{\rho R}(x_0))} \leq C \operatorname{ess\,inf}_{B_{\theta R}(x_0)} u,$$

where $0 < m < \frac{N}{N-2}$, $0 < \theta < \rho < 1$.

Lemma 2.4. *The problem*

$$\begin{aligned} \Delta^2 u - q\Delta u - \mu\alpha(x)u = \beta \quad &\text{in } \Omega, \\ u = 0 = \Delta u \quad &\text{on } \partial\Omega, \end{aligned} \tag{17}$$

has a positive solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ where $\alpha \in L^1_{\text{loc}}(\Omega)$, $\beta \in L^2(\Omega)$, $\alpha, \beta \geq 0$ a.e., $\beta \not\equiv 0$, q, μ are positive constant satisfying (10) *and α satisfy* (9)*.*

Proof. For $\beta \in L^2(\Omega)$, there exists a unique weak solution $u_1 \in W^{2,2} \cap W_0^{1,2}(\Omega)$ in which:

$$\begin{aligned} \Delta^2 u_1 - q\Delta u_1 = \beta \quad &\text{in } \Omega, \\ u_1 = 0 = \Delta u_1 \quad &\text{on } \partial\Omega. \end{aligned}$$

[see [3] Theorem 1.6.1]. Strong maximum principle (Lemma 2.1) implies that $u_1 > 0$. Define u_n $(n \geq 2)$ in which satisfy

$$\begin{aligned} \Delta^2 u_n - q\Delta u_n &= \mu\alpha(x)u_{n-1} + \beta \quad \text{in } \Omega, \\ u_n &= 0 = \Delta u_n \quad \text{on } \partial\Omega. \end{aligned} \tag{18}$$

By (9), $\mu\alpha(x)u_{n-1} \in L^2(\Omega)$. Comparison principle implies that $0 < u_1 \leq \cdots \leq u_{n-1} \leq u_n \leq \ldots$.

We claim that $\{u_n\}$ is a Cauchy sequence in $W^{2,2} \cap W_0^{1,2}(\Omega)$.

In fact $\Delta^2(u_{n+1}-u_n)-q\Delta(u_{n+1}-u_n) = \mu\alpha(x)(u_n-u_{n-1})$. Considering $(u_{n+1}-u_n)$ as a test function and using (9)

$$\begin{aligned} \int_\Omega (|\Delta(u_{n+1} - u_n)|^2 + q|\nabla(u_{n+1} - u_n)|^2)dx &= \mu \int_\Omega \alpha(x)(u_n - u_{n-1})(u_{n+1} - u_n)dx \\ &\leq \mu(\int_\Omega \alpha(x)^2(u_n - u_{n-1})^2dx)^{1/2}(\int_\Omega (u_{n+1} - u_n)^2dx)^{1/2} \\ &\leq \frac{\mu}{\sqrt{\kappa}}(\int_\Omega (|\Delta(u_{n+1} - u_n)|^2 + q|\nabla(u_{n+1} - u_n)|^2)dx)^{\frac{1}{2}} \\ &\quad \times (\int_\Omega (|\Delta(u_n - u_{n-1})|^2 + q|\nabla(u_n - u_{n-1})|^2)dx)^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_\Omega (|\Delta(u_{n+1} - u_n)|^2 + q|\nabla(u_{n+1} - u_n)|^2)dx \\ &\qquad \leq \frac{\mu}{\sqrt{\kappa}} \int_\Omega (|\Delta(u_n - u_{n-1})|^2 + q|\nabla(u_n - u_{n-1})|^2)dx \\ &\qquad \leq \cdots \leq (\frac{\mu}{\sqrt{\kappa}})^{n-1} \int_\Omega (|\Delta(u_2 - u_1)|^2 + q|\nabla(u_2 - u_1)|^2)dx. \end{aligned}$$

Then $\{u_n\}$ is a Cauchy sequence in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ since $\mu < \sqrt{\kappa}$.

Completeness of spacies implies existence $u \in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ such that $u_n \to u$ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. $u > 0$ because $u_n > u_1 > 0$ for all $n \geq 1$. As $u_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ solves (18), we have

$$\int_\Omega (\Delta u_n \Delta\psi + q\nabla u \nabla\psi)dx = \mu \int_\Omega \alpha(x)u_{n-1}\psi dx + \int_\Omega \beta\psi dx \quad \forall \psi \in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega).$$

Taking the limit as $n \to \infty$, we obtain u is a solution to (17). □

Lemma 2.5. *The equation* (1) *has a unique solution* $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ *to* (1) *which satisfies* $0 \leq u \leq \tilde{w}$ *for any supersolution* $\tilde{w} \geq 0$ *of* (1) *(respectively for* (17)*), where* $\alpha \in L^1_{\rm loc}(\Omega)$, $\beta \in L^2(\Omega)$, $h : [0,\infty) \to [0,\infty)$ *(h convex) be non-negative functions. Let* $q, \mu, \lambda > 0$, $\mu < \sqrt{\kappa}$. *Suppose that there exists a non-negative supersolution*

This u is called the *minimal nonnegative solution* of (1) (respectively for (17)). Strong maximum principle implies that $u > 0$ in Ω.

Remark 2.1. We denote the minimal positive solution of (17) by η_1 this allows us to define $G(\beta) = \eta_1$. The function $0 < u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ solving (1) (respectively (17)) is also the distributional sense solutions (see definition (1.2)).

Proof of Lemma 2.5. Proof of (1) and (17) are similar, so we do for (1). First, we show uniqueness. Let u_1 and u_2 are two solutions which satisfy $0 \le u_i \le \tilde{w}, (i = 1, 2)$ for every non-negative supersolution $\tilde{w}$. Then $u_1 \le u_2$ and $u_2 \le u_1$. So $u_1 = u_2$.

Now we show the existence of solution. Suppose that $\tilde{u} \ge 0$ is a supersolution to (1) and $u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a positive solution of

$$\begin{aligned} \Delta^2 u_0 - q\Delta u_0 &= \lambda\beta \quad \text{in } \Omega, \\ u_0 = 0 &= \Delta u_0 \quad \text{on } \partial\Omega. \end{aligned}$$

By comparison principle $0 < u_0 \le \tilde{u}$ in Ω. Using iteration method, there exists $u_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ for $n = 1, 2, \ldots$ in which u_n solves the problem

$$\begin{aligned} \Delta^2 u_n - q\Delta u_n &= \mu\alpha(x)u_{n-1} + h(u_{n-1}) + \lambda\beta(x) \quad \text{in } \Omega, \\ u_n = 0 &= \Delta u_n \quad \text{on } \partial\Omega. \end{aligned} \tag{19}$$

Since $\tilde{u}$ is a weak supersolution to (1), we have $h(\tilde{u}) \in L^2(\Omega)$. $h(u_0) \le h(\tilde{u})$ since $0 < u_0 \le \tilde{u}$ and h is convex (thus h is nondecreasing), we obtain . Moreover, $h(u_0) + \lambda\beta(x) \in L^2(\Omega)$ And by (9) so $\mu\alpha(x)u_0 \in L^2(\Omega)$. Choosing u_0 in the right hand then there is a solution u_1 . Comparison principle implies that $0 < u_0 \le u_1 \le \tilde{u}$. Using the induction method, u_n is well defined and $0 < u_0 \le u_1 \le \cdots \le u_n \le \cdots \le \tilde{u}$.

We claim that $\{u_n\}$ is uniformly bounded in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

From (19)

$$\begin{aligned} \int_\Omega (|\Delta u_n|^2 + q|\nabla u_n|^2)dx &= \int_\Omega (\mu\alpha(x)u_{n-1} + h(u_{n-1}) + \lambda\beta(x))u_n dx \\ &\le \int_\Omega (\mu\alpha(x)\tilde{u}^2 + h(\tilde{u})\tilde{u} + \lambda\beta\tilde{u})dx \\ &\le \left[\mu|\alpha(x)\tilde{u}|_{L^2(\Omega)} + |h(\tilde{u})|_{L^2(\Omega)} + \lambda|\beta|_{L^2(\Omega)}\right]|\tilde{u}|_{L^2(\Omega)} \le C. \end{aligned}$$

There is $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ such that up to a subsequence $u_n \rightharpoonup u$ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$. From (19)

$$\int_\Omega (\Delta u_n \Delta\psi + q\nabla u\nabla\psi)dx = \int_\Omega [\mu\alpha(x)u_{n-1} + h(u_{n-1}) + \lambda\beta]\psi dx, \ \forall\psi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Using Vitaly's convergence theorem and passing limit $n \to \infty$, u is a solution to (1). $u > 0$ since $u_n > u_0 > 0$ for all $n \ge 1$.

If $\tilde{w}$ is another supersolution, then by comparison principle $u_0 \le \tilde{w}$ and $u_n \le \tilde{w}$ for every $n \ge 1$. Taking the limit $n \to \infty$, $u \le \tilde{w}$. □

3. Existence results

Theorem 3.1. *Let $\alpha \in L^1_{\text{loc}}(\Omega)$, $0 \not\equiv \beta \in L^2(\Omega)$, α, β, h be non-negative functions, q is a positive constant, (9), (10), (2), (3), (4) and (5) are satisfied. Suppose that $G = (\Delta^2 - q\Delta - \mu\alpha(x))^{-1}$ and $\eta_1 = G(\beta)$, as proved in Lemma 2.4 (also see Remark 2.1). and there exists constants $\epsilon > 0$ and $C > 0$ in which*

$$h(\epsilon\eta_1) \in L^2(\Omega) \quad \text{and} \quad G(h(\epsilon\eta_1)) \le C\eta_1 \quad \text{a.e.} \tag{20}$$

Then there is $0 < \lambda_0 = \lambda_0(N, \alpha(x), \beta(x), h, \mu)$ in which for $\lambda < \lambda_0$, (1) has a minimal positive solution $u_\lambda \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $u_\lambda \ge \lambda\eta_1$.

If $\lambda > \lambda_0$ (1) has no positive solution in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

For any small $\lambda > 0$

$$\lambda\eta_1 \leq u_\lambda \leq 2\lambda\eta_1.$$

For the first time (20) is motivated from a results of Dupaigne and Nedev (see [9, Theorem 1]).

Lemma 3.2. *Suppose that* α, β *and* μ *satisfy the assumptions in Theorem 3.1,* $\eta_1 = G(\beta)$ *as in theorem 3.1 and* (2) *is satisfied. If*

$$h(2\eta_1) \in L^2(\Omega) \quad \textit{and} \quad G(h(2\eta_1)) \leq \eta_1,$$

then (1) *admits solution* $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ *for* $\lambda = 1$.

Proof. For $h(2\eta_1) \in L^2(\Omega)$ and $G(h(2\eta_1)) \leq \eta_1$. Set $v := G(h(2\eta_1)) + \eta_1$. Clearly $v > 0$ and $v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ since η_1 and $G(h(2\eta_1))$ are in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ (by Lemma 2.4). morevere,

$$v - \eta_1 = G(h(2\eta_1)), \quad v \leq 2\eta_1, \quad h(v) \in L^2(\Omega).$$

Then

$$\Delta^2(v - \eta_1) - q\Delta(v - \eta_1) - \mu\alpha(x)(v - \eta_1) = h(2\eta_1) \quad \text{in } \Omega.$$

Therefore,

$$\Delta^2 v - q\Delta v - \mu\alpha(x)v = h(2\eta_1) + \beta \geq h(v) + \beta \quad \text{in } \Omega$$

and $v = 0 = \Delta v$ on $\partial\Omega$. This shows that v is a positive supersolution of (1) but fir $\lambda = 1$. Applying Lemma 2.5 we conclude the existence of minimal positive solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ of (1) with $\lambda = 1$. □

Proposition 3.3. *Let* $(P_{\tilde{\lambda}})$ *has a positive solution* $u_{\tilde{\lambda}} \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ *for* $\tilde{\lambda} > 0$. *Then for any* $0 < \lambda < \tilde{\lambda}$, (1) *has a solution in* $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

Proof. Suppose that $u_{\tilde{\lambda}} \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a positive solution for (1) with $\tilde{\lambda}$ instead of λ. From definition (see Definition 1.1) $h(u_{\tilde{\lambda}}) \in L^2(\Omega)$. Set $v := \tilde{\lambda}\eta_1$,

$$\Delta^2\left(\frac{u_{\tilde{\lambda}}}{\tilde{\lambda}}\right) - q\Delta\left(\frac{u_{\tilde{\lambda}}}{\tilde{\lambda}}\right) - \mu\alpha(x)\left(\frac{u_{\tilde{\lambda}}}{\tilde{\lambda}}\right) = \frac{1}{\tilde{\lambda}}(h(u_{\tilde{\lambda}}) + \tilde{\lambda}\beta) = \frac{h(u_{\tilde{\lambda}})}{\tilde{\lambda}} + \beta \geq \beta \quad \text{in } \Omega.$$

Then $\frac{u_{\tilde{\lambda}}}{\tilde{\lambda}}$ is a positive supersolution to (17). Minimality of η_1 implies that $\eta_1 \leq \frac{u_{\tilde{\lambda}}}{\tilde{\lambda}}$ and so that $v \leq u_{\tilde{\lambda}}$. For $0 < \lambda < \tilde{\lambda}$ define $w := u_{\tilde{\lambda}} - v + \lambda\eta_1$. Clearly $w > 0$ and v $w \leq u_{\tilde{\lambda}}$. Convexity of h, implies that $\frac{h(t)}{t}$ is increasing and h is non-decreasing. Then $h(w) \leq h(u_{\tilde{\lambda}})$ and so $h(w) \in L^2(\Omega)$. Also,

$$\Delta^2 w - q\Delta w - \mu\alpha(x)w = h(u_{\tilde{\lambda}}) + \tilde{\lambda}\beta - (\tilde{\lambda} - \lambda)\beta = h(u_{\tilde{\lambda}}) + \lambda\beta \geq h(w) + \lambda\beta.$$

Where $w \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a positive supersolution to (1). From Lemma 2.5, there is minimal positive solution for (1). □

Proof of Theorem 3.1. We assume (20) holds.

Step I: If $\lambda > 0$ is small then (1) has a positive a solution $u_\lambda \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. We show in the spirit of [9]. Lemma 3.2, follows that (1) has a solution if

$$h(2\lambda\eta_1) \in L^2(\Omega) \quad \text{and} \quad G(h(2\lambda\eta_1)) \leq \lambda\eta_1. \tag{21}$$

$g(\frac{\epsilon}{2\lambda}) \geq \frac{h(t)}{h(t\frac{\epsilon}{2\lambda})}$ for all $t > 0$. Set $t := 2\lambda\eta_1$, $h(2\lambda\eta_1) \leq h(\epsilon\eta_1)g(\frac{\epsilon}{2\lambda})$. Applying (20), $h(2\lambda\eta_1) \in L^2(\Omega)$ and $G(h(2\lambda\eta_1))$, minimality of $G(h(2\lambda\eta_1))$ and assumption (20) implies that

$$G(h(2\lambda\eta_1)) \leq g(\frac{\epsilon}{2\lambda})G(h(\epsilon\eta_1)) \leq Cg(\frac{\epsilon}{2\lambda})\eta_1.$$

To verify (21) for small $\lambda > 0$, it is enough to check that

$$\lim_{\lambda\to 0}\frac{1}{\lambda}g(\frac{\epsilon}{2\lambda}) = 0 \quad \text{or equivalently} \quad \lim_{K\to\infty} Kg(K) = 0.$$

$s \to sg(s)$ is non-increasing so this limit valied, there is $C' \geq 0$ such that $\lim_{K\to\infty} Kg(K) = C'$. If $C' > 0$, then $g(K) \sim \frac{C}{K}$ near ∞ and this contradicts (4). Hence, $C' = 0$ and (21) holds for $\lambda > 0$ small.
Step II: Define,

$$\Lambda = \{\lambda > 0 : (P_\lambda) \text{ has a minimal positive solution } u_\lambda\},$$

From Step I and Proposition 3.3, Λ is a non-empty interval. define,

$$\lambda_0 = \sup \Lambda.$$

It is direct that, if $\lambda < \lambda_0$, (1) has a minimal positive solution and for $\lambda > \lambda_0$, (1) does not have any positive solution in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.
Step III: From $G(\beta) = \eta_1$, $G(\lambda\beta) = \lambda\eta_1$. If $\lambda < \lambda_0$ and u_λ denotes the corresponding minimal positive solution of (1), then u_λ is a supersolution to the equation satisfied and by minimality of $\lambda\eta_1$,

$$u_\lambda \geq \lambda\eta_1. \tag{22}$$

Step IV: We claim that if $\lambda > 0$ is small, then

$$\lambda\eta_1 \leq u_\lambda \leq 2\lambda\eta_1.$$

Since $\lambda > 0$ is small so (21) holds . For, $w := G(h(2\lambda\eta_1)) + \lambda\eta_1$.

$$w \leq 2\lambda\eta_1 \quad \text{and} \quad w - \lambda\eta_1 = G(h(2\lambda\eta_1)).$$

Similar to proof of Lemma 3.2, $w \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a positive supersolution of (1) and $u_\lambda \leq w \leq 2\lambda\eta_1$. This inequality and (22) implies that $\lambda\eta_1 \leq u_\lambda \leq 2\lambda\eta_1$. $\square$

For the next result we set

$$u^*(x) := \lim_{\lambda\uparrow\lambda_0} u_\lambda(x), \quad x \in \Omega. \tag{23}$$

Theorem 3.4. *Let assumptions in Theorem 3.1 satisfied, u_λ denotes the minimal positive solution of* (1) *for $0 < \lambda < \lambda_0$ and u^* be as* (23). *Morevere,*

$$\lim_{s\to\infty}\frac{sh'(s)}{h(s)} > 1. \tag{24}$$

Then $u^* \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ a solution to (1) *for λ_0 instead of λ. Moreover, $u_\lambda \to u^*$ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.*

Remark 3.1. (24) is a mild assumption, since h is convex and C^1. It is direct that if $h \in C^2$ and strictly convex, then (24) is obvious.

Proof of Theorem 3.4. Since u_λ is a solution of (1)

$$\int_\Omega (\Delta u_\lambda \Delta v + q\nabla u_\lambda \nabla v) = \mu \int_\Omega \alpha(x) u_\lambda v + \int_\Omega h(u_\lambda) v + \lambda \int_\Omega \beta(x) v \quad \forall v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega). \tag{25}$$

On the other hand $\int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2 - \mu\alpha(x)u_\lambda^2 - h'(u_\lambda)u_\lambda^2)dx \geq 0$ (see[5, Theorem 5.2]. Taking $v = u_\lambda$ in (25)

$$\int_\Omega h'(u_\lambda)u_\lambda^2 dx \leq \int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2 - \mu\alpha(x)u_\lambda^2)dx = \int_\Omega (h(u_\lambda)u_\lambda + \lambda\beta(x)u_\lambda)dx. \tag{26}$$

Using (24), for $\epsilon > 0$ there exists $C > 0$ such that

$$(1+\epsilon)h(s)s \leq h'(s)s^2 + C \quad \forall s \geq 0. \tag{27}$$

From (26) and (27)

$$(1+\epsilon)\int_\Omega (h'(u_\lambda)u_\lambda^2 - \lambda\beta(x)u_\lambda)dx \leq (1+\epsilon)\int_\Omega h(u_\lambda)u_\lambda dx \leq \int_\Omega (h'(u_\lambda)u_\lambda^2 + C)dx.$$

Then

$$\epsilon \int_\Omega h'(u_\lambda)u_\lambda^2 dx \leq C|\Omega| + (1+\epsilon)\lambda \int_\Omega \beta u_\lambda dx,$$

so

$$\int_\Omega h(u_\lambda)u_\lambda dx \leq C_1 + C_2\lambda \int_\Omega \beta u_\lambda dx, \tag{28}$$

for some constants $C_1, C_2 > 0$. From $\lambda < \lambda_0$, applying Holder inequality and (28)

$$\begin{aligned}\int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2)dx &= \mu\int_\Omega \alpha(x)u_\lambda^2 + \int_\Omega h(u_\lambda)u_\lambda + \lambda\int_\Omega bu_\lambda \\ &\leq \mu|\alpha(x)u_\lambda|_{L^2(\Omega)}|u_\lambda|_{L^2(\Omega)} + \lambda_0(1+C_2)\int_\Omega bu_\lambda dx + C_1.\end{aligned}$$

(9) and Cauchy-Schwartz inequality with $\delta > 0$ on the above estimate implies that

$$\begin{aligned}\int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2)dx &\leq \frac{\mu}{\sqrt{\kappa}}\int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2)dx + C_3|\beta|_{L^2(\Omega)}|u_\lambda|_{L^2(\Omega)} + C_1 \\ &\leq \frac{\mu}{\sqrt{\kappa}}\int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2)dx + \frac{C_3}{\sqrt{\kappa}}|\beta|_{L^2(\Omega)}|\Delta u_\lambda|_{L^2(\Omega)} + C_1 \\ &\leq \frac{\mu}{\sqrt{\kappa}}\int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2)dx + \delta|\Delta u_\lambda|^2_{L^2(\Omega)} + c(\delta)|\beta|^2_{L^2(\Omega)} + C_1 \\ &\leq \frac{\mu}{\sqrt{\kappa}}\int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2)dx + \delta\int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2)dx \\ &+ c(\delta)|\beta|^2_{L^2(\Omega)} + C_1.\end{aligned}$$

Since $\mu < \sqrt{\kappa}$ (by (10)), there is $\delta > 0$ in which $\frac{\mu}{\sqrt{\kappa}} + \delta < 1$. From this estimate

$$\int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2)dx \leq C_4|\beta|^2_{L^2(\Omega)} + C_1 \leq C',$$

for some constant $C' > 0$. Therefore, $\{u_\lambda\}$ is uniformly bounded in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ for $\lambda < \lambda_0$. From (23), $u_\lambda \rightharpoonup u^*$ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Passing to the limit $\lambda \to \lambda_0$ in (25), via Lebesgue monotone convergence theorem, it is easy to check that u^* is a solution to (1) with λ_0 instead of λ. Limiting $\lambda \to \lambda_0$, using monotone convergence theorem

$$\begin{aligned}
\|u_\lambda\|^2_{W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)} &= \int_\Omega (|\Delta u_\lambda|^2 + q|\nabla u_\lambda|^2)dx \\
&= \mu \int_\Omega \alpha(x) u_\lambda^2 + \int_\Omega h(u_\lambda)u_\lambda + \lambda \int_\Omega b u_\lambda \\
&\to \mu \int_\Omega \alpha(x) {u^*}^2 + \int_\Omega h(u^*)u^* + \lambda_0 \int_\Omega b u^* \\
&= \int_\Omega (|\Delta u^*|^2 + q|\nabla u^*|^2)dx = \|u^*\|^2_{W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)}.
\end{aligned}$$

Thus $\|u_\lambda\|_{W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)} \to \|u^*\|_{W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)}$. Combining this along with the weak convergence, $u_\lambda \to u^*$ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. □

References

[1] V. Adolfsson, L^2-integrability of second-order derivatives for Poisson's equation in nonsmooth domains, *Math. Scand.* **70** (1992), no. 1, 146–160.

[2] G. Arioli, F. Gazzola, H-C. Grunau, E. Mitidieri, A semilinear fourth order elliptic problem with exponential nonlinearity, *SIAM J. Math. Anal.* **36** (2005), no. 4, 1226–1258.

[3] M. Badiale, E. Serra, *Semilinear Elliptic Equations for Beginners: Existence Results via the Variational Approach*, Springer Science and Business Media, 2010.

[4] E. Berchio, F. Gazzola, Some remarks on biharmonic elliptic problems with positive, increasing and convex nonlinearities, *Electron. J. Differential Equations* **2005** (2005), no. 34, 1–20.

[5] M. Bhakta, Solutions to semilinear elliptic PDE's with biharmonic operator and singular potential, *Electron. J. Differential Equations* 2016 (2016), no. 261, 1–17.

[6] H. Brezis, J.L. Vázquez, Blow-up solutions of some nonlinear elliptic problems, *Rev. Mat. Univ. Complut. Madrid* **10** (1997), no. 2, 443–469.

[7] J. Dávila, L. Dupaigne, I. Guerra, M. Montenegro; Stable solutions for the bilaplacian with exponential nonlinearity, *SIAM J. Math. Anal.* **39** (2007), no. 2, 565–592.

[8] L. Dupaigne, A nonlinear elliptic PDE with the inverse square potential, *J. Anal. Math.* **86** (2002), 359–398.

[9] L. Dupaigne, G. Nedev, Semilinear elliptic PDE's with a singular potential, *Adv. Differential Equations* **7** (2002), no. 8, 973–1002.

[10] C. Escudero, F. Gazzola, I. Peral, Global existence versus blow-up results for a fourth order parabolic PDE involving the Hessian, *J. Math. Pures Appl.* **103** (2015), 924–957.

[11] F. Gazzola, H. C. Grunau, G. Sweers, *Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2010.

[12] M. Pérez-Llanos, A. Primo, Semilinear biharmonic problems with a singular term, *J. Differential Equations* **257** (2014), no. 9, 3200–3225.

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