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On lacunary statistical convergence of sequences in gradual normed linear spaces

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Abstract. In this paper, we introduce and investigate the notion of lacunary statistical convergence of sequences in gradual normed linear spaces. We study some of its basic properties and some inclusion relations. In the end, we introduce the notion of lacunary statistical Cauchy sequences and prove that it is equivalent to the notion of lacunary statistical convergence.

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1. Introduction

The idea of fuzzy sets [30] was first introduced by Zadeh in the year 1965 which was an extension of the classical set-theoretical concept. Nowadays it has wide applicability in different branches of science and engineering. The term "fuzzy number" plays a crucial role in the study of fuzzy set theory. Fuzzy numbers were basically the generalization of intervals, not numbers. Even fuzzy numbers do not obey a few algebraic properties of the classical numbers. So the term "fuzzy number" is debatable to many authors due to its different behavior. The term "fuzzy intervals" is often used by many authors instead of fuzzy numbers. To overcome the confusion among the researchers, in 2008, Fortin et.al. [8] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are mainly known by their respective assignment function which is defined in the interval $(0, 1]$. So in some sense, every real number can be viewed as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and have uses in computation and optimization problems.

In 2011, Sadeqi and Azari [23] first introduced the concept of gradual normed linear space. They studied various properties of the space from both the algebraic and topological point of view. Further progress in this direction has been occurred due to Ettefagh, Azari, and Etemad (see [6], [7]) and many others. For extensive study on gradual real numbers one may refer to ([1], [4], [18], [27]).

On the other hand, in 1993, Fridy [14] introduced the concept of lacunary statistical convergence mainly as one of the extensions of statistical convergence (for more details on statistical convergence, one may refer [9], [10], [11], [12]).

A lacunary sequence is an increasing integer sequence $\theta = (k_n)_{n\in\mathbb{N}\cup\{0\}}$ satisfying $k_0 = 0$ and $h_n = k_n - k_{n-1} \to \infty$, as $n \to \infty$.

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A real valued sequence (x_k) is said to be lacunary statistically convergent (in short $S_\theta-$convergent) to a real number l, if for any $\varepsilon > 0$,

$$\lim_n \frac{1}{h_n} \mid \{k \in I_n : |x_k - l| \geq \varepsilon\} \mid = 0,$$

where $I_n = (k_{n-1}, k_n]$. In this case, l is called the lacunary statistical limit of the sequence (x_k) and symbolically, it is written as $S_\theta - \lim(x_k) = l$ or $x_k \to l(S_\theta)$. Further, the set of all lacunary statistical convergent sequences with regard to the lacunary sequence θ is denoted by S_θ. In [14], Fridy and Orhan established the relation between statistical convergence and lacunary statistical convergence. In [13], Fridy introduced the notion of lacunary statistically Cauchy sequences and in [10], Freedman et. al. studied the relationship between the two sequence spaces $|\sigma_1|$ and N_θ defined as follows:

$$|\sigma_1| = \left\{(x_k) : \text{for some } l \in \mathbb{R}, \lim_n \frac{1}{n}\left(\sum_{k=1}^{n} |x_k - l|\right) = 0\right\}$$

$$\text{and} \quad N_\theta = \left\{(x_k) : \text{for some } l \in \mathbb{R}, \lim_n \frac{1}{h_n}\left(\sum_{k \in I_n} |x_k - l|\right) = 0\right\}.$$

For more information about of lacunary convergence and its generalizations, [5], [15], [16], [17], [19], [20], [21], [22], [24], [25], [26], [28], [29] can be addressed where many more references can be found.

Research on the convergence of sequences in gradual normed linear spaces has not yet gained much ground and it is still in its infant stage. The research carried out so far shows a strong analogy in the behavior of convergence of sequences in gradual normed linear spaces (for details one may refer to [6], [7], [23]).

Recently, the convergence of sequences in gradual normed linear spaces was introduced by Ettefagh et. al. [7]. Also, they have investigated some properties from the topological point of view [6]. In [2], Choudhury and Debnath have generalized the notion of convergence of sequences in the gradual normed linear spaces to ideal convergence. Therefore, the study of lacunary statistical convergence of sequences in gradual normed linear spaces is very natural.

2. Definitions and preliminaries

Definition 2.1. [8] A gradual real number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}} : (0, 1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual real number $\tilde{r}$ is said to be non-negative if for every $\xi \in (0, 1]$, $A_{\tilde{r}}(\xi) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.

In [8], the gradual operations between the elements of $G(\mathbb{R})$ was defined as follows:

Definition 2.2. Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$ respectively. Then $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $A_{\tilde{r}_1 * \tilde{r}_2}$ given by $A_{\tilde{r}_1 * \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) * A_{\tilde{r}_2}(\xi), \forall \xi \in (0, 1]$. Then the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r}(c \in \mathbb{R})$ are defined by

$$A_{\tilde{r}_1 + \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) + A_{\tilde{r}_2}(\xi) \quad \textit{and} \quad A_{c\tilde{r}}(\xi) = cA_{\tilde{r}}(\xi), \ \forall \xi \in (0, 1].$$

For any real number $p \in \mathbb{R}$, the constant gradual real number $\tilde{p}$ is defined by the constant assignment function $A_{\tilde{p}}(\xi) = p$ for any $\xi \in (0, 1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are the constant gradual numbers defined by $A_{\tilde{0}}(\xi) = 0$ and $A_{\tilde{1}}(\xi) = 1$ respectively. One can easily verify that $G(\mathbb{R})$ with the gradual addition and gradual scalar multiplication forms a real vector space [8].

Definition 2.3. [23] Let X be a real vector space. The function $||\cdot||_G : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X, if for every $\xi \in (0, 1]$, following three conditions are true for any $x, y \in X$:
(G_1) $A_{||x||_G}(\xi) = A_{\tilde{0}}(\xi)$ if and only if $x = 0$;
(G_2) $A_{||\lambda x||_G}(\xi) = |\lambda| A_{||x||_G}(\xi)$ for any $\lambda \in \mathbb{R}$;
(G_3) $A_{||x+y||_G}(\xi) \leq A_{||x||_G}(\xi) + A_{||y||_G}(\xi)$.

The pair $(X, ||\cdot||_G)$ is called a gradual normed linear space (GNLS).

Definition 2.4. [7] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually bounded if for every $\xi \in (0, 1]$, there exists $B = B(\xi) > 0$ such that $A_{||x_k||_G} \leq B$ for all $k \in \mathbb{N}$.

Definition 2.5. [23] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually convergent to $x \in X$, if for every $\xi \in (0, 1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that $A_{||x_k - x||_G}(\xi) < \varepsilon$, $\forall k \geq N$.

Definition 2.6. [23] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually Cauchy, if for every $\xi \in (0, 1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that $A_{||x_k - x_j||_G}(\xi) < \varepsilon$, $\forall k, j \geq N$.

Theorem 2.1. *([23], Theorem 3.6) Let $(X, ||\cdot||_G)$ be a GNLS, then every gradually convergent sequence in X is also a gradually Cauchy sequence.*

Example 2.1. [23] Let $X = \mathbb{R}^m$ and for $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m, \xi \in (0, 1]$, define $||\cdot||_G$ by

$$A_{||x||_G}(\xi) = e^\xi \sum_{i=1}^{m} |x_i|.$$

Then, $||\cdot||_G$ is a gradual norm on $\mathbb{R}^m$ and $(\mathbb{R}^m, ||\cdot||_G)$ is a GNLS.

Definition 2.7. [13] Let $\theta = (k_n)$ be a lacunary sequence and (x_k) be a real valued sequence. Then (x_k) is said to be lacunary statistically Cauchy (in short $S_\theta-$Cauchy) if there exists a subsequence $(x_{k'(n)})$ of (x_k) such that the following three conditions hold:
(i) For any n, $k'(n) \in I_n$;
(ii) $(x_{k'(n)})$ is convergent to x as $n \to \infty$;
(iii) For any $\varepsilon > 0$, $\lim\limits_n \frac{1}{h_n} \mid \{k \in I_n : |x_k - x_{k'(n)}| \geq \varepsilon\} \mid = 0$.

Theorem 2.2. [13] *The real valued sequence (x_k) is $S_\theta-$convergent if and only if it is $S_\theta-$Cauchy.*

Definition 2.8. Let $(X, ||\cdot||_G)$ be any GNLS. We define the new sequence spaces $|\sigma_1(G)|$ and $N_\theta(G)$ as follows:

$$|\sigma_1(G)| = \left\{ (x_k) : \text{for some } x \in X \text{ and for all } \xi \in (0, 1],\ \lim_n \frac{1}{n} \left(\sum_{k=1}^{n} A_{||x_k - x||_G}(\xi) \right) = 0 \right\}$$

and $\quad N_\theta(G)=\left\{(x_k): \text{for some } x\in X \text{ and for all } \xi\in(0,1],\ \lim_n \frac{1}{h_n}\left(\sum_{k\in I_n} A_{||x_k-x||_G}(\xi)\right)=0\right\}.$

Definition 2.9. Let (x_k) be a sequence in the GNLS $(X,||\cdot||_G)$. Then (x_k) is said to be gradually statistically convergent (in short $S(G)$ convergent) to $x\in X$ if for every $\xi\in(0,1]$ and $\varepsilon>0$, the set $\{k\in\mathbb{N}: A_{||x_k-x||_G}(\xi)\geq\varepsilon\}$ has natural density zero. Symbolically, $S(G)-\lim x_k=x$ or $x_k\to x(S(G))$. Further, we use $S(G)$ to denote the collection of all gradually statistical convergent sequences in X.

Throughout the paper, for simplicity we use $\mathbf{0}$ to denote the $m-$tuple $(0,0,....0,0)$.

3. Main Results

Definition 3.1. Let (x_k) be a sequence in the GNLS $(X,||\cdot||_G)$ and θ be any lacunary sequence. Then (x_k) is said to be gradually $S_\theta-$convergent (or shortly $S_\theta(G)-$convergent) to $x\in X$ if for every $\varepsilon>0$ and $\xi\in(0,1]$,

$$\lim_n \frac{1}{h_n}\mid\{k\in I_n: A_{||x_k-x||_G}(\xi)\geq\varepsilon\}\mid=0.$$

In this case, we write, $S_\theta(G)-\lim x_k=x$ or $x_k\to x(S_\theta(G))$. Further, we use $S_\theta(G)$ to denote the collection of all gradually $S_\theta-$convergent sequences in X.

Example 3.1. Let $X=\mathbb{R}^m$ and $||\cdot||_G$ be the norm defined in Example 2.1. Consider $\theta=(\theta_n)$ defined by $\theta_n=\begin{cases}0, & n=0\\ 3^n, & n\geq1\end{cases}$. Then, the sequence (x_k) in $\mathbb{R}^m$ defined as

$$x_k=\begin{cases}(0,0,...,0,m), & if\ k=p^2, p\in\mathbb{N}\\ (0,0,.....,0,0), & otherwise\end{cases}$$

is gradually $S_\theta-$convergent to $\mathbf{0}$ in $\mathbb{R}^m$.

Justification. We have,

$$\begin{aligned}\lim_n \frac{1}{h_n}\mid\{k\in I_n: A_{||x_k-\mathbf{0}||_G}(\xi)\geq\varepsilon\}\mid &= 3\lim_n\frac{1}{3^n}\mid\{k\in(3^{n-1},3^n]: A_{||x_k-\mathbf{0}||_G}(\xi)\geq\varepsilon\}\mid\\ &\leq 3\lim_n\frac{1}{3^n}\mid\{k\leq 3^n: A_{||x_k-\mathbf{0}||_G}(\xi)\geq\varepsilon\}\mid\\ &\leq 3\lim_n\frac{[\sqrt{n}]}{n},\ \textit{where } [p] \textit{ denotes the largest integer } \leq p\\ &=0.\end{aligned}$$

Hence we conclude that $x_k\to\mathbf{0}(S_\theta(G))$.

Example 3.2. Let $X=\mathbb{R}$ and for any $x\in\mathbb{R}$, let $||\cdot||_G$ be the norm defined as $A_{||x||_G}=e^\xi|x|$. Consider the sequence $\theta=(\theta_n)$ defined in Example 3.1. Then the sequence (x_k) in X defined as $x_k=k^2$ is not $S_\theta(G)-$convergent.

Justification. For any $x\in\mathbb{R}$, we have $x\leq0$ or $x>0$. Then for each of the following cases, (x_k) will not $S_\theta(G)-$convergent to x.
Case-I: If $x\leq0$, we choose $\varepsilon=\frac{1}{2}e^\xi$. Then we have,

$$\lim_n\frac{1}{h_n}\mid\{k\in I_n: A_{||x_k-x||_G}(\xi)\geq\varepsilon\}\mid=\lim_n\frac{3}{n}\mid\{k\in(3^{n-1},3^n]: A_{||k^2-x||_G}(\xi)\geq\frac{1}{2}e^\xi\}\mid=1.$$

Case-II: If $x > 0$, then there exists $k_0 \in \mathbb{N}$ such that $x_{k_0-1} \leq x \leq x_{k_0}$.

Subcase-I: If $0 < x < 1$, then choose $\varepsilon = \frac{e^{\xi}}{2}\min\{x, 1-x\}$. Then, it is easy to verify that $\lim_n \frac{1}{h_n} \mid \{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \mid = 1$.

Subcase-II: If $x \geq 1$, then choose $\varepsilon = \frac{e^{\xi}}{2}\min\{x - x_{k_0-1}, x_{k_0} - x\}$. Then, it is easy to verify that $\lim_n \frac{1}{h_n} \mid \{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \mid = 1$.

From the above case study, we can conclude that (x_k) is not $S_\theta(G)-$convergent.

Theorem 3.1. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$ such that $x_k \to x(S_\theta(G))$ for a fixed θ. Then x is unique.*

Proof. If possible suppose $x_k \to x(S_\theta(G))$ and $x_k \to y(S_\theta(G))$ for some $x \neq y$ in X. Then for any $\xi \in (0,1]$ and $\varepsilon > 0$,

$$\lim_n \frac{1}{h_n} \mid \{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \mid = 0$$

and

$$\lim_n \frac{1}{h_n} \mid \{k \in I_n : A_{||x_k - y||_G}(\xi) \geq \varepsilon\} \mid = 0.$$

Therefore, $M = \{k \in I_n : A_{||x_k - x||_G}(\xi) < \varepsilon\} \cap \{k \in I_n : A_{||x_k - y||_G}(\xi) < \varepsilon\} \neq \emptyset$. Choose $\varepsilon = A_{||\frac{x-y}{2}||_G}(\xi)$. Then, for any $p \in M$, we have

$$2\varepsilon = A_{||x-y||_G}(\xi) \leq A_{||x_p - x||_G}(\xi) + A_{||x_p - x||_G}(\xi) < \varepsilon + \varepsilon = 2\varepsilon, \text{ a contradiction.}$$

Hence we must have $x = y$. □

Theorem 3.2. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, ||\cdot||_G)$ such that $x_k \to x(S_\theta(G))$ and $y_k \to y(S_\theta(G))$. Then,*
(i) $x_k + y_k \to x + y(S_\theta(G))$ and (ii) $cx_k \to cx(S_\theta(G))$ for any $c \in \mathbb{R}$.

Proof. (i) Since $x_k \to x(S_\theta(G))$ and $y_k \to y(S_\theta(G))$, so for every $\xi \in (0,1]$ and $\varepsilon > 0$,

$$\lim_n \frac{1}{h_n} \mid C_1 \mid = 0 \text{ and } \lim_n \frac{1}{h_n} \mid C_2 \mid = 0, \tag{1}$$

where $C_1 = \{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \frac{\varepsilon}{2}\}$ and $C_2 = \{k \in I_n : A_{||y_k - y||_G}(\xi) \geq \frac{\varepsilon}{2}\}$. Now since the inclusion $\{k \in I_n : A_{||x_k + y_k - x - y||_G}(\xi) \geq \varepsilon\} \subseteq C_1 \cup C_2$ holds, we must have $\frac{1}{h_n} \mid \{k \in I_n : A_{||x_k + y_k - x - y||_G}(\xi) \geq \varepsilon\} \mid \leq \frac{1}{h_n} \mid C_1 \mid + \frac{1}{h_n} \mid C_2 \mid$ and consequently from (1) we have, $\lim_n \frac{1}{h_n} \mid \{k \in I_n : A_{||x_k + y_k - x - y||_G}(\xi) \geq \varepsilon\} \mid = 0$. This completes the proof.
(ii) Proof of this part is easy so omitted. □

Theorem 3.3. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$ and $\theta = (\theta_n)$ be a lacunary sequence. Then,*
(i) If $x_k \to x(N_\theta(G))$ then $x_k \to x(S_\theta(G))$ although the reverse is not necessarily true.
(ii) The reverse of (i) holds if (x_k) is gradually bounded.

Proof. (i) Let $\varepsilon > 0$ be arbitrary and $x_k \to x(N_\theta(G))$. Then, the proof follows directly from the following fact:

$$\sum_{k \in I_n} A_{||x_k - x||_G}(\xi) \geq \sum_{\substack{k \in I_n \\ A_{||x_k - x||_G}(\xi) \geq \varepsilon}} A_{||x_k - x||_G}(\xi) \geq \varepsilon \mid \left\{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\right\} \mid .$$

Now for the converse part, we construct a counterexample by considering the gradual normed space $(\mathbb{R}^m, ||\cdot||_G)$, where $||\cdot||_G$ is the norm defined in Example 2.1. Let θ be given and x_k be $(0,0,...,1),(0,0,...,2),...,(0,0,...,[\sqrt{h_n}])$ at the first $[\sqrt{h_n}]$ integers in I_n, and $x_k = \mathbf{0}$ otherwise. Then, we have for any $\varepsilon > 0$ with $0 < \varepsilon e^{\xi} \leq 1$,

$$\lim_n \frac{1}{h_n} \mid \{k \in I_n : A_{||x_k - \mathbf{0}||_G}(\xi) \geq \varepsilon\} \mid = \lim_n \frac{[\sqrt{h_n}]}{h_n} = 0.$$

Hence, $x_k \to \mathbf{0}(S_\theta(G))$. On the other hand,

$$\lim_n \frac{1}{h_n} \left(\sum_{k \in I_n} A_{||x_k - \mathbf{0}||_G}(\xi) \right) = \lim_n \frac{[\sqrt{h_n}]([\sqrt{h_n}]+1)}{2h_n} = \frac{1}{2} \neq 0$$

which means that $x_k \nrightarrow \mathbf{0}(S_\theta(G))$.
(ii) Suppose $x_k \to x(S_\theta(G))$ and (x_k) is gradually bounded. Then, gradual boundedness of (x_k) implies the existence of $B = B(\xi) > 0$ such that $A_{||x_k - x||_G} \leq B$ for all $k \in \mathbb{N}$. Then for given $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{h_n} \left(\sum_{k \in I_n} A_{||x_k - x||_G}(\xi) \right) \\
=&\frac{1}{h_n} \left(\sum_{\substack{k \in I_n \\ A_{||x_k - x||_G}(\xi) \geq \varepsilon}} A_{||x_k - x||_G}(\xi) \right) + \frac{1}{h_n} \left(\sum_{\substack{k \in I_n \\ A_{||x_k - x||_G}(\xi) < \varepsilon}} A_{||x_k - x||_G}(\xi) \right) \\
\leq&\frac{B}{h_n} \mid \left\{ k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon \right\} \mid + \varepsilon;
\end{aligned}$$

which consequently implies that $x_k \to x(N_\theta(G))$. □

In the following lemmas, we investigate the inclusion relations between the sets $S(G)$ and $S_\theta(G)$ under some restrictions on θ and we will use q_n to denote the ratio $\frac{k_n}{k_{n-1}}$.

Lemma 3.4. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then for any lacunary sequence θ, $S(G) - \lim x_k = x$ implies $S_\theta(G) - \lim x_k = x$ if and only if $\liminf_n q_n > 1$. Further if $\liminf_n q_n = 1$, then there exists a $S(G)-$convergent sequence which is not $S_\theta(G)-$convergent to any limit.*

Proof. Let us first assume $\liminf_n q_n > 1$. Then, for sufficiently large n, there exists an $\nu > 0$ such that $q_n > 1 + \nu$ which further implies $\frac{k_n}{h_n} \leq \frac{1+\nu}{\nu}$. Now as $S(G) - \lim x_k = x$, so for any $\varepsilon > 0$ and for sufficiently large n, the following inequation

$$\begin{aligned}
\frac{1}{h_n} \mid \left\{ k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon \right\} \mid &= \frac{k_n}{h_n} \frac{1}{k_n} \mid \left\{ k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon \right\} \mid \\
&\leq \frac{1+\nu}{\nu} \frac{1}{k_n} \mid \left\{ k \leq k_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon \right\} \mid
\end{aligned}$$

yields that $S_\theta(G) - \lim x_k = x$.

For the converse part, take $\liminf_n q_n = 1$. We intend to form a sequence which is $S(G)-$convergent but is not $S_\theta(G)-$convergent to any limit. Proceeding as in

([10], page-510 and [14], page-46), we can select a subsequence (k_{n_j}) of the lacunary sequence θ satisfying $\frac{k_{n_{j-1}}}{k_{n_j}} > \frac{j}{j+1}$ and $\frac{k_{n_{j-1}}}{k_{n_j-1}} < j$, where $n_j - n_{j-1} \geq 2$. Now we define a gradually bounded sequence (x_k) in $(\mathbb{R}^m, ||\cdot||_G)$ (where $||\cdot||_G$ is the norm defined in Example 2.1) as follows:

$$x_k = \begin{cases} (0,0,...,0,1) & k \in I_{n_j}, j = 1,2,3,... \\ \mathbf{0}, & otherwise \end{cases}.$$

Then, for any $x \in \mathbb{R}^m$, we have

$$\frac{1}{h_{n_j}}\left(\sum_{k\in I_{n_j}} A_{||x_k - x||_G}(\xi)\right) = A_{||(0,0,...,0,1)-x||_G}(\xi), j = 1,2,3,...$$

and

$$\frac{1}{h_{n_j}}\left(\sum_{k\in I_n} A_{||x_k - x||_G}(\xi)\right) = A_{||x||_G}(\xi), \text{ for } n \neq n_j$$

which as a consequence gives

$$\lim_n \frac{1}{h_n} \mid \{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}| \neq 0$$

i.e., $x_k \nrightarrow x(S_\theta(G))$.

However (x_k) is $S(G)-$convergent, since if p is an integer that is sufficiently large, we can have a unique j satisfying $k_{n_j-1} < p \leq k_{n_{j+1}-1}$ and then

$$\frac{1}{p}\sum_{k=1}^{p} A_{||x_k||_G}(\xi) \leq \frac{k_{n_{j-1}} + h_{n_j}}{k_{n_j-1}} \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}$$

as $p \to \infty$, it follows that $j \to \infty$. Hence, $(x_k) \in |\sigma_1(G)|^0$. Now by applying the technique of proof of Theorem 2.1 of [3], one can show that (x_k) is $S(G)-$convergent. □

Lemma 3.5. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, for any lacunary sequence θ, $S_\theta(G)-\lim x_k = x$ implies $S(G)-\lim x_k = x$ if and only if $\limsup_n q_n < \infty$. Further, if $\limsup_n q_n = \infty$, then there exists a $S_\theta(G)-$convergent sequence which is not $S(G)-$convergent to any limit.*

Proof. First assume $\limsup_n q_n < \infty$ along with $S_\theta(G) - \lim x_k = x$. Then there is a $B > 0$ such that $q_n < B$ holds for all n. Let N_n denote the cardinal number of the set $\{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}$. Then, by our assumption, for given $\eta > 0$, there is an $n_0 \in \mathbb{N}$ such that $\forall n > n_0$, $\frac{N_n}{h_n} < \eta$. Let $M = \max\{N_1, N_2, ..., N_{n_0}\}$ and let t be an integer satisfying $k_{n-1} < t < k_n$. Then we have,

$$\begin{aligned}
\frac{1}{t} \mid \{k \leq t : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \mid &\leq \frac{1}{k_{n-1}} \mid \{k \leq k_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \mid \\
&= \frac{1}{k_{n-1}} \{N_1 + N_2 + ... + N_{n_0} + N_{n_0+1} + ... + N_n\} \\
&\leq \frac{M}{k_{n-1}} n_0 + \frac{1}{k_{n-1}} \{h_{n_0+1} \frac{N_{n_0+1}}{h_{n_0+1}} + ... + h_n \frac{N_n}{h_n}\} \\
&\leq \frac{n_0 M}{k_{n-1}} + \frac{1}{k_{n-1}} \left(\sup_{n > n_0} \frac{N_n}{h_n} \right) \{h_{n_0+1} + ... + h_n\} \\
&\leq \frac{n_0 M}{k_{n-1}} + \eta \frac{k_n - k_{n_0}}{k_{n-1}} \\
&\leq \frac{n_0 M}{k_{n-1}} + \eta q_n \\
&\leq \frac{n_0 M}{k_{n-1}} + \eta B
\end{aligned}$$

which immediately gives $S(G) - \lim x_k = x$.

Conversely, suppose $\limsup_n q_n = \infty$. We intend to form a sequence that is $S_\theta(G)-$convergent but is not $S(G)-$convergent to any limit. Following the idea in ([10], page-511 and [14], page-47), we can construct a subsequence (k_{n_j}) of the lacunary sequence $\theta = (k_n)$ such that $q_{n_j} > j$. Now we define a gradually bounded sequence (x_k) in $(\mathbb{R}^m, || \cdot ||_G)$ (where $|| \cdot ||_G$ is the norm defined in Example 2.1) as follows:

$$x_k = \begin{cases} (0, 0, ..., 0, 1) & k_{n_j - 1} < k < 2k_{n_j - 1}, j = 1, 2, ... \\ \mathbf{0}, & otherwise \end{cases}.$$

Proceeding as in ([10], page-511) one can show $(x_k) \in N_\theta(G)$ but $(x_k) \notin |\sigma_1(G)|$. Now by virtue of Theorem 1(i) of ([14], page-44), we can say that (x_k) is $S_\theta(G)-$convergent whereas using a similar procedure of Theorem 2.1 of [3], one can easily show that (x_k) is not $S(G)-$convergent. □

The combination of the above two lemmas leads us to the following theorem:

Theorem 3.6. *Let (x_k) be a sequence in the GNLS $(X, || \cdot ||_G)$. Then, for any lacunary sequence θ, $S_\theta(G) - \lim x_k = S(G) - \lim x_k$ if and only if $1 \leq \liminf_n q_n \leq \limsup_n q_n < \infty$.*

Definition 3.2. Let $\theta = (k_n)$ be a lacunary sequence and (x_k) be any sequence in the GNLS $(X, || \cdot ||_G)$. Then (x_k) is said to be gradually lacunary statistical Cauchy (in short $S_\theta(G)-$Cauchy) if there exists a subsequence $(x_{k'(n)})$ of (x_k) such that the following three conditions hold:
(i) For any n, $k'(n) \in I_n$;
(ii) $(x_{k'(n)})$ is gradually convergent to x as $n \to \infty$ for some $x \in X$;
(iii) For any $\varepsilon > 0$ and $\xi \in (0, 1]$, $\lim_n \frac{1}{h_n} \mid \{k \in I_n : A_{||x_k - x_{k'(n)}||_G}(\xi) \geq \varepsilon\} \mid= 0$.

Theorem 3.7. *Let (x_k) be any sequence in the GNLS $(X, || \cdot ||_G)$. Then, (x_k) is $S_\theta(G)-$convergent if and only if (x_k) is $S_\theta(G)-$Cauchy.*

Proof. Suppose $x_k \to x(S_\theta(G))$ and for each $m \in \mathbb{N}$, $K(\xi, m)$ denote the set $\{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) < \frac{1}{m}\}$. Then, for any $m \in \mathbb{N}$, the inclusion $K(\xi, m+1) \subseteq K(\xi, m)$ holds and we have $\lim_n \frac{|K(\xi,m)\cap I_n|}{h_n} = 1$. Choose $r(1)$ such that $n \geq r(1)$ implies $\frac{|K(\xi,1)\cap I_n|}{h_n} > 0$ i.e., $K(\xi, 1) \cap I_n \neq \emptyset$. Next choose $r(2) > r(1)$ such that $n \geq r(2)$ implies $K(\xi, 2) \cap I_n \neq \emptyset$. Then, for each n satisfying $r(1) < n \leq r(2)$, choose $k'(n) \in I_n$ such that $k'(n) \in I_n \cap K(\xi, 1)$, i.e., $A_{||x_{k'(n)} - x||_G}(\xi) < 1$. Proceeding like this, one can choose $r(p+1) > r(p)$ such that $n > r(p+1)$ implies $K(\xi, p+1) \cap I_n \neq \emptyset$. Then, for any n satisfying $r(p) \leq n < r(p+1)$, choose $k'(n) \in I_n \cap K(\xi, p)$, i.e.,

$$A_{||x_{k'(n)} - x||_G}(\xi) < \frac{1}{p}. \tag{2}$$

Hence, we have $k'(n) \in I_n$ for any r, and Equation (2) implies that $(x_{k'(n)})$ is gradually convergent to x as $n \to \infty$.

Then we have for any $\xi \in (0, 1]$ and $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{h_n} \mid \{k \in I_n : A_{||x_k - x_{k'(n)}||_G}(\xi) \geq \varepsilon\} \mid &\leq \frac{1}{h_n} \mid \{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \frac{\varepsilon}{2}\} \mid \\ &\quad + \frac{1}{h_n} \mid \{k \in I_n : A_{||x_{k'(n)} - x||_G}(\xi) \geq \frac{\varepsilon}{2}\} \mid .\end{aligned}$$

Now using the assumption that $x_k \to x(S_\theta(G))$ and the fact that $(x_{k'(n)})$ is gradually convergent to x, the above inequation yields the result.

Now for the converse part, we assume (x_k) is an $S_\theta-$Cauchy sequence. Then, for any $\xi \in (0, 1]$ and $\varepsilon > 0$,

$$\begin{aligned}\mid \{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \mid &\leq \mid \{k \in I_n : A_{||x_k - x_{k'(n)}||_G}(\xi) \geq \frac{\varepsilon}{2}\} \mid \\ &\quad + \mid \{k \in I_n : A_{||x_{k'(n)} - x||_G}(\xi) \geq \frac{\varepsilon}{2}\} \mid\end{aligned}$$

which as a consequence implies that $x_k \to x(S_\theta(G))$. □

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